Variable Annuities with State-Dependent Fees

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Abstract In this paper we consider a variable annuity with guarantees at death and maturity financed through the application of state-dependent fees. We define a general valuation model for them, and propose to apply the LSMC approach in order to analyse the interaction between fee rates, death/maturity guarantees, fee thresholds and surrender penalties under alternative model assumptions and policyholder behaviours. However, special care is needed in the numerical implementation of this approach, due to the shape of the surrender region. We can stem the numerical errors arising in the regression step by using suitable arrangements of the LSMC valuation algorithm.

Keywords Variable annuities · State-dependent fees · Surrender option · LSMC

1 Introduction

Variable annuities are very flexible life insurance contracts that can package living and death benefits with a number of possible guarantees against financial or biometric risks. Typically, a lump sum premium is paid at inception, and is invested in well diversified mutual funds. This initial investment sets up a reference portfolio (*policy account*) and all guarantees are financed by periodical proportional deductions (*fees*) from this account.

Guarantees are often set in such a way that at least the lump sum premium is totally recouped. Then, when the account value is high, the policyholder has an incentive to surrender the contract, stopping to pay high fees for an out-of-the-money guarantee. Conversely, when the account value is low, the policyholder

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pays a low fee for an in-the-money guarantee. Summing up, there is an unfair misalignment between costs incurred by the insurer and premiums (fees) to cover them, and a great incentive, for policyholders, to abandon their contracts when they become uneconomical. To eliminate this misalignment and reduce the surrender incentive insurers can adopt a *threshold expense structure*, or *state-dependent fees*, according to which the fees, still proportional to the account value, are paid only if this value is below a given threshold.

In this paper we consider a variable annuity which provides guarantees at death and maturity financed through the application of a state-dependent fee structure, as defined first in [3] and extensively analysed in [4] and [5]. We define a general valuation model for such guarantees, along the lines of [2], and test the application of Least Squares Monte Carlo methods (LSMC), that allow to analyse numerically the interaction between fee rates, death/maturity guarantees, fee thresholds and surrender penalties under alternative model assumptions and policyholder behaviours. In particular, special care is needed when applying these techniques, due to the shape of the surrender region. We can stem the numerical errors arising in the regression step by using suitable arrangements of the LSMC valuation algorithm based on a theoretical result.

The paper is structured as follows. In Sect. 2 we describe the structure of the contract. In Sect. 3 we present our valuation framework. Section 4 is devoted to a discussion of the problems encountered in the numerical implementation of the model and the settlements to overcome them.

2 The Structure of the Contract

Consider a single premium variable annuity contract which provides guarantees at death and maturity. We denote by P the single premium, 0 the time of issuance, T the contract maturity, and assume that the death benefit is paid upon death within the contract maturity. The single premium is invested in a well diversified mutual fund with unit price process S, and the (net) value of the accumulated investments in this fund is referred to as the *policy account value*. We denote by A_t this value at time t. The cost of the guarantees is recouped through the application of a proportional deduction from this account, at a rate denoted by φ (fee rate). However, this deduction is assumed to be made only when the account value is below a given threshold, denoted by β, i.e., we adopt a *state-dependent fee* structure. Of course, in the degenerate case of $\beta = \infty$ (no barrier) we recover a *constant fee* structure.

Both death and maturity benefits contain a guarantee of the *roll-up* type, with the same roll-up rate δ. The death benefit is given by $b_\tau^D = \max\{A_\tau, Pe^{\delta\tau}\}$, $\tau \le T$, and the survival benefit is $b_T^M = \max\{A_T, Pe^{\delta T}\}$, $\tau > T$, where we have denoted by τ the residual lifetime of the policyholder.

We assume that the contract can be surrendered at any time before maturity, if the insured is alive, and that, in case of surrender at time $\lambda < T \wedge \tau$, the policyholder receives a cash amount, called surrender value, given by $b_\lambda^S = A_\lambda(1 - p_\lambda)$, where p_λ is a penalty rate, possibly time dependent and such that $0 \le p_\lambda < 1$ for any λ.

3 Valuation Framework

A key-element in the valuation of the contract from the insurer's point of view is constituted by the behavioral risk. The policyholder, in fact, can choose among a set of possible actions, such as partial or total withdrawal (i.e., surrender), selection of new guarantees, switch between different reference funds, and so on. In particular, in [2] the possible policyholder behaviours are classified, with respect to the aspect concerning partial or total withdrawals, into three categories, characterized by an increasing level of rationality: *static*, *mixed* and *dynamic*. Although in principle partial withdrawals from the account value may be admitted also within the specific contract analysed in this paper, the most relevant valuation approaches for it are the first two, static and mixed.

