Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data

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Abstract. We consider the inverse boundary value problem of determining the potential q in the equation $\Delta u + qu = 0$ in $\Omega \subset \mathbb{R}^n$, from local Cauchy data. A result of global Lipschitz stability is obtained in dimension $n \geqslant 3$ for potentials that are piecewise linear on a given partition of Ω. No sign, nor spectrum condition on q is assumed, hence our treatment encompasses the reduced wave equation $\Delta u + k^2c^{-2}u = 0$ at fixed frequency k.

Keywords: Lipschitz stability, Cauchy data, Green's function, Full Waveform Inversion

1. Introduction

The purpose of this paper is to achieve good stability estimates in the determination of the coefficient $q = \frac{1}{c^2}$ ($c =$ wavespeed) in the Helmholtz type equation

$$\Delta u + k^2 qu = 0 \tag{1.1}$$

in a domain Ω, when all possible Cauchy data $u|_\Sigma$, $\frac{\partial u}{\partial \nu}|_\Sigma$ are known on an open portion Σ of $\partial\Omega$, at a single given frequency k. In view of the well-known exponential ill-posedness of this problem [31] we shall introduce the rather strong *a-priori* assumption on the unknown coefficient q of being piecewise linear. The precise formulation will be given later on in Section 2. Uniqueness of the inverse boundary value problem associated with the Helmholtz equation in dimension $n \geqslant 3$ was established by Sylvester and Uhlmann [41] assuming that the wavespeed is a bounded measurable function.

The inverse boundary value problem associated with the Helmholtz equation has been extensively studied from an optimization point of view primarily using computational experiments. In reflection seismology, iterative methods for this inverse boundary value problem have been collectively referred to as full waveform inversion (FWI). (The term 'full waveform inversion' was supposedly introduced by Pan, Phinney and Odom in [34] with reference to the use of full seismograms information.) Lailly [25]

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and Tarantola [43,44] introduced the formulation of the seismic inverse problem as a local optimization problem using a misfit functional. The misfit functional was originally based on a least-squares criterion, but it needs to be carefully designed to fit the analysis of the inverse boundary value problem on the one hand – possibly, using Hilbert–Schmidt operators as the data – and match actual data acquisition on the other hand. Furthermore, we mention the original work of Bamberger, Chavent & Lailly [8,9] in the one-dimensional case. Initial computational experiments in the two-dimensional case were carried out by Gauthier [21].

The time-harmonic formulation was initially promoted by Pratt and his collaborators [35,37]; he also emphasized the importance of available wide-angle reflection data in [36]. In this line of research, we mention the more recent work of Ben-Hadj-Ali, Operto and Virieux [11].

A key question, namely, that of convergence of such iterative schemes in concert with a well-understood regularization, was left open for almost three decades. It appears that Lipschitz stability estimates and conditional Lipschitz stability estimates provide a platform for convergence analyses of the Landweber iteration [18] and projected steepest descent method [19], respectively, with natural extensions to Newton-type methods.

As mentioned above, the exponential character of instability of the inverse boundary value problem associated with the Helmholtz equation cannot be avoided. However, conditional Lipschitz stability estimates can be obtained: Including discontinuities in the coefficient, Beretta, De Hoop and Qiu [12] showed that such an estimate holds if the unknown coefficient is a piecewise constant function with a known underlying domain partition. Beretta, De Hoop, Qiu and Scherzer [13] also give a quantitative estimate for the stability constant revealing the precise exponential growth with the number of subdomains in the partition. We generalize the conditional Lipschitz stability estimate to piecewise linear functions. Moreover, we use Cauchy data rather than the Dirichlet-to-Neumann map. If we view to (1.1) as the reduced wave equation, the coefficient $q(x)$ equals $\frac{1}{c^2(x)}$, where c is the variable speed of propagation and thus $q > 0$. This implies that 0 might be a Dirichlet (or Neumann) eigenvalue, and even if not, it might be close to an eigenvalue. Therefore it is not convenient for the purpose of stability estimates, to express the boundary data in terms of the well-known D-N map (or the N-D one). We have chosen to express errors on the boundary data in terms of the so-called angle (or distance) between the spaces of Cauchy data viewed as subspaces of a suitable Hilbert space (see below Section 2).

Moreover, since the present estimates are obtained for measurements at one fixed frequency $k > 0$ we convene from now on to set $k = 1$ and we shall also admit that q may be real valued but of variable sign, thus our analysis encompasses more generally the stationary Schrödinger equation

$$\Delta u + qu = 0.$$

Let us emphasize also that Cauchy data are a proxy to data obtained from advanced marine acquisition systems. Here, airgun arrays excite waves underneath the sea surface that is accounted for by a Dirichlet boundary condition, which are detected in possibly variable depth towed dual sensor streamers positioned (on some hypersurface) below the airgun arrays. Dual sensors provide both the pressure and the normal particle velocity forming Cauchy data. To allow favorable paths of (parallel) streamers, in so-called full-azimuth acquisition, one uses two recording vessels with their own sources and two separate source vessels[1]

[1] See, for example, the dual coil shooting full-azimuth acquisition by WesternGeco.

One can exploit conditional Lipschitz stability estimates, via a Fourier transform, in the corresponding time-domain inverse boundary value problem with bounded frequency data. Datchev and De Hoop [17] showed how, via resolvent estimates for the Helmholtz equation, the prerequisites for application of projected steepest descent and Newton-type iterative reconstruction methods to inverse wave problems can be satisfied.

With the objective of obtaining approximate reconstructions in the class of models for which conditional Lipschitz stability estimates hold, from a starting model – the error of which can be estimated as well in this context [18] – compression plays an important role. This is elucidated in the work of De Hoop, Qiu and Scherzer [19] using multi-level schemes based on successive refinement and arises in mitigating the growth of the stability constants with the number of subdomains in the partition or, simply, the number parameters. Indeed, the class of piecewise linear functions provide an excellent way to achieve compression in the presence of discontinuities. The application of wavelet bases in compressing the successive models in iterative methods has been considered by [29,30] in wave-equation tomography in a the framework of sparsity promoting optimization and in FWI by Lin, Abubakar & Habashy [26] for the purpose of reducing the size of the Jacobian.

In FWI one commonly applies a 'nonlinear' conjugate gradient method, a Gauss–Newton method, or a quasi-Newton method (L-BFGS; for a review, see Brossier [14]). For the application of multi-scale Newton methods, see Akcelik [1]. In the gradient method, the step length is typically estimated by a simple line search for which a linearization of the direct problem is used (Gauthier, Virieux & Tarantola [21]). This estimation is challenging in practice and may lead to a failure of convergence. For this purpose, research based on trust region has been studied by Eisenstat & Walker in [20], for FWI see Métivier and others in [32], the method is detailed in [16]. In various approaches based on the Gauss–Newton scheme one accounts just for the diagonal of the Hessian [39]. In certain earthquake seismology applications, one builds the Fréchet derivative or Jacobian (sensitivity, for example by Chen *et al.* [15]) explicitly and then applies LSQR. We also mention the work of Métivier and others in [32] based on Hessian vector multiplication techniques to reduce the cost of a dense Hessian computation. The use of complex frequencies was studied in [23,38].

Coming back to the object of the present paper, let us point out that various new aspects appear in comparison to prior results of stability under assumptions of piecewise constant or piecewise linear coefficients [4,7,12] which require novel arguments.

I) *Singular solutions.* Since we admit that the underlying equation may be in the eigenvalue regime for the Dirichlet or the Neumann boundary value problem we need to construct Green's functions for a boundary of mixed type which is of Dirichlet type on part of the boundary and of complex valued Robin-type on the remaining part. This construction relies on quantitative estimates of unique continuation which take inspiration from an idea of Bamberger and Hua Duong [10] and an iterative procedure for the approximation with the standard fundamental solution of Laplace's equation.
II) *Asymptotics.* In order to determine values of the potential q and its gradient we need singular solutions whose blow up rate is of the order up to $|x|^{-n}$, for this purpose we have to estimate the asymptotic behaviour of the previously found Green's function up to its second derivatives.
III) *Stability at the boundary.* The typical initial step in stability from inverse boundary value problems is the stability at the boundary of the unknown coefficient. For Calderón's problem it is well-known that the stability at the boundary for the conductivity coefficient is of Lipschitz type [3,42]. For the potential coefficient q the situation is different. In fact we are able in general to obtain only Hölder type stability. This fact is related to the different dimensionality of the boundary energy $\int_{\partial\Omega} u \frac{\partial u}{\partial \nu}$

and of the volume integral $\int_\Omega qu^2$ whereas in the conductivity case $((\gamma\nabla u) = 0)$ the equality $\int_{\partial\Omega} u\gamma\frac{\partial u}{\partial\nu} = \int_\Omega \gamma|\nabla u|^2$ provides the right balance.
Nevertheless, under the piecewise linear assumption, using the kind of bootstrap argument introduced in [7] we eventually achieve the desired global Lipschitz stability.

The outline of the paper is as follows. In Section 2 we provide the basic set up. We introduce the spaces of local Cauchy data and their metric structure as subspaces of a Hilbert space in Section 2.1. Next, in Section 2.2 we present the *a-priori* assumptions on the domain and on the potential and we state our main stability result theorem 2.2. In Section 3.1 we state the main propositions which provide the main tools for the proof of theorem 2.2 which is completed in Section 3.2. Section 4 contains the proofs of the various propositions previously stated. Section 4.1 contains the construction of the Green's function for the mixed Dirichlet–Robin boundary value problem and the estimates of its asymptotic behavior. Section 4.2 contains the quantitative estimates of unique continuation adapted for the singular solution $\tilde{S}_{\mathcal{U}_k}$ introduced in Section 3. We conclude in Section 4.3 with the stability estimates at the boundary for the potential q and its normal derivative.

2. Main result

2.1. *Definitions and preliminaries*

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geqslant 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we shall denote with $B_r(x)$, $B'_r(x)$ the open balls in $\mathbb{R}^n$, $\mathbb{R}^{n-1}$ respectively centred at x with radius r and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r\left(x'\right) \times (x_n - r, x_n + r).$$

In the sequel, we shall make a repeated use of quantitative notions of smoothness for the boundaries of various domains. Let us introduce the following notation and definitions.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants r_0, L if there exists $P \in \Sigma$ and there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \left\{x \in Q_{r_0} | x_n > \varphi\left(x'\right)\right\},$$
$$\Sigma = \left\{x \in Q_{r_0} | x_n = \varphi\left(x'\right)\right\},$$

where φ is a Lipschitz function on B'_{r_0} satisfying

$$\varphi(0) = \left|\nabla_{x'}\varphi(0)\right| = 0; \qquad \|\varphi\|_{C^{0,1}(B'_{r_0})} \leqslant Lr_0.$$

It is understood that $\partial\Omega$ is of Lipschitz class with constants r_0, L if it is finite union of portions of Lipschitz class with constants r_0, L.

Definition 2.2. Let Ω be a domain in $\mathbb{R}^n$. We say that a portion Σ of $\partial\Omega$ is a flat portion of size r_0 if there exists $P \in \Sigma$ and there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\begin{aligned}
&\Sigma \cap Q_{r_0/3} = \{x \in Q_{r_0/3} | x_n = 0\} \\
&\Omega \cap Q_{r_0/3} = \{x \in Q_{r_0/3} | x_n > 0\} \\
&\left(\mathbb{R}^n \setminus \Omega\right) \cap Q_{r_0/3} = \{x \in Q_{r_0/3} | x_n < 0\}.
\end{aligned} \tag{2.1}$$

Let us define the space of *local Cauchy data* on Σ for $H^1(\Omega)$ solutions to

$$\Delta u + qu = 0 \quad \text{in } \Omega, \tag{2.2}$$

having zero trace on $\partial\Omega \setminus \overline{\Sigma}$.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and Σ a non-empty open portion of $\partial\Omega$. Let us introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H^{\frac{1}{2}}_{co}(\Sigma) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) | \operatorname{supp} f \subset \Sigma \right\}. \tag{2.3}$$

We recall that its closure with respect to the $H^{\frac{1}{2}}(\partial\Omega)$-norm is the space $H^{\frac{1}{2}}_{00}(\Sigma)$ (see [27,45]).

