



LYAPUNOV STABILITY OF AN SIRS EPIDEMIC MODEL WITH VARYING POPULATION: ECOLOGICAL VS PHYSICAL APPROACHES

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Abstract

We consider a general SIRS epidemiological transmission model for an endemic infection with variable population including a general assumption of density-dependent mortality and vertical transmission. We analyse global stability by the Lyapunov approach, proposing two alternative Lyapunov functions. The first one is based on traditional ecological arguments, while the second one is a 'physical' Lyapunov function based on the energy of an appropriate second order equation. Both functions allow to prove the global stability of the (unique) endemic state under broad - not equivalent - parametric conditions.

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This suggests that using different Lyapunov functions provide useful complementary information on the parametric regions where global stability prevails.

1. Introduction

Since the seminal work initiated a century ago by Kermack and McKendrick (see [5], [8]), Mathematical Epidemiology has undertaken an extraordinary development to the point that mathematical models nowadays represent a key support of public policies aimed to control infectious diseases. All this is based on refinements of a few basic deterministic models, in continuous time, and often expressed by a system of nonlinear ordinary differential equation (ODEs). These basic models share a simple structure, where the population is divided into states or “compartments” representing the status with respect to the infection such as e.g., susceptible, infectious, and immune, as in classical SIR and SIRS models (see [5]), which individuals can visit according to simple transition rules. These simple structures can be generalized allowing to (i) model any type of infectious diseases, ranging from vaccine preventable to vertically or sexually transmitted and to vector-borne [1], to (ii) include any type of intermediate determinants of epidemiological outputs such as e.g., the role of individual's age in transmission, geographic structures, population dynamics, and control variables [1], such as vaccination or treatments, up to the psychological dimension of human behavior [17], and to (iii) include different kinds of nonlinearities in social contacts and transmission processes [5].

Epidemiological models are typically categorised in models for *epidemic* infections, describing the course of a single epidemic outbreak, and models for *endemic* infections, i.e., infection capable to persist in their host's population in the very long run. The main dynamic features of the latter models are the existence of a *disease-free equilibrium* and, provided an appropriate parameter representing the reproduction of infection is above a threshold, of an *endemic equilibrium* where the infection persists in the population in the long term. A key issue regards the (local and/or global) stability of these equilibria.

The involved mathematics follows two alternative approaches. The first one is based on Poincaré-Bendixon theory for planar systems and its recent multidimensional extension (see [3] and [15] for an application). The second one uses the classical Lyapunov direct method, still widely applied (see [9-14], [16], [19] and [20]).

In this article we consider a fairly general SIRS epidemiological model for an endemic infectious disease including vertical transmission. The model extends, by including a general specification of density-dependent mortality, a classical model whose global stability was recently analysed in [19] under - however - very restrictive conditions of structural parameters. We investigate the global stability of the unique endemic equilibrium by a Lyapunov approach, using two alternative Lyapunov functions. The first one is based on traditional epidemiological arguments, while the second one is a 'physical' Lyapunov function based on the total energy of an appropriate second order Newtonian equation associated to the original system.

The outline of this paper is the following: in Section 2 we introduce an SIRS model and discuss the existence and unicity of the endemic equilibrium; in Section 3 we prove the global asymptotic stability of the endemic equilibrium by an ecological Lyapunov function and in Section 4 a similar (but weaker) result is obtained by a physical approach. Concluding remarks and an Appendix containing the analysis of the disease-free equilibrium complete our work.

2. The SIRS Epidemiological Model and its Equilibria

In [19], the following SIRS model was proposed

$$\begin{cases} X' = b(N - pY) - \mu X - \beta X \frac{Y}{N} + \alpha Z \\ Y' = bpY + \beta X \frac{Y}{N} - (\mu + \nu + \delta)Y \\ Z' = \nu Y - (\alpha + \mu)Z, \end{cases} \quad (1)$$

where $b > 0$ and $\mu > 0$ are the birth and death rates, $\beta > 0$ is the transmission rate, $\nu > 0$ is the rate of recovery from infection, $\delta > 0$ is the disease-specific mortality rate, $p \in (0, 1)$ is the fraction of vertically infected newborn and $\alpha > 0$ is the rate of return to susceptibility by loss of immunity. Still in [19] the stability of the endemic equilibrium was analysed by Lyapunov direct method under the very special assumption of constant population size N . However, it is easy to verify that $N = X + Y + Z$ is constant only under the following highly special condition on model parameters

$$\beta(b - \mu)(\alpha + \mu + \nu) = \delta(\alpha + \mu)((pb + \beta) - (\mu + \nu + \delta)).$$