3.1 The Static Approach

Under this approach it is assumed that the policyholder keeps her contract until its natural termination, that is death or maturity, without making any partial or total withdrawal from her policy account value.

The instantaneous evolution of the account value while the contract is still in force can be formally described as follows:

$$\frac{dA_t}{A_t} = \frac{dS_t}{S_t} - \varphi 1_{\{A_t < \beta\}} dt,$$

where $A_0 = P$ and 1_C denotes the indicator of the event C. Then, the return on the account value is that of the reference fund, adjusted for fees that are applied, according to the fixed rate φ, only when A_t is below the barrier β.

The contract value at time $t < T$, on the set $\{\tau > t\}$, is thus given by

$$V_t = E\left[b_\tau^D \frac{B(t)}{B(\tau)} 1_{\{\tau \le T\}} + b_T^M \frac{B(t)}{B(T)} 1_{\{\tau > T\}} \middle| \mathscr{F}_t \right],$$

where $B(u) = e^{\int_0^u r_v dv}$ defines the bank account value accumulated with the risk-free rate r, the expectation is taken under a given risk-neutral measure and the filtration $\mathbb{F} \doteq (\mathscr{F}_t)_{t \ge 0}$ carries knowledge on all financial and biometric variables.

3.2 *The Mixed Approach*

Under this approach it is assumed that, at any time of contract duration, the policyholder chooses whether or not to exercise the surrender option, and her decision is aimed at maximizing the current value of the contract payoff.

The instantaneous evolution of the account value is the same as in the static approach, while the contract value at time $t < T$, on the set $\{\tau > t,\ \lambda \geq t\}$, is the solution of the following optimal stopping problem: $V_t = \sup_{\lambda \in \mathbb{T}_t} V_t(\lambda)$, where

$$V_t(\lambda) = E\left[b_\tau^D \frac{B(t)}{B(\tau)} 1_{\{\tau \leq T \wedge \lambda\}} + b_T^M \frac{B(t)}{B(T)} 1_{\{\tau > T,\, \lambda \geq T\}} + b_\lambda^S \frac{B(t)}{B(\lambda)} 1_{\{\lambda < \tau \wedge T\}} \middle| \mathcal{F}_t \right]$$

is the contract value given the surrender time λ, and $\mathbb{T}_t$ is the set of stopping times taking values in $[t, +\infty)$.

Note that the contract value V_t can also be expressed as $V_t = \max\{V_t^c, b_t^S\}$, with V_t^c denoting the continuation value, given by $V_t^c = \sup_{\lambda \in \mathbb{T}_t^c} V_t(\lambda)$, where $\mathbb{T}_t^c$ is now the set of stopping times taking values in $(t, +\infty)$.

In particular, in [5] it is proven, under the assumption of lognormality for the price process S and deterministic mortality intensity, that surrender is never optimal (i.e., the continuation value is higher than the surrender benefit) if the account value is above the fee threshold. The intuition behind this result is clear: when $A_t \geq \beta$ the guarantees at death and maturity are offered for free, hence there is no incentive for the policyholder to surrender the contract. We are able to generalize this result, just requiring that the discounted price process is a martingale (under the risk-neutral measure) and financial related variables are independent of mortality (see [1]).

Finally, we note that the contract value under the mixed approach is not less than the corresponding value under the static approach (American versus European-style contract), and the difference between them is the surrender option value.

4 Numerical Implementation

The optimal stopping problem giving the contract value under the mixed approach needs to be tackled numerically. In [4] it is claimed that the Least Squares Monte Carlo techniques are unsuitable to solve it, due to the shape of the surrender region, that is like a corridor (even very strict), thus implying too significant numerical errors. Since we believe that the intrinsic flexibility of Monte Carlo methods is a very important feature, we have tested their application to the solution of the problem. Doing this, we have actually verified that a straightforward application of them is a bit problematic, specially for relatively low levels of the fee, i.e., when very likely the surrender incentive has been completely eliminated leading to a valueless surrender option. In these cases, in fact, the contract value under the static approach turns out to be higher than that under the mixed approach, contradicting the theoretical relation and confirming the claim by Bernard et al. [4]. Given this appears to happen only for low levels of the fee, a possible explanation is that the regression tends to underestimate the continuation value, thus inducing surrender even when this is not the optimal decision. We have observed this behaviour by comparing the residuals plots printed at each regression step for the cases of constant and state-dependent fees. While in the constant case the residuals appeared to be balanced between positive and negative values for all regression steps, in the state-dependent case they tended to shift towards positive values in the last few steps. Since the LSMC algorithm proceeds backward, this means that at the very first surrender decision dates the real continuation values were generally much greater than the predicted ones, leading to earlier and sub-optimal terminations of the contract. Therefore, in the attempt to improve the regression, we have tried several methods, such as changing type and number of basis functions, or using different regression techniques, which however have not brought substantial enhancements. In contrast, the theoretical result mentioned before, according to which the regression step can be skipped when $A_t \geq \beta$, has allowed us to significantly reduce the numerical error.