We define the *Cauchy data* associated to q with first component vanishing on $\partial\Omega \setminus \overline{\Sigma}$ to be the space $\mathcal{C}_q^{\Sigma}(\partial\Omega)$ defined by

$$\begin{aligned}
\mathcal{C}_q^{\Sigma}(\partial\Omega) = \Big\{ (f, g) \in H^{\frac{1}{2}}_{00}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega) | \exists u \in H^1(\Omega) \text{ weak solution to} \\
\Delta u + qu = 0 \text{ in } \Omega, \\
u|_{\partial\Omega} = f, \partial_\nu u|_{\partial\Omega} = g \Big\}.
\end{aligned} \tag{2.4}$$

Analogously, we consider the subspace of $H^{\frac{1}{2}}(\partial\Omega)$,

$$H^{\frac{1}{2}}_{00}(\partial\Omega \setminus \overline{\Sigma})$$

and the closed subspace of $H^{-\frac{1}{2}}(\partial\Omega)$ of functionals vanishing on $H^{\frac{1}{2}}_{00}(\Sigma)$ functions

$$H^{-\frac{1}{2}}_{00}(\partial\Omega \setminus \overline{\Sigma}) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) | \langle \psi, \varphi \rangle = 0, \text{ for any } \varphi \in H^{\frac{1}{2}}_{00}(\Sigma) \right\}. \tag{2.5}$$

Here $\langle \psi, \varphi \rangle$ denotes the duality between the complex valued spaces $H^{-\frac{1}{2}}(\partial\Omega)$, $H^{\frac{1}{2}}(\partial\Omega)$ based on the L^2 inner product

$$\langle \psi, \varphi \rangle = \int_{\partial\Omega} \psi \overline{\varphi}.$$

We denote by $H^{\frac{1}{2}}(\partial\Omega)|_\Sigma$ and $H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$ the *restrictions* of $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ to Σ respectively. Note that $H^{\frac{1}{2}}(\partial\Omega)|_\Sigma$ can be interpreted as the quotient space of $H^{\frac{1}{2}}(\partial\Omega)$ through the equivalence relation

$$\varphi \sim \psi \quad \text{iff} \quad \varphi - \psi \in H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \overline{\Sigma}).$$

The same reasoning applies to $H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$, that is

$$H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma = H^{-\frac{1}{2}}(\partial\Omega)/H_{00}^{-\frac{1}{2}}(\partial\Omega \setminus \overline{\Sigma}).$$

We can now define the local Cauchy data that will be considered here.

Definition 2.4. The *local Cauchy data* associated to q *having zero first component* on $\partial\Omega\setminus\overline{\Sigma}$ are defined by

$$\begin{aligned}\mathcal{C}_q^\Sigma(\Sigma) = \Big\{(f,g) \in H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma | \exists u \in H^1(\Omega) \text{ weak solution to} \\ \Delta u + qu = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f, \\ \langle \partial_\nu u|_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\rangle, \forall \varphi \in H_{00}^{\frac{1}{2}}(\Sigma)\Big\}.\end{aligned}$$

For the sake of completeness, let us also introduce the general local Cauchy data $\mathcal{C}_q(\Sigma)$ with no zero Dirichlet condition on $\partial\Omega \setminus \overline{\Sigma}$

$$\begin{aligned}\mathcal{C}_q(\Sigma) = \Big\{(f,g) \in H^{\frac{1}{2}}(\partial\Omega)|_\Sigma \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma | \exists u \in H^1(\Omega) \text{ weak solution to} \\ \Delta u + qu = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} - f \in H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \overline{\Sigma}), \\ \langle \partial_\nu u|_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\rangle, \forall \varphi \in H_{00}^{\frac{1}{2}}(\Sigma)\Big\}.\end{aligned}$$

Observe also that $H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$ is a Hilbert space with the norm

$$\left\|(f,g)\right\|_{H_{00}^{\frac{1}{2}}(\Sigma) \oplus H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma} = \left(\|f\|^2_{H_{00}^{\frac{1}{2}}(\Sigma)} + \|g\|^2_{H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}\right)^{\frac{1}{2}}. \tag{2.6}$$

We recall that given closed subspaces S_1, S_2 of a Hilbert space $(H, \|\cdot\|)$, the *distance* (*aperture*) between S_1, S_2 is defined as

$$d(S_1, S_2) = \max\left\{ \sup_{h \in S_2, h\neq 0} \inf_{k \in S_1} \frac{\|h-k\|_H}{\|h\|_H}, \sup_{k \in S_1, k\neq 0} \inf_{h \in S_2} \frac{\|h-k\|_H}{\|k\|_H} \right\}, \tag{2.7}$$

see for instance [2]. From now on, given two potential q_i, $i = 1, 2$, we will simply denote the local Cauchy data $\mathcal{C}_{q_i}^\Sigma(\Sigma)$ with $\mathcal{C}_i$, $i = 1, 2$. We recall also that it is known that when $d(S_1, S_2) < 1$, then the two quantities within the maximum in (2.7) coincide (see [24]). Then, since we are interested in Cauchy data spaces $\mathcal{C}_1, \mathcal{C}_2$ corresponding to potentials q_1, q_2 respectively, when $\mathcal{C}_1$ and $\mathcal{C}_2$ are close to each other, it is sensible to set

$$d(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(f_2,g_2)\in\mathcal{C}_2\setminus\{(0,0)\}} \inf_{(f_1,g_1)\in\mathcal{C}_1} \frac{\|(f_1, g_1) - (f_2, g_2)\|_{H_{00}^{\frac{1}{2}}(\Sigma)\oplus H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}}{\|(f_2, g_2)\|_{H_{00}^{\frac{1}{2}}(\Sigma)\oplus H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}}. \tag{2.8}$$

Note also that properly speaking the above quantity is a distance between the closures $\overline{\mathcal{C}_1}$ and $\overline{\mathcal{C}_2}$. We notice that it could be proved that the subspaces $\mathcal{C}_1$ and $\mathcal{C}_2$ are indeed closed, but this fact is not much relevant in the present context.

Let us also recall some more or less well-known calculations. Let $u_i \in H^1(\Omega)$ be solutions to (2.2) when $q = q_i$, $i = 1, 2$ respectively and such that $u_i|_{\partial\Omega} \in H_{co}^{\frac{1}{2}}(\Sigma)$. Green's identity yields

$$\int_\Omega (q_1 - q_2)u_1u_2 = \langle \partial_\nu \overline{u_2}, u_1\rangle - \langle \partial_\nu u_1, \overline{u_2}\rangle. \tag{2.9}$$

Notice that a complex valued function u_i is a solution to (2.2) when $q = q_i$ (real valued) if and only if so is $\overline{u_i}$. Notice also that, if v_i is any other such solution with $q = q_i$, we have

$$\langle \partial_\nu v_i, \overline{u_i}\rangle - \langle \partial_\nu \overline{u_i}, v_i\rangle = 0, \quad \text{for every } i = 1, 2. \tag{2.10}$$

Hence, for any v_2 solving $\Delta v_2 + q_2v_2 = 0$ in Ω,

$$\int_\Omega (q_1 - q_2)u_1u_2 = \big\langle \partial_\nu \overline{u_2}, (u_1 - v_2)\big\rangle - \langle \partial_\nu u_1 - \partial_\nu v_2, \overline{u_2}\rangle, \tag{2.11}$$

from such an identity one easily deduces

$$\left|\int_\Omega (q_1 - q_2)u_1u_2\, dx\right| \leqslant d(\mathcal{C}_1, \mathcal{C}_2)\big\|(u_1, \partial_\nu u_1)\big\|_{H_{00}^{\frac{1}{2}}(\Sigma)\oplus H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma} \tag{2.12}$$

$$\times \big\|(u_2, \partial_\nu \overline{u_2})\big\|_{H_{00}^{\frac{1}{2}}(\Sigma)\oplus H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}. \tag{2.13}$$

2.2. *Conditional Lipschitz stability*

2.2.1. *Assumptions about the domain Ω*

1. We assume that Ω is a domain in $\mathbb{R}^n$ and that there is a positive constant B such that

$$|\Omega| \leqslant Br_0^n, \tag{2.14}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

2. We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local Cauchy data are taken).
3.
$$\bar{\Omega} = \bigcup_{j=1}^{N} \bar{D}_j,$$
where D_j, $j = 1, \ldots, N$ are known open sets of $\mathbb{R}^n$, satisfying the conditions below.
   (a) D_j, $j = 1, \ldots, N$ are connected and pairwise nonoverlapping polyhedrons.
   (b) ∂D_j, $j = 1, \ldots, N$ are of Lipschitz class with constants r_0, L.
   (c) There exists one region, say D_1, such that $\partial D_1 \cap \Sigma$ contains a *flat* portion Σ_1 of size r_0 and for every $i \in \{2, \ldots, N\}$ there exists $j_1, \ldots, j_K \in \{1, \ldots, N\}$ such that
$$D_{j_1} = D_1, \qquad D_{j_K} = D_i. \tag{2.15}$$
   In addition we assume that, for every $k = 1, \ldots, K$, $\partial D_{j_k} \cap \partial D_{j_{k-1}}$ contains a *flat* portion Σ_k of size r_0 (here we agree that $D_{j_0} = \mathbb{R}^n \setminus \Omega$), such that
$$\Sigma_k \subset \Omega, \quad \text{for every } k = 2, \ldots, K.$$
   Let us emphasise that under such an assumption, for every $k = 1, \ldots, K$, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates (depending on k) under which we have $P_k = 0$ and
$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n = 0\}, \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n > 0\}, \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n < 0\}. \end{aligned} \tag{2.16}$$

2.2.2. A-priori information on the potential q

We shall consider a real valued function $q \in L^\infty(\Omega)$, with
$$\|q\|_{L^\infty(\Omega)} \leqslant E_0, \tag{2.17}$$
for some positive constant E_0 and of type
$$q(x) = \sum_{j=1}^{N} q_j(x)\chi_{D_j}(x), \quad x \in \Omega, \tag{2.18a}$$
$$q_j(x) = a_j + A_j \cdot x, \tag{2.18b}$$
where $a_j \in \mathbb{R}$, $A_j \in \mathbb{R}^n$ and D_j, $j = 1, \ldots, N$ are the given subdomains introduced in Section 2.2.1.

Definition 2.5. Let B, N, r_0, L, E_0 be given positive numbers with $N \in \mathbb{N}$. We will refer to this set of numbers, along with the space dimension n, as to the *a-priori data*. Several constants depending on the *a-priori data* will appear within the paper. In order to simplify our notation, any quantity denoted by $C, C_1, C_2, \ldots$ will be called a *constant* understanding in most cases that it only depends on the a priori data.

Remark 2.1. Observe that the class of functions of the form (2.18a)–(2.18b) is a finite dimensional linear space. The L^∞-norm $\|q\|_{L^\infty(\Omega)}$ is equivalent to the norm

$$|||q||| = \max_{j=1,\dots,N}\left\{|a_j| + |A_j|\right\}$$

modulo constants which only depend on the a-priori data.

Theorem 2.2. *Let Ω, D_j, $j = 1, \dots, N$ and Σ be a domain, N subdomains of Ω and a portion of $\partial\Omega$ as in Section 2.2.1 respectively. Let $q^{(i)}$, $i = 1, 2$ be two potentials satisfying (2.17) and of type*

$$q^{(i)} = \sum_{j=1}^{N} q_j^{(i)}(x)\chi_{D_j}(x), \quad x \in \Omega, \tag{2.19}$$

where

$$q_j^{(i)}(x) = a_j^{(i)} + A_j^{(i)} \cdot x,$$

with $a_j^{(i)} \in \mathbb{R}$ and $A_j^{(i)} \in \mathbb{R}^n$, then we have

$$\left\|q^{(1)} - q^{(2)}\right\|_{L^\infty(\Omega)} \leqslant C d(\mathcal{C}_1, \mathcal{C}_2), \tag{2.20}$$

where $\mathcal{C}_i$ denotes the space of Cauchy data $\mathcal{C}_{q_i}^{\Sigma}(\Sigma)$, for $i = 1, 2$ and C is a positive constant that depends on the a-priori data only.

Remark 2.3. An analogous result to the one obtained in theorem 2.2 could be similarly obtained if the local Cauchy data (with vanishing condition on $\partial\Omega \setminus \overline{\Sigma}$ on the first component) considered here are to be replaced by the general local data with no vanishing condition on the first component as introduced in definition 2.4. Our present choice is motivated by the fact that the solutions that shall be actually used for the purpose of the stability estimates do indeed satisfy such zero trace condition.