The aim of this paper is therefore to re-analyze the stability of steady states of model (1) under non trivial conditions. To do so, in this section we first reformulate the dynamics of the population by making a general assumption of density-dependent mortality, and then (in the forthcoming sections) we demonstrate the global stability of the (unique) endemic state under broad parametric conditions by using two alternative Lyapunov functions.

Following [6, 7], we assume that $\mu = \mu(N)$ is strictly increasing, with $b > \mu(0)$ and $b < \mu(+\infty)$. The resulting population dynamics is

$$N' = (b - \mu(N))N - \delta Y. \quad (2)$$

Note that, since the equation $y' = (b - \mu(y))y$ admits a unique globally asymptotically stable (GAS) equilibrium y^* , our variant to model (1) can be studied on the positively invariant set

$$\mathcal{D} = \{(X, Y, Z, N) : X \geq 0, Y \geq 0, Z \geq 0, X + Y + Z = N \leq N^*\}.$$

Since N is not constant, it is convenient to pass to fractions $S = X/N$, $I = Y/N$ and $R = Z/N$, obtaining the same epidemiological structure of system (1):

$$\begin{cases} S' = b(1 - S - pI) - (\beta - \delta)SI + \alpha R \\ I' = (\beta S - ((1 - p)b + v + \delta) + \delta I)I \\ R' = vI - (b + \alpha - \delta I)R \end{cases} \quad (3)$$

completed by the following equation for the population:

$$N' = N(b - \mu(N) - \delta I). \quad (4)$$

It is simple to verify that the region

$$\mathcal{D}^{fra} = \{(S, I, R) : S \geq 0, I \geq 0, R \geq 0, S + I + R = 1\} \quad (5)$$

is positive invariant and attractive. Moreover, model (3) always admits the disease-free equilibrium $E_0 = (1, 0, 0) \in \mathcal{D}^{fra}$. Consider the basic reproduction number for system (3):

$$R_0 = \frac{\beta}{\gamma}, \quad \gamma = (1 - p)b + v + \delta.$$

It is possible to show by standard arguments (see the Appendix) that E_0 is the unique equilibrium of (3) when $R_0 \leq 1$, and it is GAS in \mathcal{D}^{fra} if and only if $R_0 \leq 1$, and unstable for $R_0 > 1$.

Next, we prove that, if $R_0 > 1$, a unique endemic equilibrium $E_1 = (S_e, I_e, R_e) \in \mathcal{D}^{fra}$ for (3) exists. Since $S + I + R = 1$, we can consider the reduced system

$$\begin{cases} I' = (\beta(1 - I - R) - \gamma + \delta I)I \\ R' = vI - \delta(\rho - I)R, \end{cases} \quad (6)$$

where $\rho = (b + \alpha)/\delta$, on the invariant set

$$\overline{\mathcal{D}}_0^{fra} = \{(I, R) : I \geq 0, R \geq 0, I + R \leq 1\} \setminus \{(0, 0)\}$$

and show that (6) has a unique positive solution $\hat{E}_1 = (I_e, R_e) \in \overline{\mathcal{D}}_0^{fra}$ when

$R_0 > 1$ (equivalent to $\beta > \gamma$). Note preliminarily that condition $R_0 > 1$ implies $\beta > \delta$ and it can occur $R' = 0$ only if $I < \rho$. By solving $I' = R' = 0$, we obtain the quadratic equation in I

$$P(I) = -\delta(\beta - \delta)I^2 + (\delta\rho(\beta - \delta) + (\beta - \gamma)\delta + \beta\nu)I - \delta\rho(\beta - \gamma) = 0.$$

Since $P(0) < 0$ and

$$\lim_{I \rightarrow \rho^-} P(I) = \beta\nu\rho > 0$$

there is only one solution of $P(I) = 0$ smaller than ρ . Therefore, if $\rho \leq 1$, the proof immediately follows, while if $\rho > 1$, note that

$$P(1) = \delta(\rho - 1)(\gamma - \delta) + \beta\nu > 0,$$

and again the claim follows.

