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Article

Continuous Multi-Utility Representations of Preorders and the Chipman Approach

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Abstract: Chipman contended, in stark contrast to the conventional view, that, utility is not a real number but a vector, and that it is inherently lexicographic in nature. On the other hand, in recent years continuous multi-utility representations of a preorder on a topological space, which proved to be the best kind of continuous representation, have been deeply studied. In this paper, we first state a general result, which guarantees, for every preordered topological space, the existence of a lexicographic order-embedding of the Chipman type. Then, we combine the Chipman approach and the continuous multi-utility approach, by stating a theorem that guarantees, under certain general conditions, the coexistence of these two kinds of continuous representations.

Keywords: Hausdorff space; continuous multi-utility representation; order-embedding; semi-closed preorder

MSC: 62H05, 62H12, 62H20

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1. Introduction

The first author attended and presented a contribution at an international conference that took place in Essen (Germany) in October 1997. It was the case of a very impressive conference, which concerned mathematical utility theory and brought together economists, mathematicians, and psychologists. On that occasion, the first author knew personally, for the first time, Professor Gerhard Herden, an extremely fertile and intelligent mathematician, who was the principal organizer of the conference. Professor Herden personally compiled the list of all the invited speakers (there were no contributed talks). Professor Herden, who became a frequent coauthor of the first author, passed away on 30 January 2019. His enthusiasm remains unforgettable, as well as his capacity to formulate problems and to furnish extremely sharp definitions and axiomatizations. His work inspired the present contribution. In addition, the first author met in Essen Professor Chipman, who was also an invited speaker. He was a very kind person, whose contributions were invasive.

1.1. The Chipman Approach

Let $(X, \precsim)$ be some arbitrarily chosen non-empty preordered set. Using $\mathbb{R}$ as the codomain of a utility function (order-preserving function) on $(X, \precsim)$ is of almost universal practice in mathematical utility theory. But, as it has been shown by many illustrating examples in Herden and Mehta [1], there are cogent economic (and mathematical) reasons for not insisting on real-valued utility representations. In order to approach the general (continuous) utility representation problem, we, therefore, follow, in a first step, the views of Chipman. Chipman explicitly and wittingly studies utility functions with values in a set of lexicographically ordered transfinite sequences of length equal to an ordinal number λ. Indeed, in an elegant paper, Chipman [2] contends, in stark contrast to the conventional view, that utility is not a real number but a vector, and that it is inherently lexicographic in nature. Chipman also says that the concept of utility as a vector is easier to understand than that of utility as a real number.

In the same paper, he argues that, even if there is a real-valued utility function, it is preferable from an economic point of view to use a utility representation of the commodity space (which always exists) into a Dedekind-complete chain $(\{0,1\}^{\lambda}, \leq_{lex})$ of transfinite sequences of length λ, where λ is an appropriately chosen ordinal number and $\leq_{lex}$ is the lexicographic ordering on $\{0,1\}^{\lambda}$. In addition, Chipman [3] argues that requiring the existence of a real-valued utility representation implies the commodity space to have a countable order-dense subset (this is a necessary condition in the case of a total preorder). But countability does not have any intuitive appeal from an economic point of view.

In order to now mathematically approximate the Chipman approach, let $(X, \precsim)$ be an arbitrarily chosen preordered set. We denote, for every point $x \in X$, by $l(x)$ the set of all points $y \in X$ such that $y \prec x$ and by $r(x)$ the set of all points $z \in X$ such that $x \prec z$. In addition, we still note that $d(x)$ is, for every point $x \in X$, the set of all points $y \in X$ such that $y \precsim x$, and $i(x)$ is, for every point $x \in X$, the set of all points $z \in X$ such that $x \precsim z$. Then, the order topology $t^{\precsim}$ on X is the coarsest topology on X for which the sets $l(x)$ and $r(x)$ are open. In order to avoid artificial and superfluous considerations, we assume, for the moment, that $t^{\precsim}_{|\sim}$ is a Hausdorff topology on $X_{|\sim}$. Recall that $\sim$ is the indifference relation associated to the preorder $\precsim$ on X (i.e., for all points $x, y \in X$, $x \sim y$ is equivalent to the assertion that $x \precsim y$ and $y \precsim x$). For underlining the importance of this assumption and for later use, we still notice that in case $X_{|\sim}$ contains at least two elements, the following necessary conditions for $t^{\precsim}_{|\sim}$ to be Hausdorff hold. For the sake of brevity, the straightforward proofs of these conditions may be omitted. Nevertheless, the reader should notice that the validity of condition **LR** is based upon the validity of condition **lSB**.