3. Proof of the main result

The proof of our main result (theorem 2.2) is based on an argument that combines asymptotic type of estimates for a Green's function of the third kind for the operator

$$L = \Delta + q(x) \quad \text{in } \Omega \tag{3.1}$$

(Propositions 3.4, 3.5), with q satisfying (2.17)–(2.18b), together with a result of unique continuation (Proposition 3.6) for solutions to

$$Lu = 0, \quad \text{in } \Omega.$$

Our idea in estimating $q^{(1)} - q^{(2)}$ exploits this singular behaviour on one hand estimating from below the blow up of some singular solutions (we shall introduce below) $S_{\mathcal{U}_k}$ and some of its derivatives if

$q^{(1)} - q^{(2)}$ is large at some point. On the other hand we use estimates of propagation of smallness to show that $S_{\mathcal{U}_k}$ needs to be small if $q_1 - q_2$ is small. We shall give the precise formulation of these results in what follows.

3.1. *Piecewise linear potential*

We collect here a series of auxiliary results. Most of the proofs are postponed to the following section.

3.1.1. *Asymptotics at interfaces*

We will find convenient to introduce Green's function not precisely for the physical domain Ω but for an augmented domain Ω_0.

We recall that by assumption 3(c) of Section 2.2.1 we can assume that there exists a point P_1 such that up to a rigid transformation of coordinates we have that $P_1 = 0$ and (2.1) holds with $\Sigma = \Sigma_1$.

Denoting by

$$D_0 = \left\{ x \in \left(\mathbb{R}^n \setminus \Omega\right) \cap B_{r_0} | |x_i| < \frac{2}{3} r_0, i = 1, \ldots, n-1, \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\},$$

it turns out that the augmented domain $\Omega_0 = \Omega \cup D_0$ is of Lipschitz class with constants $\frac{r_0}{3}$ and $\widetilde{L}$, where $\widetilde{L}$ depends on L only. Given $r > 0$, we set

$$\Sigma_0 = \left\{ x \in \Omega_0 | |x_i| \leqslant \frac{2}{3} r_0, x_n = -\frac{2}{3} r_0 \right\}, \tag{3.2}$$

$$(\Omega_0)_r = \left\{ x \in \Omega_0 | \operatorname{dist}(x, \partial\Omega_0) > r \right\}. \tag{3.3}$$

In this section we shall introduce a mixed boundary value problem for $\Delta + q$ in Ω_0 which is always well posed, independently of any *a-priori* condition on q, besides the assumption of being real valued and bounded. This shall enable us to construct a Green'n function for $\Delta + q$ in Ω_0. The underlying ideas of such construction can be traced back to an idea by Bamberger and Hua Duong [10].

The main result is as follows.

Proposition 3.1. *Let $q \in L^\infty(\Omega_0)$, for any $y \in \Omega_0$ there exists a unique distributional solution $G(\cdot, y)$ to the problem*

$$\begin{cases} \Delta G(\cdot, y) + q(\cdot) G(\cdot, y) = -\delta(\cdot - y), & \text{in } \Omega_0, \\ G(\cdot, y) = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_\nu G(\cdot, y) + i G(\cdot, y) = 0, & \text{on } \Sigma_0. \end{cases} \tag{3.4}$$

In particular we have

$$\left| G(x, y) \right| \leqslant C |x - y|^{2-n}, \quad \text{for any } x, y \in \Omega, x \neq y, \tag{3.5}$$

where $C > 0$ is a constant depending on the a priori data only.

In what follows we shall make a repeated use of the solution to (3.4) in the special case $q = 0$.

Definition 3.1. We denote by $G_0 = G_0(x, y)$ the Green's function for the Laplacian and mixed boundary conditions which solves in the distributional sense

$$\begin{cases} \Delta G_0(\cdot, y) = -\delta(\cdot - y), & \text{in } \Omega_0, \\ G_0(\cdot, y) = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_\nu G_0(\cdot, y) + i G_0(\cdot, y) = 0, & \text{on } \Sigma_0. \end{cases} \tag{3.6}$$

Remark 3.2. The existence and uniqueness of a distributional solution $G_0 \in L^1(\Omega_0)$ to (3.6) is a consequence of the standard theory on boundary value problem for the Laplace equation. It is also well-known that

$$G_0(\cdot, y) \in H^1\big(\Omega_0 \setminus B_\epsilon(y)\big), \quad \text{for every } \epsilon > 0 \tag{3.7}$$

and

$$\big|G_0(x, y)\big| \leqslant C|x - y|^{2-n}, \quad \text{for any } x, y \in (\Omega_0)_{\frac{r_0}{4}}, x \neq y, \tag{3.8}$$

where $C > 0$ is a constant depending on the a priori data only.

Estimate (3.8) holds also when x and y approach $\partial\Omega$. Indeed, by adapting the change of variable arguments in Lemma 4.5 in [40], we may consider an homogeneous Neumann condition on Σ_0 instead. The latter, together with the techniques carried over in [6, Prop. 8.3], enables to perform an even extension $\widetilde{G}_0$ of G_0 across Σ_0. We define the set $\Sigma_0^\perp = \{x \in \mathbb{R}^n \,|\, |x_i| = \frac{2}{3}r_0, -\frac{19}{24}r_0 \leqslant x_n \leqslant -\frac{13}{24}r_0\}$ and we observe that the extension $\widetilde{G}_0$ satisfies an homogeneous Dirichlet condition on $\Sigma_0^\perp$ and it is Lipschitz continuous up to $\Sigma_0^\perp$. Finally, the results in [28] allow us to conclude that

$$\big|G_0(x, y)\big| \leqslant C|x - y|^{2-n}, \quad \text{for any } x, y \in \Omega_0, x \neq y, \tag{3.9}$$

where $C > 0$ is a constant depending on the a priori data only.

Proposition 3.3. *For any $y \in \Omega_0$ and every $r > 0$ we have that*

$$\int_{\Omega_0 \setminus B_r(y)} \big|\nabla G(\cdot, y)\big|^2 \leqslant C r^{2-n}, \tag{3.10}$$

where $C > 0$ is a constant depending on n, r_0, L and on $\|q\|_{L^\infty(\Omega_0)}$.

Proof. The proof can be obtained in a straightforward fashion by combining Caccioppoli inequality with (3.5). $\square$

Proposition 3.4. *For every $x, y \in (\Omega_0)_{r_0}$, $x \neq y$ we have that*

$$\big|G(x, y) - \Gamma(x, y)\big| \leqslant \begin{cases} C, & \text{if } n = 3, \\ C(|\log|x - y|| + 1), & \text{if } n = 4, \\ C|x - y|^{4-n}, & \text{if } n \geqslant 5, \end{cases} \tag{3.11}$$

and

$$\left|\nabla_y G(x, y) - \nabla_y \Gamma(x, y)\right| \leqslant \begin{cases} C(|\log|x-y||+1), & \text{if } n = 3, \\ C|x-y|^{3-n}, & \text{if } n \geqslant 4. \end{cases} \tag{3.12}$$

Our next goal is to analyze the asymptotics of $\nabla_y^2 G(x, y)$ (∇_y^2 = Hessian matrix). To this purpose it is necessary to use accurately the fine structure of q as a piecewise linear function. We are especially interested to the case when x, y are near to an interface of q. More precisely, we examine the case when x and y are on the opposite sides of the interface and y approaches orthogonally the interface.

Proposition 3.5. *Let Q_{l+1} be a point such that $Q_{l+1} \in B_{\frac{r_0}{8}}(P_{l+1}) \cap \Sigma_{l+1}$ with $l \in \{1, \dots, N-1\}$. There exist a constant $\overline{C} > 0$ depending on n, r_0, L and $\|q\|_{L^\infty(\Omega_0)}$ such that following inequality hold true for every $x \in B_{\frac{r_0}{16}}(Q_{l+1}) \cap D_{j_{l+1}}$ and every $= Q_{l+1} - re_n$, where $r \in (0, \frac{r_0}{16})$*

$$\left|\nabla_y^2\big(G(x, y) - \Gamma(x, y)\big)\right| \leqslant \overline{C} r^{2-n}. \tag{3.13}$$

3.1.2. Quantitative unique continuation

We consider the operator L_i given by

$$L_i = \Delta + \widetilde{q}^{(i)}(x) \quad \text{in } \Omega_0, i = 1, 2, \tag{3.14}$$

where $\widetilde{q}^{(i)}$ is the extension on Ω_0 of $q^{(i)}$ obtained by setting $\widetilde{q}^{(i)}|_{D_0} = 1$, for $i = 1, 2$. For every $y \in \Omega_0$, we shall denote with $G_i(\cdot, y)$ the Green function solution to (3.4) when $q = \widetilde{q}^{(i)}$ for $i = 1, 2$. We also define the set

$$\widehat{D}_0 = \left\{x \in D_0 | \operatorname{dist}(x, \Sigma_1) \geqslant \frac{r_0}{3}\right\}.$$

Let $K \in 1, \dots, N$ be such that

$$E = \left\|q^{(1)} - q^{(2)}\right\|_{L^\infty(\Omega)} = \left\|q^{(1)} - q^{(2)}\right\|_{L^\infty(D_K)} \tag{3.15}$$

and recall that there exist $j_1, \dots, j_K \in 1, \dots, N$ such that

$$D_{j_1} = D_1, \dots, D_{j_K} = D_K,$$

with $D_{j_1}, \dots, D_{j_K}$ satisfying assumption 3(c). For simplicity, let us rearrange the indices of these subdomains so that the above mentioned chain is simply denoted by $D_1, \dots, D_K$, $K \leqslant N$. We also denote

$$\mathcal{W}_k = \bigcup_{i=0}^{k} D_i, \qquad \mathcal{U}_k = \Omega_0 \setminus \overline{\mathcal{W}_k}, \quad \text{for } k = 1, \dots, K \tag{3.16}$$

and for any $y, z \in \mathcal{W}_k$

$$\tilde{S}_{\mathcal{U}_k}(y, z) = \int_{\mathcal{U}_k} \big(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\big) \widetilde{G}_1(\cdot, y) \widetilde{G}_2(\cdot, z), \quad \text{for } k = 1, \dots, K. \tag{3.17}$$

It is a relatively straightforward matter to see that for every $y, z \in \mathcal{W}_k$ with $k = 0, \ldots, K$ we have that $\widetilde{S}_{\mathcal{U}_k}(\cdot, z), \widetilde{S}_{\mathcal{U}_k}(y, \cdot) \in H^1_{loc}(\mathcal{W}_k)$ are weak solutions to

$$\left(\Delta + q^{(1)}(x)\right)\widetilde{S}_{\mathcal{U}_k}(\cdot, z) = 0, \quad \text{in } \mathcal{W}_k, \tag{3.18}$$

$$\left(\Delta + q^{(2)}(x)\right)\widetilde{S}_{\mathcal{U}_k}(y, \cdot) = 0, \quad \text{in } \mathcal{W}_k. \tag{3.19}$$

It is expected that some derivatives of $\widetilde{S}_{\mathcal{U}_k}(y, z)$ blow up as y, z approach simultaneously one point of $\partial\mathcal{U}_k$. The following estimate for $\widetilde{S}_{\mathcal{U}_k}(y, z)$ holds true, for $k = 1, \ldots, K$.

Proposition 3.6 (Estimates of unique continuation)**.** *If, for a positive number ε_0, we have*

$$\left|\widetilde{S}_{\mathcal{U}_k}(y, z)\right| \leqslant r_0\varepsilon_0, \quad \text{for every } (y, z) \in \widehat{D}_0 \times \widehat{D}_0, \tag{3.20}$$

then the following inequalities hold true for every $r \leqslant 2r_1$

$$\left|\widetilde{S}_{\mathcal{U}_k}(y_{k+1}, y_{k+1})\right| \leqslant C(E + \varepsilon_0)\left(\frac{\varepsilon_0}{E + \varepsilon_0}\right)^{r^2\beta^{2N_1}}\left(1 + r^{2\gamma}\right), \tag{3.21}$$

$$\left|\partial_{y_j}\partial_{z_i}\widetilde{S}_{\mathcal{U}_k}(y_{k+1}, y_{k+1})\right| \leqslant C(E + \varepsilon_0)\left(\frac{\varepsilon_0}{E + \varepsilon_0}\right)^{r^2\beta^{2N_1}}\left(1 + r^{2\gamma}\right)r^{-2}, \tag{3.22}$$

$$\left|\partial^2_{y_j}\partial^2_{z_i}\widetilde{S}_{\mathcal{U}_k}(y_{k+1}, y_{k+1})\right| \leqslant C(E + \varepsilon_0)\left(\frac{\varepsilon_0}{E + \varepsilon_0}\right)^{r^2\beta^{2N_1}}\left(1 + r^{2\gamma}\right)r^{-4}, \tag{3.23}$$

for any $i, j = 1, \ldots, n$, where $y_{k+1} = P_{k+1} - 2r\nu(P_{k+1})$, ν is the exterior unit normal to ∂D_k at P_{k+1}, $\beta = \frac{\ln(8/7)}{\ln 4}$, $r_1 = \frac{r_0}{16}$, $\gamma = 2 - \frac{n}{2}$, and the constants $N_1, C > 0$ depend on the a-priori data only.