3. Global Stability of the Endemic State: An Ecological Lyapunov Function

We show now that, via Lyapunov's second method based on an ecological Lyapunov function, the endemic equilibrium \hat{E}_1 of system (6) is GAS.

Theorem 1. *If $R_0 > 1$, then the unique endemic equilibrium \hat{E}_1 of system (6) is GAS on \overline{D}_0^{fra} .*

Proof. We first observe that system (6) can be expressed as

$$\begin{cases} I' = (\gamma(R_0 - 1) - (\beta - \delta)I - \beta R)I \\ R' = \nu I + \delta IR - \rho\delta R. \end{cases} \quad (7)$$

Since, by definition, at \hat{E}_1 it holds:

$$\begin{cases} \gamma(R_0 - 1) = (\beta - \delta)I_e + \beta R_e \\ \rho\delta = \nu \frac{I_e}{R_e} + \delta I_e \end{cases} \quad (8)$$

substituting in (6), we can write:

$$\begin{cases} I' = -((\beta - \delta)(I - I_e) - \beta(R - R_e))I \\ R' = \nu \left(I - \frac{I_e}{R_e} R \right) + \delta R(I - I_e). \end{cases} \quad (9)$$

Let us now consider the positive functions on $\overline{D}_0^{f^a} \setminus \{(I_e, R_e)\}$

$$L_1(t) = \omega_1(I - I_e - I_e \ln I); \quad L_2(t) = \omega_2(R - R_e - R_e \ln R),$$

where ω_1 and ω_2 are two positive constants to be determined.

Along the solutions of (9) we have

$$\frac{dL_1(t)}{dt} = \omega_1 \frac{I - I_e}{I} I' = \omega_1(-(\beta - \delta)(I - I_e)^2 - \beta(R - R_e)(I - I_e))$$

and, since $I - \frac{I_e}{R_e} R = -\frac{I}{R_e}(R - R_e) + \frac{R}{R_e}(I - I_e)$,

$$\begin{aligned} \frac{dL_2(t)}{dt} &= \omega_2 \frac{R - R_e}{R} R' \\ &= \omega_2 \left(-\frac{\nu I}{R_e R} (R - R_e)^2 + \left(\frac{\nu}{R_e} + \delta \right) (I - I_e)(R - R_e) \right). \end{aligned}$$

Therefore, function $L_{EE} = L_1 + L_2$ is positive for $(I, R) \neq (I_e, R_e)$ and

$$\begin{aligned} L'_{EE}(t) &= -\omega_1(\beta - \delta)(I - I_e)^2 - \omega_2 \frac{\nu I}{R_e R} (R - R_e)^2 \\ &\quad + \left(\omega_2 \left(\frac{\nu}{R_e} + \delta \right) - \omega_1 \beta \right) (R - R_e)(I - I_e). \end{aligned}$$

Since $\beta > \delta$, by setting

$$\omega_1 = \frac{\omega_2}{\beta} \left(\frac{\nu}{R_e} + \delta \right).$$

L_{EE} is a Lyapunov function and the endemic equilibrium is GAS. \square

The result just obtained straightforwardly extends to the endemic equilibrium E_1 of system (3).

Remark 2. The present characterization of the stability of equilibria allows clear conclusions about the effects of endemicity on the dynamics of the population described by equation (4). Indeed, if the endemic state is GAS, then for large t it holds

$$N' \rightarrow (b - \delta I_e - \mu(N))N$$

which implies that the disease will bring the extinction of the population under the condition:

$$b \leq \delta I_e + \mu(0).$$

Conversely, if the previous condition is not fulfilled, then the stability of the endemic state will force the population to reach the equilibrium state $N_e = \mu^{-1}(b - \delta I_e)$, where the persistent presence of the disease will regulate the population size [4], [6, 7], [18].