In the following tables we report some results for the contract value under the mixed approach obtained with Monte Carlo simulation and alternative regression techniques (Least Squares, LS; Generalized Linear Models, GLM; Ridge regression; the Lasso) without skipping the regression step, as well as those obtained with LS by skipping this step (Adj LS), and the values under the static approach (Static). Although, as previously mentioned, we advocate the use of Monte Carlo methods for their flexibility, and hence for the absence of model constraints, in these examples we show results under very simple assumptions, i.e., a constant interest rate, a deterministic mortality intensity and a Geometric Brownian Motion (GBM) for the assets evolution. This is because in this framework we have a benchmark, that is the contract value obtained by using the PDE approach as described in [5]. We refer instead to [1] for a wide range of numerical results under alternative, and rather complex, model assumptions.

In Table 1 we have tried to reproduce some results by MacKay et al. [5], that use the PDE approach to compute the contract values and determine also the fair fee rate, that is a fee rate making the contract value equal to the initial premium. More in detail, we fix a fee rate exactly equal to the fair level reported in [5] for different policyholder ages at inception. The contract parameters are as follows: $P = 100$, $T = 10$, $\delta = 0$, $p_t = 1 - e^{-0.008(10-t)}$, $\beta = 150$. Moreover, the risk-free rate is $r = 0.03$, the assets volatility is $\sigma = 0.165$, and the mortality intensity follows a Makeham law: $\mu_y = 10^{-4}(1 + 3.5 \cdot 1.075^y)$. The number of simulations is 20,000.

Table 1 Contract values estimated with PDE and Monte Carlo methods

Age	Fee	PDE	LS	GLM	Ridge	Lasso	Adj LS	Static
50	0.0167	100.01	99.22	99.08	98.69	98.98	99.44	99.50
60	0.0179	100.00	99.26	99.17	99.09	98.99	100.04	99.83
70	0.0204	100.01	99.35	99.02	98.88	98.85	99.49	99.45

Table 2 Contract values estimated with PDE and Monte Carlo methods

Fee	PDE	LS	GLM	Ridge	Lasso	Adj LS	Static
2%	113.89	112.07	111.73	112.28	112.17	113.14	113.04
6%	101.82	101.05	101.06	101.11	100.97	101.51	101.39
7%	100.52	99.95	100.02	99.87	99.95	100.01	98.70
9%	99.08	98.27	98.27	98.18	98.19	98.40	95.35

In Table 2 we consider contracts with different fee rates, both under the fair level (contract value higher than P), for which the improvement obtained with the LSMC adjustment is more remarkable, and over. We fix now $T = 15$, $\delta = 2\%$, $\beta = 134.98588$, $\sigma = 0.2$, take a policyholder aged 50 years, a constant surrender penalty of 2%, a Weibull mortality intensity $\mu_y = 10.002 \cdot 88.14778^{-10.002} y^{9.002}$, and the same values as before for P, r and the number of simulations.

References

1. Bacinello, A.R., Zoccolan, I.: Variable Annuities with State-Dependent Fees: Valuation, Numerical Implementation, Comparative Static Analysis and Model Risk. DEAMS Research Paper Series, n.1/2017 (2017). Available online at http://hdl.handle.net/10077/15141
2. Bacinello, A.R., Millossovich, P., Olivieri, A., Pitacco, E.: Variable annuities: a unifying valuation approach. Insur. Math. Econ. **49**, 285–297 (2011)
3. Bae, T., Ko, B.: Pricing maturity guarantee under a refracted Brownian motion. Lobachevskii J. Math. **34**, 234–247 (2013)
4. Bernard, C., Hardy, M.R., MacKay, A.: State-dependent fees for variable annuity guarantees. ASTIN Bull. **44**, 559–585 (2014)
5. MacKay, A., Augustyniak, M., Bernard, C., Hardy, M.R.: Risk management of policyholder behavior in equity-linked life insurance. J. Risk Insur. **84**, 661–690 (2017)