Note that $\gamma < 0$ only when $n > 4$, hence the term $r^{2\gamma}$ becomes irrelevant if $n = 3, 4$.

3.2. Lipschitz stability for piecewise linear potentials

Proof of theorem 2.2. Let D_K be the subdomain of Ω satisfying (3.15) and let $D_1, \ldots, D_K$ be the chain of domains satisfying assumption 3(c). For any $k = 1, \ldots, K$ we shall denote by $D_T f$ and $\partial_\nu f$ the $n - 1$ dimensional vector of the tangential partial derivatives of a function f on Σ_k and the normal partial derivative of f on Σ_k respectively. We denote by C_i the local Cauchy data, for $i = 1, 2$. Let us also simplify our notation by replacing $C^{\Gamma}_{q^{(i)}}$ with We shall also denote

$$\varepsilon_0 = d(C_1, C_2), \qquad \delta_l = \left\|\tilde{q}^{(1)} - \tilde{q}^{(2)}\right\|_{L^\infty(\mathcal{W}_l)}$$

and introduce for any number $b > 0$, the concave non decreasing function $\omega_b(t)$, defined on $(0, +\infty)$,

$$\omega_b(t) = \begin{cases} 2^b e^{-2}|\log t|^{-b}, & t \in (0, e^{-2}), \\ e^{-2}, & t \in [e^{-2}, +\infty). \end{cases} \tag{3.24}$$

We recall (see (4.34) and (4.35) in [7]) that for any $\beta \in (0,1)$ we have that

$$(0,+\infty) \ni t \to t\omega_b\left(\frac{1}{t}\right) \quad \text{is a nondecreasing function} \tag{3.25}$$

and

$$\omega_b\left(\frac{t}{\beta}\right) \leqslant \left|\log e\beta^{-1/2}\right|^b \omega_b(t), \qquad \omega_b\left(t^\beta\right) \leqslant \left(\frac{1}{\beta}\right)^b \omega_b(t). \tag{3.26}$$

Furthermore, we set $\omega_\alpha^{(0)}(t) = t^\alpha$ with $0 < \alpha < 1$ and we shall denote the iterated compositions

$$\omega_b^{(1)} = \omega_b, \qquad \omega_b^{(j)} = \omega_b \circ \omega_b^{(j-1)}, \quad j = 2, 3, \ldots. \tag{3.27}$$

We begin by noticing that for each $l = 1, 2, \ldots \|\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\|_{L^\infty(D_l)}$ can be evaluated in terms of the quantities

$$\left\|\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))}, \tag{3.28}$$

$$\left|\partial_\nu\left(\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right)(P_l)\right|, \tag{3.29}$$

where $r_0 > 0$ is the constant introduced in Section 2.1. In fact, let us denote

$$\alpha_l + \beta_l \cdot x = \left(\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right)(x), \quad x \in D_l \tag{3.30}$$

and choose $\{e_j\}_{j=1,\ldots,n-1}$ orthonormal vectors starting at P_l and generating the hyperplane containing the flat part of Σ_l. By computing $\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}$ on the points P_l, $P_l + \frac{r_0}{5}e_j$, $j = 1, \ldots, n-1$ and taking their differences we obtain

$$|\alpha_l + \beta_l \cdot P_l| + \sum_{j=1}^{n-1} |\beta_l \cdot e_j| \leqslant C\left\|\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))}. \tag{3.31}$$

Next we notice that

$$|\beta_l \cdot \nu| = \left|\partial_\nu\left(q_l^{(1)} - q_l^{(2)}\right)(P_l)\right|. \tag{3.32}$$

Hence each of the components of β_l can be estimated and eventually also $|\alpha_l|$. In conclusion

$$|\alpha_l| + |\beta_l| \leqslant C\left(\left\|\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} + \left|\partial_\nu\left(q_l^{(1)} - q_l^{(2)}\right)(P_l)\right|\right). \tag{3.33}$$

Hence our task will be to estimate the quantities

$$\left\|\widetilde{q}_l^{(1)} - \widetilde{q}_l^{(2)}\right\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_l))} \quad \text{and} \quad \left|\partial_\nu\left(q_l^{(1)} - q_l^{(2)}\right)(P_l)\right|$$

iteratively with respect to l (see for example [4]). When $l = 1$ this corresponds to a stability estimate at the boundary for the potential q and its normal derivative of the following type

$$\|\widetilde{q}_1^{(1)} - \widetilde{q}_1^{(2)}\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_1))} + \left|\partial_\nu \left(q_1^{(1)} - q_l^{(2)}\right)(P_1)\right| \leqslant C(\varepsilon_0 + E)\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\eta_1}, \tag{3.34}$$

for some constant η_1, $0 < \eta_1 < 1$, resulting in

$$\delta_1 \leqslant C(\varepsilon_0 + E)\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\eta_1}, \quad 0 < \eta_1 < 1.$$

The bound (3.34) is proven in Section 4.3. We proceed to estimate δ_2 by proving

$$\|\widetilde{q}_2^{(1)} - \widetilde{q}_2^{(2)}\|_{L^\infty(\Sigma_l \cap B_{\frac{r_0}{4}}(P_2))} + \left|\partial_\nu \left(q_2^{(1)} - q_2^{(2)}\right)(P_2)\right| \leqslant C(\varepsilon_0 + E)\omega_{\eta_2}\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right), \tag{3.35}$$

for some constant η_2, $0 < \eta_2 < 1$, where ω_b is the function defined in (3.24). We recall that for every $y, z \in D_0$ we have

$$\begin{aligned} &\int_{\partial\Omega} \left(\widetilde{G}_1(x, y)\partial_\nu \widetilde{G}_2(x, z) - \widetilde{G}_2(x, z)\partial_\nu \widetilde{G}_1(x, y)\right) dS(x) \\ &\quad = \widetilde{S}_{\mathcal{U}_1}(y, z)\, dx + \int_{\mathcal{W}_1} \left(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\right)(x)\widetilde{G}_1(x, y)\widetilde{G}_2(x, z)\, dx, \end{aligned} \tag{3.36}$$

and for every $i = 1, \ldots, n$

$$\begin{aligned} &\int_{\partial\Omega} \left(\partial_{y_i} \widetilde{G}_1(x, y)\partial_\nu \partial_{z_i} \widetilde{G}_2(x, z) - \partial_{z_i} \widetilde{G}_2(x, z)\partial_\nu \partial_{y_i} \widetilde{G}_1(x, y)\right) dS(x) \\ &\quad = \partial_{y_i}\partial_{z_i} \widetilde{S}_{\mathcal{U}_1}(y, z) + \int_{\mathcal{W}_1} \left(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\right)(x)\partial_{y_i} \widetilde{G}_1(x, y)\partial_{z_i} \widetilde{G}_2(x, z)\, dx, \end{aligned} \tag{3.37}$$

where $dS(x)$ denotes surface integration with respect to the variable x. Here an argument of propagation of smallness allows to get estimates on Σ_2. In order to estimate $\widetilde{q}^{(1)} - \widetilde{q}^{(2)}$ we can repeat the argument already used in [12] to prove (3.37) and obtain (omitting the details)

$$\|\widetilde{q}_2^{(1)} - \widetilde{q}_2^{(2)}\|_{L^\infty(\Sigma_2 \cap B_{\frac{r_0}{4}}(P_2))} \leqslant C(\varepsilon_0 + E)\left|\ln\left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)\right|^{-\frac{1}{4}}, \tag{3.38}$$

where $r_0 > 0$ is the constant introduced in Section 2.1. In comparison to [12], we have the additional task to estimate

$$\left|\partial_\nu \left(q_2^{(1)} - q_2^{(2)}\right)(P_2)\right|.$$

In this case we detail the procedure. Let us consider for any $i, j = 1, \dots, n$

$$\int_{\partial\Omega} \big(\partial^2_{y_i y_j} \widetilde{G}_1(x,y)\partial_\nu \partial^2_{z_i z_j} \widetilde{G}_2(x,z) - \partial^2_{z_i z_j} \widetilde{G}_2(x,z)\partial_\nu \partial^2_{y_i y_j} \widetilde{G}_1(x,y)\big)\, dS(x)$$
$$= \partial^2_{y_i y_j}\partial^2_{z_i z_j} \widetilde{S}_{\mathcal{U}_1}(y,z) + \int_{\mathcal{W}_1} \big(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\big)(x)\partial^2_{y_i y_j} \widetilde{G}_1(x,y) \cdot \partial^2_{z_i z_j} \widetilde{G}_2(x,z)\, dx. \tag{3.39}$$

From (3.36) we obtain

$$\begin{aligned}\big|\widetilde{S}_{\mathcal{U}_1}(y,z)\big| &\leqslant \varepsilon_0\Big(\big\|\widetilde{G}_1(\cdot,y)\big\|_{H^{\frac{1}{2}}_{00}(\Sigma)} + \big\|\partial_\nu \widetilde{G}_2(\cdot,z)\big\|_{H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}\Big)\\ &\qquad \times \Big(\big\|\widetilde{G}_2(\cdot,z)\big\|_{H^{\frac{1}{2}}_{00}(\Sigma)} + \big\|\partial_\nu \widetilde{G}_1(\cdot,y)\big\|_{H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}\Big)\\ &\qquad + \delta_1 \big\|\widetilde{G}_1(\cdot,y)\big\|_{L^2(\mathcal{W}_1)}\big\|\widetilde{G}_2(\cdot,z)\big\|_{L^2(\mathcal{W}_1)}\\ &\leqslant C(\varepsilon_0+\delta_1)r_0^{2-n}, \quad \text{for every } y,z \in \widehat{D}_0.\end{aligned} \tag{3.40}$$

Let $\rho_0 = \frac{r_0}{\overline{C}}$, where $\overline{C}$ is the constant introduced in Proposition 3.5, let $r \in (0, 2r_1)$ (where r_1 was introduced in Proposition 3.6) and denote

$$y_2 = P_2 + r\nu,$$

then for every $i, j = 1, \dots, n$

$$\partial^2_{y_i y_j}\partial^2_{z_i z_j} \widetilde{S}_{\mathcal{U}_1}(y_2,y_2) = I_1^{ij}(y_2) + I_2^{ij}(y_2), \tag{3.41}$$

where

$$I_1^{ij}(y_2) = \int_{B_{\rho_0}(P_2)\cap D_2} \big(q^{(1)} - q^{(2)}\big)(\cdot)\partial^2_{y_i y_j} \widetilde{G}_1(\cdot,y_2)\partial^2_{z_i z_j} \widetilde{G}_2(\cdot,y_2),$$
$$I_2^{ij}(y_2) = \int_{\Omega\setminus(B_{\rho_0}(P_2)\cap D_2)} \big(q^{(1)} - q^{(2)}\big)(\cdot)\partial^2_{y_i y_j} \widetilde{G}_1(\cdot,y_2)\partial^2_{z_i z_j} \widetilde{G}_2(\cdot,y_2).$$