4. An ‘energy-like’ Lyapunov Function

In this section we propose an approach yielding an ‘energy-like’ Lyapunov function for the stability of the endemic state. The approach is based on the derivation of an appropriate Newtonian dynamics described by a second-order ODE associated to the original epidemiological system. Setting as time unit $1/\beta$ (i.e., setting $\beta = 1$), the reduced system (6) reads as follows:

$$\begin{cases} I' = I((1 - I - R) - \gamma + \delta I) \\ R' = \nu I - \delta(\rho - I)R. \end{cases} \quad (10)$$

Since we are interested to the behaviour for $I > 0$ in the case where $R_0 > 1$, we can adopt the following change of variables

$$y = \ln(I) \quad (11)$$

with $y \leq 0$, as $I \in (0, 1]$. The first equation of (10) becomes as follows:

$$y' = 1 - \gamma - (1 - \delta)e^y - R \quad (12)$$

i.e.,

$$R = 1 - \gamma - (1 - \delta)e^y - y'. \quad (13)$$

Substituting in the second equation of (10) one gets the following Newtonian equation with unit mass [2]:

$$y'' = -\zeta(y)y' + F(y) \quad (14)$$

i.e.,

$$\begin{cases} y' = v \\ v' = -\zeta(y)v + F(y) \end{cases} \quad (15)$$

on the set

$$\overline{\mathcal{D}}^{New} = \{(y, v) : y \leq 0, -\gamma + \delta e^y \leq v \leq 1 - \gamma - (1 - \delta)e^y\}.$$

Notice that, since $\gamma > \delta$ it follows $-\gamma + \delta e^y < 0$. The variable damping $\zeta(y)$ reads as follows:

$$\zeta(y) = \rho\delta + (1 - 2\delta)e^y = \delta(\rho - e^y) + (1 - \delta)e^y \quad (16)$$

and the force $F(y)$ reads as follows:

$$F(y) = -ve^y + \delta(\rho - e^y)(1 - \gamma - (1 - \delta)e^y) = a_0 - a_1e^y + a_2e^{2y}$$

with

$$a_0 = \rho\delta(1 - \gamma);$$

$$a_1 = \delta\rho(1 - \delta) + (1 - \gamma)\delta + v;$$

$$a_2 = \delta(1 - \delta).$$

Since $1 - \delta > 0$ and $1 - \gamma > 0$, all the three previous coefficients are positive. System (15) has an equilibrium in $\overline{\mathcal{D}}^{New}$ if and only if $y' = v' = 0$ that is if and only if $v = 0$ and $F(y) = 0$. Then, a necessary condition for

the existence of an equilibrium is

$$1 - \gamma - (1 - \delta)e^y > 0$$

that is

$$y < y^* := \ln \frac{1 - \gamma}{1 - \delta} < 0,$$

where the second inequality follows from $\gamma > \delta$. Then we look for equilibria on the set

$$\overline{D}_1^{New} = \{(y, v) : y < y^*, -\gamma + \delta e^y \leq v \leq 1 - \gamma - (1 - \delta)e^y\} \subset \overline{D}^{New}.$$

We demonstrate that there exists a unique $y_e < y^*$ such that $F(y^*) = 0$, that is, $E_2 = (y_e, 0)$ is the unique equilibrium of (15). Since:

$$(1) \lim_{y \rightarrow -\infty} F(y) = a_0^- \text{ as } a_1 > 0;$$

$$(2) F'(y) > 0 \text{ if and only if } y > y_m := \ln\left(\frac{a_1}{2a_2}\right), \text{ where}$$

$$F(y_m) = -\frac{1}{4a_2}(a_1^2 - 4a_0a_2) < 0$$

the existence and unicity of y_e follows by

$$\lim_{y \rightarrow (y^*)^-} F(y) = -v \frac{1 - \gamma}{1 - \delta} < 0.$$

As a consequence the associated potential energy $U(y)$, defined as

$$U(y) = -\int_{y_e}^y F(z) dz = -a_0y + a_1e^y - \frac{a_2}{2}e^{2y} - \left(-a_0y_e + a_1e^{y_e} - \frac{a_2}{2}e^{2y_e}\right)$$

has a minimum at y_e .