SB: In order for $t^{\precsim}_{|\sim}$ to be Hausdorff, it is necessary that the sets $l(x)$ and $r(x)$, where x runs through X, constitute a sub-basis of $t^{\precsim}$.

LR: In order for $t^{\precsim}_{|\sim}$ to be Hausdorff, it is necessary that, for all points $x \in X$ and $y \in X$, the validity of the following implication holds for all $z \in l(x)$ and for all $u \in r(x)$:

$$(y \in r(z)) \wedge (y \in d(u)) \Rightarrow x \sim y.$$

In Lemma 2, it will be proved that, when $\precsim$ is *semi-closed*, i.e., $d(x)$ and $i(x)$ are closed subsets of $(X, t^{\precsim})$ for every $x \in X$, the validity of the conditions **SB** and **LR** already implies that $t^{\precsim}_{|\sim}$ is Hausdorff. The general case, however, is difficult. No simple solution can be expected.

Let us proceed by considering an ordinal number $0 < \lambda$. Then, the triplet $(\{0,1\}^{\lambda}, \leq_{lex}, t^{\leq_{lex}})$ is the preordered topological space that consists of the lexicographically ordered set $(\{0,1\}^{\lambda}, \leq_{lex})$ endowed with its order topology $t^{\leq_{lex}}$. Now, we set $\kappa := | X_{|\sim} |$, i.e., κ is the cardinality of the set $X_{|\sim}$ of indifference classes $[x]$ of $\precsim$, and consider the particular triplet $(\{0,1\}^{\kappa}, \leq_{lex}, t^{\leq_{lex}})$. As it is at least implicitly well-known, there exists a natural order-embedding $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex})$. But for some arbitrarily chosen topology t on X that is finer than $t^{\precsim}$, in general, there exists no order preserving function $\gamma : (\psi(X), \leq_{lex}) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex})$ such that the composition

$$\gamma \circ \psi : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex}, t^{\leq_{lex}})$$

is continuous. The reader may notice that this observation principally excludes the existence of some continuous order-embedding $\vartheta : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex}, t^{\leq_{lex}})$. Therefore, using a construction that is due to Beardon [4], an effort has been made in coarsening $\leq_{lex}$ to some preorder $\lesssim$ in such a way that the indifference classes of $\lesssim$ are (in some sense) the smallest possible closed intervals of $(\{0,1\}^{\kappa}, \leq_{lex})$ with respect to the property of guaranteeing a continuous order-embedding $\vartheta : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \leq, t^{\leq})$. This is the content of Theorem 1.

In the literature, various Dedekind-complete chains $(D, \leq)$ have been considered for possibly appropriate codomains of a utility function (cf. the papers that are quoted in Herden and Mehta [1]).

In order to examine these possible codomains of a utility function more closely, let $(D, \leq)$ be a fixed chosen preordered set, and consider an order-embedding $\phi : (X, \precsim) \longrightarrow (D, \leq)$. Let, furthermore, λ run through the class of all ordinal numbers. Then, the universal character of $(\{0,1\}^{\lambda}, \leq_{lex})$ as codomain of a utility function is underlined by the observation that there exist an ordinal number λ and order-embeddings $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$ and $\eta : (D, \leq) \longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$ such that the following diagram commutes

$$\begin{array}{ccc} (\{0,1\}^{\lambda}, \leq_{lex}) & \overset{\eta}{\leftarrow} & (D, \leq) \\ \uparrow \psi & \nearrow \phi & \\ (X, \precsim) & & \end{array} .$$