If for $k = 1, 2$ we denote by $|I_k(y_2)|$ the Euclidean norm of matrix $I_k(y_2) = \{I_k^{ij}(y_2)\}_{i,j=1,\dots,n}$, we have

$$\big|I_2(y_2)\big| \leqslant CE\rho_0^{-n}, \tag{3.42}$$

where C depends on λ and n only (see [4]) and

$$\big|I_1(y_2)\big|$$
$$\geqslant \frac{1}{n}\sum_{i,j=1}^n \Big\{\Big|\int_{B_{\rho_0}(P_2)\cap D_2} \big(\partial_\nu \big(q_2^{(1)} - q_2^{(2)}\big)(P_2)\big)(x-P_2)_n \partial^2_{y_i y_j} \widetilde{G}_1(x,y_2)\partial^2_{z_i z_j} \widetilde{G}_2(x,y_2)\, dx\Big|$$

$$
- \int_{B_{\rho_0}(P_2)\cap D_2} \big| \big(D_T \big(q_2^{(1)} - q_2^{(2)}\big)(P_2)\big) \cdot (x - P_2)' \big| \big| \partial^2_{y_i y_j} \tilde{G}_1(x, y_2) \big| \big| \partial^2_{z_i z_j} \widetilde{G}_2(x, y_2)\, dx \big|
$$

$$
- \int_{B_{\rho_0}(P_2)\cap D_2} \big| \big(q_2^{(1)} - q_2^{(2)}\big)(P_2) \big| \big| \partial^2_{y_i y_j} \widetilde{G}_1(x, y_2) \big| \big| \partial^2_{z_i z_j} \widetilde{G}_2(x, y_2)\, dx \big| \Big\}
$$

and, noticing that up to a transformation of coordinates we can assume that P_2 coincides with the origin O of the coordinates system and by Theorem 3.5, this leads to

$$
\begin{aligned}
\big| I_1(y_2) \big| \geqslant C \Big\{ &\big| \partial_\nu \big(q_2^{(1)} - q_2^{(2)}\big)(O) \big| \int_{B_{\rho_0}(O)\cap D_2} \big| \nabla_y^2 \Gamma(x, y_2) \big|^2 |x_n|\, dx \\
&- E \int_{B_{\rho_0}(O)\cap D_2} \big| \nabla_y^2 \Gamma(x, y_2) \big| |x - y_2|^{2-n} |x_n|\, dx \\
&- E \int_{B_{\rho_0}(O)\cap D_2} |x - y_2|^{4-2n} |x_n|\, dx \Big\} \\
&- \int_{B_{\rho_0}(O)\cap D_2} \big| D_T \big(q_2^{(1)} - q_2^{(2)}\big) \big| |x'| \big| \nabla_y^2 \widetilde{G}_1(x, y_2) \big| \big| \nabla_z^2 \widetilde{G}_2(x, y_2) \big|\, dx \\
&- \int_{B_{\rho_0}(O)\cap D_2} \big| \big(q_2^{(1)} - q_2^{(2)}\big)(O) \big| \big| \nabla_y^2 \widetilde{G}_1(x, y_2) \big| \big| \nabla_z^2 \widetilde{G}_2(x, y_2) \big|\, dx.
\end{aligned} \tag{3.43}
$$

Therefore, by combining (3.43) together with (3.41) and (3.42), we obtain

$$
\begin{aligned}
\big| I_1(y_2) \big| \geqslant C \Big\{ &\big| \partial_\nu \big(q_2^{(1)} - q_2^{(2)}\big)(O) \big| \int_{B_{\rho_0}(O)\cap D_2} |x - y_2|^{-2n} |x_n|\, dx \\
&- 2E \int_{B_{\rho_0}(O)\cap D_2} |x - y_2|^{3-2n}\, dx \\
&- (\varepsilon_0 + E) \left| \ln\left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right) \right|^{-\frac{1}{4}} \int_{B_{\rho_0}(O)\cap D_2} |x - y_2|^{1-2n}\, dx \\
&- (\varepsilon_0 + E) \left| \ln\left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right) \right|^{-\frac{1}{4}} \int_{B_{\rho_0}(O)\cap D_2} |x - y_2|^{-2n}\, dx \Big\},
\end{aligned}
$$

which leads to

$$
\big| \partial_\nu \big(q_2^{(1)} - q_2^{(2)}\big) \big| r^{1-n} \leqslant \big| I_1(y_2) \big| + C \left((\varepsilon_0 + E) \left| \ln\left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right) \right|^{-\frac{1}{4}} r^{-n} + 2E r^{3-n} \right), \tag{3.44}
$$

where

$$
\big| I_1(y_2) \big| \leqslant \big| \partial^2_{y_n} \partial^2_{z_n} \widetilde{S}_{\mathcal{U}_1}(y_2, y_2) \big| + C E \rho_0^{-n}. \tag{3.45}
$$

Thus by combining the last two inequalities we get

$$\left|\partial_\nu\left(q_2^{(1)}-q_2^{(2)}\right)\right|r^{1-n} \leqslant \left|\partial^2_{y_n}\partial^2_{z_n}\tilde{S}_{\mathcal{U}_1}(y_2,y_2)\right| + C\Bigg\{E\left(\rho_0^{-n}+r^{3-n}\right) + (\varepsilon_0+E)\left|\ln\left(\frac{\varepsilon_0}{E+\varepsilon_0+\delta_1}\right)\right|^{-\frac{1}{4}} r^{-n}\Bigg\} \tag{3.46}$$

and by recalling that by Proposition 3.6 we have

$$\left|\partial^2_{y_j}\partial^2_{z_i}\tilde{S}_{\mathcal{U}_1}(y_2,y_2)\right| \leqslant C\left(\frac{\varepsilon_0+\delta_1}{E+\varepsilon_0+\delta_1}\right)^{r^2\beta^{2N_1}}(\varepsilon_0+E)\left(1+r^{4-n}\right)r^{-4},$$

where β, N_1 are the constants introduced in Proposition 3.6, we obtain

$$\left|\partial_\nu\left(q_2^{(1)}-q_2^{(2)}\right)\right| \leqslant C\Bigg\{\left(\frac{\varepsilon_0+\delta_1}{E+\varepsilon_0+\delta_1}\right)^{r^2\beta^{2N_1}}(\varepsilon_0+E+\delta_1)r^{p_n}+Er^2 + (\varepsilon_0+E)\left|\ln\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)\right|^{-\frac{1}{4}} r^{-1}\Bigg\}, \tag{3.47}$$

for any r, $0<r<2r_1$, where r_1 is as above and

$$p_n = \begin{cases} -1 & n\geqslant 4,\\ n-5 & n=3.\end{cases}$$

By minimizing (3.47) with respect to r we obtain

$$\left|\partial_\nu\left(q_2^{(1)}-q_2^{(2)}\right)\right| \leqslant C(\varepsilon_0+\delta_1+E)\omega_{\eta_2}\left(\frac{\varepsilon_0+\delta_1}{\varepsilon_0+\delta_1+E}\right), \quad \text{for some } \eta_2,\ 0<\eta_2<1 \tag{3.48}$$

and by combining (3.48) together with

$$\frac{\varepsilon_0+\delta_1}{\varepsilon_0+\delta_1+E} \leqslant C\omega^{(0)}_{\eta_1}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right) \tag{3.49}$$

and by the properties of the modulus ω_b, we obtain (3.35). By proceeding by iteration on l in order to estimate $q_l^{(1)}-q_l^{(2)}$ for $l=1,\dots,K$, we replace (3.36), (3.37) and (3.39) by

$$\int_{\partial\Omega}\left(\widetilde{G}_1(x,y)\partial_\nu\widetilde{G}_2(x,z)-\widetilde{G}_2(x,z)\partial_\nu\widetilde{G}_1(x,y)\right)dS(x) = \widetilde{S}_{\mathcal{U}_{l-1}}(y,z)+\int_{\mathcal{W}_{l-1}}\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\widetilde{G}_1(x,y)\widetilde{G}_2(x,z)\,dx, \tag{3.50}$$

$$\int_{\partial\Omega}\big(\partial_{y_i}\widetilde{G}_1(x,y)\partial_\nu\partial_{z_i}\widetilde{G}_2(x,z)-\partial_{z_i}\widetilde{G}_2(x,z)\partial_\nu\partial_{y_i}\widetilde{G}_1(x,y)\big)\,dS(x)$$
$$=\partial_{y_i}\partial_{z_i}\widetilde{S}_{\mathcal{U}_{l-1}}(y,z)+\int_{\mathcal{W}_{l-1}}\big(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\big)(x)\partial_{y_i}\widetilde{G}_1(x,y)\partial_{z_i}\widetilde{G}_2(x,z)\,dx \tag{3.51}$$

for any $i=1,\dots,n$ and

$$\int_{\partial\Omega}\big(\partial^2_{y_iy_j}\widetilde{G}_1(x,y)\partial_\nu\partial^2_{z_iz_j}\widetilde{G}_2(x,z)-\partial^2_{z_iz_j}\widetilde{G}_2(x,z)\partial_\nu\partial^2_{y_iy_j}\widetilde{G}_1(x,y)\big)\,dS(x)$$
$$=\partial^2_{y_iy_j}\partial^2_{z_iz_j}\widetilde{S}_{\mathcal{U}_{l-1}}(y,z)+\int_{\mathcal{W}_{l-1}}\big(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\big)(x)\partial^2_{y_iy_j}\widetilde{G}_1(x,y)\cdot\partial^2_{z_iz_j}\widetilde{G}_2(x,z)\,dx \tag{3.52}$$

for any $i,j=1,\dots,n$ respectively. Note that (3.50) leads to

$$\big|\widetilde{S}_{\mathcal{U}_{l1}}(y,z)\big|\leqslant C(\varepsilon_0+\delta_{l-1})r_0^{2-n},\quad \text{for every } y,z\in\widehat{D}_0, \tag{3.53}$$

where C depends on L, λ, n. By repeating the same argument applied for the special case $l=2$ we obtain

$$\big\|\widetilde{q}_l^{(1)}-\widetilde{q}_l^{(2)}\big\|_{L^\infty(\Sigma_l\cap B_{\frac{r_0}{4}}(P_l))}+\big|\partial_\nu\big(q^{(1)}-q^{(2)}\big)(P_l)\big|\leqslant C(\varepsilon_0+E)\omega_{\eta_l}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right), \tag{3.54}$$

for some η_l, $0<\eta_l<1$ and observing that

$$\delta_l\leqslant\delta_{l-1}+\big\|q_l^{(1)}-q_l^{(2)}\big\|_{L^\infty(D_l)},$$

we obtain for every $l=2,3,\dots$

$$\delta_l\leqslant\delta_{l-1}+C(\varepsilon_0+\delta_{l-1}+E)\omega_{\eta_l}\left(\frac{\varepsilon_0+\delta_{l-1}}{\varepsilon_0+\delta_{l-1}+E}\right),$$

hence trivially

$$\frac{\varepsilon_0+\delta_l}{\varepsilon_0+\delta_l+E}\leqslant C\omega_{\eta_l}\left(\frac{\varepsilon_0+\delta_{l-1}}{\varepsilon_0+\delta_{l-1}+E}\right). \tag{3.55}$$

Using the properties of the logarithmic moduli ω_b, (3.54) and the induction step (3.55) we arrive at

$$\big\|q^{(1)}-q^{(2)}\big\|_{L^\infty(\Omega)}\leqslant C(\varepsilon_0+E)\omega^{(K-1)}_{\eta_K}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right),$$

therefore

$$E\leqslant C(\varepsilon_0+E)\omega^{(K-1)}_{\eta_K}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right). \tag{3.56}$$

Assuming that $E > \varepsilon_0 e^2$ (if this is not the case then the theorem is proven) we obtain

$$E \leqslant C\left(\frac{E}{e^2} + E\right)\omega_{\eta_K}^{(K-1)}\left(\frac{\varepsilon_0}{E}\right),$$

which leads to

$$\frac{1}{C} \leqslant \omega_{\eta_K}^{(K-1)}\left(\frac{\varepsilon_0}{E}\right),$$

therefore

$$E \leqslant \frac{1}{\omega_{\eta_K}^{(-(K-1))}\left(\frac{1}{C}\right)}\varepsilon_0,$$

where here, with a slight abuse of notation, $\omega_{\frac{1}{C}}^{(-(K-1))}$ denotes the inverse function of $\omega_{\eta_K}^{(K-1)}$. $\square$

4. Proof of technical propositions

4.1. Asymptotic estimates

Proof of Proposition 3.1. Let us consider the following inhomogeneous boundary value problem. Given $f \in L^2(\Omega_0)$, we wish to find $v \in H^1(\Omega_0)$ such that

$$\begin{cases} \Delta v + qv = f, & \text{in } \Omega_0, \\ v = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_\nu v + iv = 0, & \text{on } \Sigma_0. \end{cases} \tag{4.1}$$

in the weak sense. Let us also consider the adjoint problem (recall that q is real valued).

$$\begin{cases} \Delta w + qw = g, & \text{in } \Omega_0, \\ w = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_\nu w - iw = 0, & \text{on } \Sigma_0. \end{cases} \tag{4.2}$$

It is well-known [22] that the Fredholm alternative applies and we have existence for (4.1) if and only if we have uniqueness for (4.2) and viceversa. In fact we can prove uniqueness for either of the two. We consider the homogeneous problem

$$\begin{cases} \Delta z + qz = 0, & \text{in } \Omega_0, \\ z = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_\nu z \pm iz = 0, & \text{on } \Sigma_0. \end{cases} \tag{4.3}$$