We show now that, under suitable parameters configurations the energy function [2]

$$E(t) = \frac{v^2}{2} + U(y)$$

is a Lyapunov function on the set

$$\overline{\mathcal{D}}_2^{New} = \{(y, v) : y < \ln(\rho), -\gamma + \delta e^y \leq v \leq 1 - \gamma - (1 - \delta)e^y\} \subset \overline{\mathcal{D}}^{New}.$$

In fact, $E(t)$ is definite positive and:

$$E'(t) = -v^2 \zeta(y).$$

Observe that (i) if $0 < \delta \leq 1/2$, then by the first representation of $\zeta(y)$ in (16), we have $\zeta(y) > 0$ for every $y \leq 0$; in this case the global stability of $(y_e, 0)$ holds true on $\overline{\mathcal{D}}^{New}$; (ii) if $1/2 < \delta < 1$ and $\rho \geq 1$ by the second representation of $\zeta(y)$ in (16), we obtain $\zeta(y) > 0$ for each $y \leq 0$ and again $(y_e, 0)$ is GAS on $\overline{\mathcal{D}}^{New}$; (iii) when $1/2 < \delta < 1$ and $\rho < 1$ with an argument similar to (ii) we obtain the global stability of $(y_e, 0)$ just on the set

$$\overline{\mathcal{D}}_2^{New} = \{(y, v) : y < \ln(\rho), -\gamma + \delta e^y \leq v \leq 1 - \gamma - (1 - \delta)e^y\} \subset \overline{\mathcal{D}}^{New}.$$

5. Concluding Remarks

When facing a new epidemiological system having a unique endemic state - as is very often the case - a key task of mathematical epidemiology remains that of demonstrating the global stability of this endemic state. In this paper we have considered a fairly general 'simple' epidemiological SIRS model for an endemic infectious disease including vertical transmission and a general density-dependent mortality rate. After having proven the uniqueness of its endemic state, we have investigated the GAS of the endemic state by Lyapunov second method, using two alternative types of Lyapunov functions, the first one resembling the classical 'ecological' Lyapunov

functions, while the second one was based on the total energy of a Newtonian dynamics associated to the original SIRS system. In particular, the Newtonian Lyapunov - applied here for the first time to an epidemiological system (to the best of our knowledge) - can be a very useful tool that can be applied when other approaches fail, or can be used to complement other approaches e.g., to expand the size of the parametric region where global stability prevails. Clearly other approaches can be used to investigate problems like this, departing from the Poincaré-Bendixson machinery. Nonetheless, providing a range of alternative tools that expand the catalog of possible approaches, as done here, can prove useful to solve future, harder, problems.

6. Appendix

Following [4], we show that E_0 is the unique equilibrium for model (6) when $R_0 \leq 1$. To this aim, observe that condition $I' = 0$, for $I > 0$, can be rewritten as

$$(\beta - \delta)I = -\gamma(1 - R_0) - \beta R.$$

Multiplying term by term this equality by $S > 0$, and substituting the expression on the right into equation $S' = 0$, one obtains the equality

$$\gamma(1 - R_0)S + (\alpha + \beta S)R + b(1 - S - pI) = 0$$

that, as $1 - S - I > 0$, shows that no solutions can exist when $R_0 \leq 1$.

The following theorem states the global stability of the disease-free equilibrium E_0 .

Theorem 3. *The disease-free equilibrium $E_0 \in \mathcal{D}^{fra}$ is GAS in \mathcal{D}^{fra} if and only if $R_0 \leq 1$, and unstable for $R_0 > 1$.*

Proof. By linearization it is easy to see that E_0 is locally asymptotically stable when $R_0 < 1$, and unstable when $R_0 > 1$. We assume in the following $R_0 \leq 1$ and we show that $L_{DFE}(t) = I(t)$ is a Lyapunov function. In fact, as

$S = 1 - I - R$, we can write

$$I' = (\gamma(R_0 - 1) - \beta R - (\beta - \delta)I)I.$$

Then, by $R_0 \leq 1$ and $\beta > \delta$, we immediately obtain $I' \leq 0$. If $\delta \geq \beta$, for $R_0 < 1$, as $\gamma(R_0 - 1) = \beta - \gamma$, it follows

$$I' = (-\beta R - (\delta - \beta)(1 - I) - ((1 - p)b + \nu))I \leq 0.$$

Since the DFE is the only positively invariant subset of $\{(S, I, R) : I' = 0\}$, by LaSalle Invariance Principle, we conclude that E_0 is GAS for $R_0 \leq 1$. \square

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