Using $\bar{z}$ as a test function we obtain

$$\int_{\Omega_0} |\nabla z|^2 - \int_{\Omega_0} q|z|^2 \pm i \int_{\Sigma_0} |z|^2 = 0, \tag{4.4}$$

which in turn implies $z = 0$ on Σ_0. Consequently also $\partial_\nu z = 0$ on Σ_0. From the uniqueness for the Cauchy problem, it follows that $z \equiv 0$ in Ω_0. Hence the solution to (4.1) exists and is unique. We now show that (4.1) is also well-posed in $H^1(\Omega_0)$. From the weak formulation of the problem (4.1) we have

$$\int_{\Sigma_0} |v|^2 = -\operatorname{Im}\left(\int_{\Omega_0} f\bar{v}\right), \tag{4.5}$$

$$\int_{\Omega_0} |\nabla v|^2 = -\operatorname{Re}\left(\int_{\Omega_0} f\bar{v}\right) + \int_{\Omega_0} q|v|^2. \tag{4.6}$$

We define

$$\varepsilon^2 = \int_{\Sigma_0} |v|^2 + \int_{\Sigma_0} |\partial_\nu v|^2, \tag{4.7}$$

$$\eta^2 = \int_{\Omega_0} |f|^2, \tag{4.8}$$

$$\delta^2 = \int_{\Omega_0} |v|^2, \tag{4.9}$$

$$E^2 = \int_{\Omega_0} |\nabla v|^2. \tag{4.10}$$

From (4.5) and the Schwartz inequality it follows that

$$\int_{\Sigma_0} |v|^2 \leqslant \eta\delta \tag{4.11}$$

which, combined with the impedance condition and Poincaré inequality, implies that

$$\varepsilon^2 \leqslant 2\eta E. \tag{4.12}$$

Moreover, by (4.6) we have that

$$E^2 \leqslant \eta\delta + \|q\|_{L^\infty(\Omega_0)}\delta^2. \tag{4.13}$$

We claim that there exists a constant $C_1 > 0$ depending on n, r_0, L and $\|q\|_{L^\infty(\Omega_0)}$ such that

$$E^2 \leqslant C_1\eta^2. \tag{4.14}$$

In order to prove our claim, we distinguish two cases. If $\delta^2 \leqslant \eta^2$, then (4.14) follows from (4.13). Otherwise, we observe that by well-known estimates for the Cauchy problem (see for instance [5]) it follows that

$$\delta^2 \leqslant \left(E^2+\varepsilon^2+\eta^2\right)\omega\left(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\right), \tag{4.15}$$

where $\omega(t) \leqslant C|\log(t)|^{-\mu}$ with C, $0<\mu<1$ depending on n, r_0, L and $\|q\|_{L^\infty(\Omega_0)}$. By (4.12) and (4.13) the above inequality leads to

$$\delta^2 \leqslant \left(3\eta\delta+\|q\|_{L^\infty(\Omega_0)}\delta^2+\eta^2\right)\omega\left(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\right). \tag{4.16}$$

From (4.16) we have that

$$1 \leqslant \left(4+\|q\|_{L^\infty(\Omega_0)}\right)\omega\left(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\right), \tag{4.17}$$

which leads to

$$\omega^{-1}\left(\frac{1}{4+\|q\|_{L^\infty(\Omega_0)}}\right)\left(E^2+\varepsilon^2+\eta^2\right) \leqslant \varepsilon^2+\eta^2. \tag{4.18}$$

Again from (4.12) we have that

$$\omega^{-1}\left(\frac{1}{4+\|q\|_{L^\infty(\Omega_0)}}\right)E^2 \leqslant 2\eta E+\eta^2. \tag{4.19}$$

By Schwartz inequality we readily obtain

$$E^2 \leqslant C\eta^2 \tag{4.20}$$

which, together with (4.12), gives

$$\|v\|_{H^1(\Omega_0)} \leqslant C\|f\|_{L^2(\Omega_0)}. \tag{4.21}$$

We set $J=[\frac{n-1}{2}]$ and we define iteratively the following kernels

$$\begin{cases} R_0(x,y)=G_0(x,y), \\ R_j(x,y)=\int_{\Omega_0} G_0(x,z)q(z)R_{j-1}(z,y)\,dz, \end{cases} \tag{4.22}$$

for every $j=1,\dots,J$. Note that for every $j=1,\dots,J$ we have

$$\begin{cases} \Delta R_j(\cdot,y)=-q(\cdot)R_{j-1}(\cdot,y), & \text{in } \Omega_0 \\ R_j(\cdot,y)=0, & \text{on } \partial\Omega_0\setminus\Sigma_0 \\ \partial_\nu R_j(\cdot,y)+iR_j(\cdot,y)=0 & \text{on } \Sigma_0 \end{cases} \tag{4.23}$$

and also by standard estimate [33]

$$\left|R_j(x,y)\right| \leqslant C|x-y|^{2j+2-n}, \quad \text{for every } j=0,\dots,J-1. \tag{4.24}$$

Now, if n is even

$$\left|R_J(x,y)\right| \leqslant C\left(\left|\log|x-y|\right|+1\right), \tag{4.25}$$

where if n is odd

$$\left|R_J(x,y)\right| \leqslant C. \tag{4.26}$$

In either case $\|R_J(\cdot,y)\|_{L^p(\Omega_0)} \leqslant C$ for every $p<\infty$. We let $R_{J+1}(\cdot,y)=v(\cdot)$ the solution to (4.1), when $f(\cdot)=-q(\cdot)R_J(\cdot,y)$. Therefore

$$\left\|R_{J+1}(\cdot,y)\right\|_{H^1(\Omega_0)} \leqslant C \tag{4.27}$$

and by relatively standard regularity estimates in the interior and at the boundary (see the arguments in Remark 3.2)

$$\left|R_{J+1}(x,y)\right| \leqslant C, \quad \text{for any } x,y\in\Omega_0, x\neq y. \tag{4.28}$$

If we form

$$G(x,y)=G_0(x,y)+\sum_{j=1}^{J+1} R_j(x,y), \tag{4.29}$$

then we end up with the desired solution to (3.4). □

Proof of Proposition 3.4. It is evident that $G_0(x,y)-\Gamma(x-y)$ is harmonic in either variables $x\in\Omega_0$, $y\in\Omega_0$. Hence standard interior regularity yields

$$\left|G_0(x,y)-\Gamma(x-y)\right|+\left|\nabla_y\left(G_0(x,y)-\Gamma(x-y)\right)\right| \leqslant C, \tag{4.30}$$

for any $x,y\in(\Omega_0)_{r_0}$. In order to prove (3.11) and (3.12) it then suffices to estimate $R(x,y)=G(x,y)-G_0(x,y)$ and its y-derivatives. Note that a crude estimate of R could be derived from (4.24), (4.25), (4.26), (4.28) and (4.29). Finer estimates are obtained as follows. By Green's identity we have

$$R(x,y)=\int_{\Omega_0} G(x,z)q(z)G_0(z,y)\,dz. \tag{4.31}$$

By combining (4.31) together with (3.5) and [33, Chap. 2] we obtain

$$\left|R(x,y)\right| \leqslant \begin{cases} C, & \text{if } n=3,\\ C(|\log|x-y||+1), & \text{if } n=4,\\ C|x-y|^{4-n}, & \text{if } n\geqslant 5. \end{cases} \tag{4.32}$$

We now estimate $\nabla_y R$. When $x \neq y$ we can differentiate under the integral sign and obtain

$$\begin{aligned}\left|\partial_{y_i} R(x, y)\right| &= \left|\int_{\Omega_0} G(x, z) q(z) \partial_{y_i} G_0(z, y)\, dz\right| \\ &\leqslant \int_{\Omega_0 \setminus B_{r_0}(x)} \left|G(x, z) q(z) \partial_{y_i} G_0(z, y)\right| dz \\ &\quad + C \int_{B_{r_0}(x)} |x - z|^{2-n} |z - y|^{1-n}\, dz, \end{aligned} \tag{4.33}$$

hence we have used the pointwise bound on G achieved in (3.5) and the above stated asymptotics on $\nabla_y G_0$. The first integral is bounded by a constant, the second one can be estimated (see [33, Chap. 2]) by

$$C\big(|\log|x - y|| + 1\big) \quad \text{if } n = 3 \tag{4.34}$$

and by

$$C|x - y|^{3-n} \quad \text{if } n \geqslant 4. \tag{4.35}$$

□

Proof of Theorem 3.5. We fix $l \in \{1, \ldots, N-1\}$. Furthermore, we observe that up to a transformation of coordinates we can assume that Q_{l+1} coincides with the origin 0 of the coordinates system. We denote

$$R(x, y) = G(x, y) - \Gamma(x - y),$$

then we have

$$\begin{cases} \Delta_x R(x, y) + q(x) R(x, y) = -q(x)\Gamma(x - y), & \text{in } \Omega_0, \\ R(x, y) = -\Gamma(x, y), & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \partial_{\nu_x} R(x, y) + i R(x, y) = -\partial_{\nu_x}\Gamma(x - y) - i\Gamma(x - y), & \text{on } \Sigma_0. \end{cases} \tag{4.36}$$

By Green's identity we arrive at

$$\begin{aligned} R(x, y) &= \int_{\Sigma_0} (\partial_{\nu_z} + i)\Gamma(z - y) G(z, x)\, d\sigma(z) \\ &\quad + \int_{\partial\Omega_0 \setminus \Sigma_0} \Gamma(z - y) G(z, x) + \int_{\Omega_0} \Gamma(z - y) q(z) G(z, x)\, dz. \end{aligned} \tag{4.37}$$

We denote

$$\widehat{R}(x, y) = \int_{B_{\frac{r_0}{8}}} \Gamma(z - y) q(z) G(z, x)\, dz. \tag{4.38}$$

With the stated assumptions on x and y, it is a straightforward matter to show that

$$\left|\nabla_y^2\big(R(x,y)-\widehat{R}(x,y)\big)\right| \leqslant C. \tag{4.39}$$

Let us investigate $\widehat{R}$. We set

$$B = B_{\frac{r_0}{8}}, \qquad B' = B'_{\frac{r_0}{8}}, \tag{4.40}$$

$$B^+ = \{x \in B : x_n > 0\}, \qquad B^- = \{x \in B : x_n < 0\}, \tag{4.41}$$

$$q^+ = q\big|_{B^+}, \qquad q^- = q\big|_{B^-}, \qquad [q] = \big(q^+ - q^-\big)|_{B'}. \tag{4.42}$$

We compute

$$\begin{aligned}
\partial_{y_i}\widehat{R}(x,y) &= -\int_B \partial_{z_i}\Gamma(z-y)q(z)G(z,x)\,dz \\
&= -\int_{\partial B}\Gamma(z-y)q(z)G(z,x)e_i\cdot\nu\,dS(z) \\
&\quad + \int_{B'}\Gamma\big(z'-y\big)\big[q\big(z'\big)\big]G\big(z',x\big)e_i\cdot e_n\,dz' \\
&\quad + \int_B \Gamma(z-y)\big(\partial_{z_i}q(z)G(z,x)+q(z)\partial_{z_i}G(z,x)\big)\,dz.
\end{aligned} \tag{4.43}$$

Note that $\partial_{z_i}q$ is well-defined on $B\setminus B'$. By further differentiation

$$\begin{aligned}
&\partial^2_{y_i y_j}\widehat{R}(x,y) \\
&\quad= -\int_{\partial B}\partial_{y_j}\Gamma(z-y)q(z)G(z,x)\,dS(z) \\
&\qquad + \int_{B'}\partial_{y_j}\Gamma\big(z'-y\big)\big[q\big(z'\big)\big]G\big(z',x\big)e_i\cdot e_n\,dz' \\
&\qquad + \int_B \partial_{y_j}\Gamma(z-y)\big(\partial_{z_i}q(z)G(z,x)+q(z)\partial_{z_i}G(z,x)\big)\,dz.
\end{aligned} \tag{4.44}$$

The first integral on the right hand side of the above equality is readily seen to be bounded. The third one is dominated by

$$\int_B \frac{1}{|z-y|^{n-1}|z-x|^{n-1}}\,dz \leqslant C|x-y|^{2-n} \tag{4.45}$$

and, since $|x-y|^2 = |x_n + r|^2 + |x'|^2 \geqslant r^2$, we can bound it as follows

$$\int_B \frac{1}{|z-y|^2|z-x|^2}\,dz \leqslant Cr^{2-n}. \tag{4.46}$$

The second integral, the one on B', is nontrivial only when $i = j = n$. Therefore,

$$\left|\partial^2_{y_i y_j} R(x, y)\right| \leqslant C|x - y|^{2-n} \qquad \text{for all } (i, j) \neq (n, n). \tag{4.47}$$

Now

$$\partial^2_{y_n} R(x, y) = \Delta_y R(x, y) - \Delta'_y R(x, y), \tag{4.48}$$

where $\Delta'_y R(x, y) = \sum_{i=1}^{n-1} \partial^2_{y_i} R(x, y)$. By the symmetry of G, we also have $R(x, y) = R(y, x)$, hence

$$\Delta_y R(x, y) + q(y)R(x, y) = -q(y)\Gamma(x - y). \tag{4.49}$$

Therefore,

$$\partial^2_{y_n} R(x, y) = -q(y)\big(R(x, y) + \Gamma(x, y)\big) - \Delta' R(x, y) \tag{4.50}$$

and consequently

$$\left|\partial^2_{y_n} R(x, y)\right| \leqslant C|x - y|^{2-n}. \tag{4.51}$$

Combining (4.47) and (4.51) together we get the desired bound for the full Hessian of R and the thesis follows. □

4.2. Propagation of smallness

Lemma 4.1. *Let v be a weak solution to*

$$\Delta v + qv = 0 \quad \text{in } \mathcal{W}_k, \tag{4.52}$$

where q is either equal to $\widetilde{q}^{(1)}$ or equal to $\widetilde{q}^{(2)}$. Assume that for given positive numbers ϵ_0, E_0 and real number γ, v satisfies

$$\left|v(x)\right| \leqslant \epsilon_0 \quad \text{for every } x \in \widehat{D}_0 \tag{4.53}$$

and

$$\left|v(x)\right| \leqslant C(\epsilon_0 + E_0)\big(1 + \mathrm{dist}(x, \mathcal{U}_k)\big)^{\gamma}, \quad \text{for every } x \in \mathcal{W}_k, \tag{4.54}$$

for some positive constant C. Then the following inequality holds true for every $0 < r < r_1$

$$\left|v(y_{k+1})\right| \leqslant C\left(\frac{\epsilon_0}{\epsilon_0 + E_0}\right)^{r\beta^{N_1}} (\epsilon_0 + E_0)\big(1 + r^{\gamma}\big), \tag{4.55}$$

where $y_{k+1} = P_{k+1} - 2r\nu(P_{k+1})$, where ν is the exterior unit normal to ∂D_k at P_{k+1}, $\beta = \frac{\ln(8/7)}{\ln 4}$, $r_1 = \frac{r_0}{16}$ and the constants C, N_1 depend on the a priori data only.

Proof. By Proposition 3.9 in [12], which is based on an iterated use of the three spheres inequality for elliptic equations, we infer that

$$\left|v(y_{k+1})\right| \leqslant C\left(\frac{\epsilon_0}{\epsilon_0+E_0}\right)^{r\beta^{N_1}}(\epsilon_0+E_0)\left(1+r^{(1-\tau_r)\gamma}\right), \tag{4.56}$$

where $\tau_r = \frac{\ln(\frac{12r_1-2r}{12r_1-3r})}{\ln(\frac{6r_1-r}{2r_1})} \in (0,1)$. By noticing that there exist positive constants C_1 and C_2 such that

$$C_1 r^{\gamma} \leqslant r^{(1-\tau_r)\gamma} \leqslant C_2 r^{\gamma}, \tag{4.57}$$

the thesis follows. □

Proof of Theorem 3.6. We observe that for any $y, z \in \mathcal{W}_k$ we have

$$\left|S_{\mathcal{U}_k}(y,z)\right| \leqslant C\left\|\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right\|_{L^\infty(\Omega_0)}\left(\mathrm{dist}(y,\mathcal{U}_k)\,\mathrm{dist}(z,\mathcal{U}_k)\right)^{2-\frac{n}{2}}, \tag{4.58}$$

for any $y, z \in \mathcal{W}_k$. By (4.58) and applying twice lemma 4.1 first to $v(\cdot) = S_{\mathcal{U}_k}(\cdot, z)$, with $z \in (D_0)_{\frac{r_0}{4}}$ and then to $v(\cdot) = S_{\mathcal{U}_k}(y,\cdot)$, with $y \in B_{3r_1-r}(x_{k+1})$, we find that

$$\left|\widetilde{S}_{\mathcal{U}_k}(y,z)\right| \leqslant C\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)^{r^2\beta^{2N_1}}\left(1+r^{2\gamma}\right), \tag{4.59}$$

for any $y, z \in B_{3r_1-r}(x_{k+1})$, where $x_{k+1} = P_{k+1} - 3r_1\nu(P_{k+1})$, ν is the exterior unit normal to ∂D_k and $\gamma = 2 - \frac{n}{2}$.

By considering $\widetilde{S}_{\mathcal{U}_k}(y,z)$ as a function of $2n$ variables where $(y,z) \in \mathbb{R}^{2n}$, (4.59) leads to

$$\left|\widetilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,z_1,\ldots,z_n)\right| \leqslant C\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)^{r^2\beta^{2N_1}}\left(1+r^{2\gamma}\right), \tag{4.60}$$

for any $x = (y_1,\ldots,y_n,z_1,\ldots,z_n) \in B_{3r_1-r}(x_{k+1}) \times B_{3r_1-r}(x_{k+1})$. Now observing that $\tilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,$ $z_1,\ldots,z_n)$ is a solution in $D_k \times D_k$ of the elliptic equation

$$(\Delta_y + \Delta_z)\widetilde{S}_{\mathcal{U}_k}(y,z) + \left(q_1(y)+q_2(z)\right)\tilde{S}_{\mathcal{U}_k}(y,z) = 0, \tag{4.61}$$

we have that given $y_{k+1} = P_{k+1} - 2r\nu(P_{k+1})$ by Schauder interior estimates

$$\begin{aligned}
&\left\|\partial_{y_i}\partial_{z_j}\tilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,z_1,\ldots,z_n)\right\|_{L^\infty(B_{\frac{r}{2}}(y_{k+1})\times\in B_{\frac{r}{2}}(y_{k+1})} \\
&\quad\leqslant \frac{C}{r^2}\left\|\tilde{S}_{\mathcal{U}_k}(y_1,\ldots,y_n,z_1,\ldots,z_n)\right\|_{L^\infty(B_r(y_{k+1})\times\in B_r(y_{k+1})}
\end{aligned}$$

and

$$\left\|\partial^2_{y_i}\partial^2_{z_j}\tilde{S}_{\mathcal{U}_k}(y_1,\dots,y_n,z_1,\dots,z_n)\right\|_{L^\infty(B_{\frac{r}{4}}(y_{k+1})\times\in B_{\frac{r}{4}}(y_{k+1}))}$$

$$\leqslant \frac{C}{r^2}\left\|\partial_{y_i}\partial_{z_j}\tilde{S}_{\mathcal{U}_k}(y_1,\dots,y_n,z_1,\dots,z_n)\right\|_{L^\infty(B_{\frac{r}{2}}(y_{k+1})\times\in B_{\frac{r}{2}}(y_{k+1}))},$$

for any $i, j = 1,\dots,n$, where $C > 0$ is a constant depending on the a priori data only. □

4.3. *Stability at the boundary*

Proof of estimate (3.34). We choose a coordinate system $\{x_1,\dots,x_n\}$ centred at P_1 with x_n in the direction of the normal ν and recall that for every $y, z \in D_0$ we have

$$\int_{\partial\Omega}\left(\partial_{y_n}\widetilde{G}_1(x,y)\partial_{x_n}\partial_{z_n}\widetilde{G}_2(x,z)-\partial_{z_n}\widetilde{G}_2(x,z)\partial_{x_n}\partial_{y_n}\widetilde{G}_1(x,y)\right)dS(x)$$

$$=\int_\Omega\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\partial_{y_n}\widetilde{G}_1(x,y)\partial_{z_n}\widetilde{G}_2(x,z)\,dx=\partial_{y_n}\partial_{z_n}\widetilde{S}_{\mathcal{U}_0}(y,z). \tag{4.62}$$

and

$$\int_{\partial\Omega}\left(\partial^2_{y_n}\widetilde{G}_1(x,y)\partial_{x_n}\partial^2_{z_n}\widetilde{G}_2(x,z)-\partial^2_{z_n}\widetilde{G}_2(x,z)\partial_{x_n}\partial^2_{y_n}\widetilde{G}_1(x,y)\right)dS(x)$$

$$=\int_\Omega\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\partial^2_{y_n}\widetilde{G}_1(x,y)\partial^2_{z_n}\widetilde{G}_2(x,z)\,dx=\partial^2_{y_n}\partial^2_{z_n}\widetilde{S}_{\mathcal{U}_0}(y,z). \tag{4.63}$$

By combining (4.62) together with (2.6), (2.12), we obtain

$$\left|\int_{\partial\Omega}\left(\partial_{y_n}\widetilde{G}_1(x,y)\partial_{x_n}\partial_{z_n}\widetilde{G}_2(x,z)-\partial_{z_n}\widetilde{G}_2(x,z)\partial_{x_n}\partial_{y_n}\widetilde{G}_1(x,y)\right)dS(x)\right|$$

$$\leqslant C\varepsilon_0\big(d(y)\,d(z)\big)^{-\frac{n}{2}},\qquad \text{for every}\, y,z\in D_0, \tag{4.64}$$

where $d(y)$ denotes the distance of y from Ω and C is a constant that depends on L, λ and n only. Let $\rho_0 = \frac{r_0}{C}$, where C is the constant introduced in Proposition 3.4, let $r \in (0, r_1)$, where r_1 has been introduced in Proposition 3.6 and denote

$$y_1 = P_1 + r\nu.$$

We set $y = z = y_1$ and obtain

$$\int_\Omega\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\partial_{y_n}\widetilde{G}_1(x,y)\partial_{z_n}\widetilde{G}_2(x,z)\,dx$$

$$=\int_{B_{\rho_0}(P_1)\cap D_1}\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\partial_{y_n}\widetilde{G}_1(x,y)\partial_{z_n}\widetilde{G}_2(x,z)\,dx$$

$$+\int_{\Omega\setminus(B_{\rho_0}(P_1)\cap D_1)}\left(\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right)(x)\partial_{y_n}\widetilde{G}_1(x,y)\partial_{z_n}\widetilde{G}_2(x,z)\,dx, \tag{4.65}$$

which leads to

$$\varepsilon_0 r^{-n} \geqslant \left| \int_{B_{\rho_0}(P_1)\cap D_1} \big(q_1^{(1)} - q_1^{(2)}\big)(\cdot)\partial_{y_n}\widetilde{G}_1(x, y_1)\partial_{z_n}\widetilde{G}_2(x, y_1) \right|$$
$$- \left| \int_{\Omega\setminus(B_{\rho_0}(P_1)\cap D_1)} \big(q^{(1)} - q^{(2)}\big)(\cdot)\partial_{y_n}\widetilde{G}_1(x, y_1)\partial_{z_n}\widetilde{G}_2(x, y_1) \right|. \tag{4.66}$$

Let $x^0 \in \overline{\Sigma_1 \cap B_{\frac{r_0}{4}(P_1)}}$ such that $(q_1^{(1)} - q_1^{(2)})(x^0) = \|\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})}$ and recall that $(q_1^{(1)} - q_1^{(2)})(x) = \alpha_1 + \beta_1 \cdot x$, therefore we obtain

$$\varepsilon_0 r^{-n} \geqslant \left| \int_{B_{\rho_0}(P_1)\cap D_1} \big(q_1^{(1)} - q_1^{(2)}\big)\big(x^0\big)\partial_{y_n}\widetilde{G}_1(x, y_1)\partial_{z_n}\widetilde{G}_2(x, y_1) \right|$$
$$- \left| \int_{B_{\rho_0}(P_1)\cap D_1} \beta_1 \cdot \big(x - x^0\big)\partial_{y_n}\widetilde{G}_1(x, y_1)\partial_{z_n}\widetilde{G}_2(x, y_1) \right| - CE\rho_0^{2-n} \tag{4.67}$$

and then

$$\|\widetilde{q}_1^{(1)} - \widetilde{q}_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))} \left| \int_{B_{\rho_0}(P_1)\cap D_1} \partial_{y_n}\widetilde{G}_1(x, y_1)\partial_{z_n}\widetilde{G}_2(x, y_1)\, dx \right|$$
$$\leqslant \int_{B_{\rho_0}(P_1)\cap D_1} |\beta_1|\big|x - x^0\big|\big|\partial_{y_n}\widetilde{G}_1(x, y_1)\big|\big|\partial_{z_n}\widetilde{G}_2(x, y_1)\big|\, dx + CE\rho_0^{2-n} + \varepsilon_0 r^{-n}. \tag{4.68}$$

For $n = 3$, by combining (4.68) together with (3.12), we obtain

$$\|\widetilde{q}_1^{(1)} - \widetilde{q}_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})} \int_{B_{\rho_0}(P_1)\cap D_1} \big|\nabla\Gamma(x - y_1)\big|^2\, dx$$
$$\leqslant C\Bigg\{ E\int_{B_{\rho_0}(P_1)\cap D_1} \big|\nabla\Gamma(x - y_1)\big| \log|x - y_1|\, dx + E\int_{B_{\rho_0}(P_1)\cap D_1} \big(\log|x - y_1|\big)^2\, dx$$
$$+ E\int_{B_{\rho_0}(P_1)\cap D_1} \big|x - x^0\big||x - y_1|^{-4}\, dx + E\rho_0^{-1} + \varepsilon_0 r^{-3} \Bigg\}, \tag{4.69}$$

which leads to

$$\|\widetilde{q}_1^{(1)} - \widetilde{q}_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}(P_1)})} \int_{B_{\rho_0}(P_1)\cap D_1} |x - y_1|^{-4}\, dx$$
$$\leqslant C\Bigg\{ E\int_{B_{\rho_0}(P_1)\cap D_1} |x - y_1|^{-3}\, dx + E\int_{B_{\rho_0}(P_1)\cap D_1} |x - y_1|^{-2}\, dx$$
$$+ E\int_{B_{\rho_0}(P_1)\cap D_1} |x - y_1|^{-3}\, dx + E\rho_0^{-1} + \varepsilon_0 r^{-3} \Bigg\}, \tag{4.70}$$

therefore

$$\left\|\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\leqslant C\left\{Er\log\left(\frac{\rho_0}{r}\right)+Er^2+E\rho_0^{-1}r+\varepsilon_0 r^{-2}\right\}$$
$$\leqslant C\left\{Er^{\theta}+\varepsilon r^{-2}\right\}, \tag{4.71}$$

for some θ, $0<\theta<1$ and by optimizing with respect to r

$$\left\|\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\leqslant C\varepsilon_0^{\frac{\theta}{\theta+2}}(E+\varepsilon_0)^{\frac{2}{\theta+2}}. \tag{4.72}$$

For $n\geqslant 4$, by combining (4.68) together (3.12), we obtain

$$\left\|\widetilde{q}_1^{(1)}-\widetilde{q}_1^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\int_{B_{\rho_0}(P_1)\cap D_1}\left|\nabla\Gamma(x-y_1)\right|^2 dx$$
$$\leqslant C\left\{E\int_{B_{\rho_0}(P_1)\cap D_1}\left|\nabla\Gamma(x-y_1)\right||x-y_1|^{3-n}\,dx\right.$$
$$+E\int_{B_{\rho_0}(P_1)\cap D_1}|x-y_1|^{6-2n}\,dx$$
$$\left.+E\int_{B_{\rho_0}(P_1)\cap D_1}\left|x-x^0\right||x-y_1|^{2-2n}\,dx+E\rho_0^{2-n}+\varepsilon_0 r^{-n}\right\}, \tag{4.73}$$

which leads to

$$\left\|\widetilde{q}_1^{(1)}-\widetilde{q}_1^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\int_{B_{\rho_0}(P_1)\cap D_1}|x-y_1|^{2-2n}\,dx$$
$$\leqslant C\left\{E\int_{B_{\rho_0}(P_1)\cap D_1}|x-y_1|^{4-2n}\,dx+E\int_{B_{\rho_0}(P_1)\cap D_1}|x-y_1|^{6-2n}\,dx\right.$$
$$\left.+E\int_{B_{\rho_0}(P_1)\cap D_1}|x-y_1|^{3-2n}\,dx+E\rho_0^{2-n}+\varepsilon_0 r^{-n}\right\}, \tag{4.74}$$

therefore

$$\left\|\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\leqslant C\left\{Er^2+Er^4+Er+E\rho_0^{2-n}r+\varepsilon_0 r^{-2}\right\}$$
$$\leqslant C\left\{Er+\varepsilon_0 r^{-2}\right\} \tag{4.75}$$

and by optimizing with respect to r

$$\left\|\widetilde{q}^{(1)}-\widetilde{q}^{(2)}\right\|_{L^{\infty}(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))}\leqslant C\varepsilon_0^{\frac{1}{3}}(E+\varepsilon_0)^{\frac{2}{3}}. \tag{4.76}$$

We proceed by estimating $\partial_\nu(\widetilde{q}^{(1)} - \widetilde{q}^{(2)})(P_1)$. By combining (4.63) together with (2.6), (2.12), we obtain

$$\left|\int_{\partial\Omega}\left(\partial^2_{y_n}\widetilde{G}_1(x,y)\partial_{x_n}\partial^2_{z_n}\widetilde{G}_2(x,z) - \partial^2_{z_n}\widetilde{G}_2(x,z)\partial_{x_n}\partial^2_{y_n}\widetilde{G}_1(x,y)\right)dS(x)\right|$$
$$\leqslant C\varepsilon_0\big(d(y)d(z)\big)^{-\frac{n}{2}-1}, \quad \text{for every } y, z \in D_0, \tag{4.77}$$

and setting $y = z = y_1$ in (4.77), we obtain

$$\int_{\partial\Omega}\left(\partial^2_{y_n}\widetilde{G}_1(x,y)\partial_{x_n}\partial^2_{z_n}\widetilde{G}_2(x,y_1) - \partial^2_{z_n}\widetilde{G}_2(x,y_1)\partial_{x_n}\partial^2_{y_n}\widetilde{G}_1(x,y_1)\right)dS(x)$$
$$= I_1(y_1) + I_2(y_1), \tag{4.78}$$

where

$$I_1(y_1) = \int_{B_{\rho_0}(P_1)\cap D_1}\big(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\big)(x)\partial^2_{y_n}\widetilde{G}_1(x,y_1)\partial^2_{z_n}\widetilde{G}_2(x,y_1)\,dx,$$
$$I_2(y_1) = \int_{\Omega\setminus(B_{\rho_0}(P_1)\cap D_1)}\big(\widetilde{q}^{(1)} - \widetilde{q}^{(2)}\big)(x)\partial^2_{y_n}\widetilde{G}_1(x,y_1)\partial^2_{z_n}\widetilde{G}_2(x,y_1)\,dx,$$

and

$$\big|I_2(w)\big| \leqslant CE\rho_0^{-n-2}. \tag{4.79}$$

We have

$$\big|I_1(y_1)\big|$$
$$\geqslant \left|\int_{B_{\rho_0}(P_1)\cap D_1}\big(\partial_{x_n}\big(q_1^{(1)} - q_1^{(2)}\big)(P_1)\big)(x-P_1)_n\partial^2_{y_n}\widetilde{G}_1(x,y_1)\partial^2_{z_n}\widetilde{G}_2(x,y_1)\right|dx$$
$$-\int_{B_{\rho_0}(P_1)\cap D_1}\big|\big(D_T\big(q_1^{(1)} - q_1^{(2)}\big)(P_1)\big)\cdot(x-P_1)'\big|\big|\partial^2_{y_n}\widetilde{G}_1(x,y_1)\big|\big|\partial^2_{z_n}\widetilde{G}_2(x,y_1)\big|\,dx$$
$$-\int_{B_{\rho_0}(P_1)\cap D_1}\big|\big(q_1^{(1)} - q_1^{(2)}\big)(P_1)\big|\big|\partial^2_{y_n}\widetilde{G}_1(x,y_1)\big|\big|\partial^2_{z_n}\widetilde{G}_2(x,y_1)\big|\,dx.$$

Noticing that up to a transformation of coordinates we can assume that P_1 coincides with the origin O of the coordinates system and recalling Theorem 3.5, this leads to

$$\big|I_1(y_1)\big| \geqslant \big|\partial_{x_n}\big(q_1^{(1)} - q_1^{(2)}\big)(O)\big|C\int_{B_{\rho_0}(O)\cap D_1}\big|\partial^2_{y_n}\Gamma(x,y_1)\big|^2|x_n|\,dx$$
$$-C\Bigg\{E\int_{B_{\rho_0}(O)\cap D_1}\big|\partial^2_{y_n}\Gamma(x,y_1)\big||x-y_1|^{2-n}|x_n|\,dx$$

$$- E \int_{B_{\rho_0}(O)\cap D_1} |x-y_1|^{4-2n} |x_n|\, dx \Big\}$$

$$- \int_{B_{\rho_0}(O)\cap D_1} \left| \left(D_T \left(q_1^{(1)} - q_1^{(2)} \right)(O) \right) \right| |x'| \left| \partial^2_{y_n} \widetilde{G}_1(x, y_1) \right| \left| \partial^2_{z_n} \widetilde{G}_2(x, y_1) \right| dx$$

$$- \int_{B_{\rho_0}(O)\cap D_1} \left| \left(q_1^{(1)} - q_1^{(2)} \right)(O) \right| \left| \partial^2_{y_n} \widetilde{G}_1(x, y_1) \right| \left| \partial^2_{z_n} \widetilde{G}_2(x, y_1) \right| dx. \tag{4.80}$$

Therefore, by combining (3.43) together with (3.41) and (3.42), we obtain

$$\begin{aligned} \left| I_1(y_1) \right| \geqslant & \left| \partial_{x_n} \left(q_1^{(1)} - q_1^{(2)} \right)(O) \right| C \int_{B_{\rho_0}(P_1)\cap D_1} |x-y_1|^{1-2n}\, dx \\ & - C \Big\{ E \int_{B_{\rho_0}(O)\cap D_1} |x-y_1|^{3-2n}\, dx \\ & - E \int_{B_{\rho_0}(O)\cap D_1} |x-y_1|^{5-2n}\, dx \\ & - (\varepsilon_0 + E) \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\eta_1} \int_{B_{\rho_0}(O)\cap D_1} |x-y_1|^{1-2n}\, dx \\ & - (\varepsilon_0 + E) \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\eta_1} \int_{B_{\rho_0}(O)\cap D_1} |x-y_1|^{-2n}\, dx \Big\}, \end{aligned}$$

which implies

$$\left| \partial_{x_n} \left(\gamma_1^{(1)} - \gamma_1^{(2)} \right)(O) \right| \sigma^{1-n} \leqslant \left| I_1(y_1) \right| + C \left\{ E r^{3-n} + (\varepsilon_0 + E) \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\eta_1} r^{-n} \right\}, \tag{4.81}$$

and

$$\begin{aligned} \left| I_1(y_1) \right| \leqslant & \left| \int_{\partial\Omega} \left(\partial^2_{y_n} \widetilde{G}_1(x, y_1) \partial_{x_n} \partial^2_{z_n} \widetilde{G}_2(x, y_1) - \partial^2_{z_n} \widetilde{G}_2(x, y_1) \partial_{x_n} \partial^2_{y_n} \widetilde{G}_1(x, y_1) \right) dS(x) \right| \\ & + C E \rho_0^{-n-2}. \end{aligned} \tag{4.82}$$

Thus by combining together the last two inequalities we get

$$\begin{aligned} \left| \partial_{x_n} \left(q_1^{(1)} - q_1^{(2)} \right)(O) \right| r^{1-n} \leqslant C \Big(& \varepsilon_0 r^{-n-2} + E \rho_0^{-n-2} \\ & + E r^{3-n} + (\varepsilon_0 + E) \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\eta_1} r^{-n} \Big), \end{aligned} \tag{4.83}$$

therefore

$$\left| \partial_{x_n} \left(q_1^{(1)} - q_1^{(2)} \right)(O) \right| \leqslant C \left\{ \varepsilon_0 r^{-3} + E r^2 + (E + \varepsilon_0) \left(\frac{\varepsilon_0}{E + \varepsilon_0} \right)^{\eta_1} r^{-1} \right\} \tag{4.84}$$

and by optimizing with respect to r we get

$$\left|\partial_{x_n}\left(q_1^{(1)}-q_1^{(2)}\right)(O)\right| \leqslant C(E+\varepsilon_0)\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)^{\frac{2\eta_1}{5}}. \tag{4.85}$$

□

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