Moreover, (total) preorders $\leq \subset \trianglelefteq$ on D and $\leq_{lex} \subset \precsim$ on $\{0,1\}^{\lambda}$, and order-embeddings

$$\nu : (\phi(X), \leq) \longrightarrow (D, \trianglelefteq)$$

and

$$\rho : (\psi(X), \leq_{lex}) \longrightarrow (\{0,1\}^{\lambda}, \precsim),$$

can be chosen in such a way that the compositions

$$\theta := \nu \circ \phi : (X, \precsim, t) \longrightarrow (D, \trianglelefteq, t^{\trianglelefteq}),$$

$$\vartheta := \rho \circ \psi : (X, \precsim, t) \longrightarrow (\{0,1\}^{\lambda}, \precsim, t^{\precsim})$$

and

$$\chi := \rho \circ \eta : (D, \trianglelefteq, t^{\trianglelefteq}) \longrightarrow (\{0,1\}^{\lambda}, \precsim, t^{\precsim})$$

are continuous and the following diagram commutes

$$\begin{array}{ccc} (\{0,1\}^{\lambda}, \precsim, t^{\precsim}) & \overset{\chi}{\leftarrow} & (D, \trianglelefteq, t^{\trianglelefteq}) \\ \uparrow \vartheta & \nearrow \theta & \\ (X, \precsim, t) & & \end{array}$$

These are the contents of Theorem 2.

1.2. The Continuous Multi-Utility Representation Theorem

In this section, we want to compare the Chipman approach to mathematical utility theory with the standard real-valued approach to mathematical utility theory.

We, thus, choose a preordered topological space $(X, \precsim, t)$, whose topology t is finer than $t^{\precsim}$. Then, the reader may recall that $\precsim$ is said to have a *continuous multi-utility representation* if there exists a family $\mathcal{F}$ of increasing and continuous functions $f : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that

$$\precsim = \{(x, y) \in X \times X \mid \forall f \in \mathcal{F}\, (f(x) \leq f(y))\}$$

or, equivalently, if there exists, for every pair $(x, y) \in X \times X$ such that *not*$(x \precsim y)$, some continuous and increasing function $f_{xy} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(y) < f_{xy}(x)$.

In addition, the reader may recall that $\precsim$ is said to be *closed* if $\precsim$ is a closed subset of $X \times X$ with respect to the product topology $t \times t$ on $X \times X$ that is induced by t.

We recall that multi-utility representations were introduced by Levin [5]. A full study of this kind of representation, compatible with nontotal preorders, was provided by Evren and Ok [6], after a seminal paper by Ok [7]. Contributions to this topic were presented by Bosi and Herden [8,9], Minguzzi [10,11], and Pivato [12]. Very interesting applications to expected utility theory are due to Dubra et al. [13], Evren [14], and Galaabaatar and Karni [15]. We also mention that continuous representations of interval orders by a pair of functions were axiomatized by Bosi et al. [16]. Hack et al. [17] studied, in a very recent paper, the classification of preordered spaces in terms of their possible multi-utility representations.

With the help of this notation, we shall prove that the assumption $t^{\precsim}_{|\sim}$ to be Hausdorff guarantees the equivalence of the assertions $\precsim$ to admit a continuous multi-utility representation, $\precsim$ to be semi-closed and $\precsim$ to be closed (cf. Theorem 3).

Let, for the moment, ω be the cardinality of the set $\mathbf{N}(\precsim)$ of all pairs $([x],[y]) \in X_{|\sim} \times X_{|\sim}$ such that $not(x \precsim y)$. Then, we arbitrarily choose a bijective function $\sigma : [0,\omega[\longrightarrow \mathbf{N}(\precsim)$ in order to consider the direct sum $(\mathbb{R},\leq,t_{nat}) \times ([0,\omega[,<)$ of ordered topological spaces $(\mathbb{R},\leq,t_{nat})$.

We proceed by assuming $t^{\precsim}_{|\sim}$ to be Hausdorff, and $\precsim$ to satisfy the equivalent assertions of Theorem 3, which, in particular, means that $\precsim$ admits a continuous multi-utility representation. Let now $(X,\precsim,t) \times ([0,\omega[,<)$ be the direct sum of preordered topological spaces $(X,\precsim,t)$. Since $\precsim$ has a continuous multi-utility representation, the definition of $\sigma : [0,\omega[\longrightarrow \mathbf{N}(\precsim)$ implies the existence of a continuous order-preserving function $\Phi : (X,\precsim,t) \times ([0,\omega[,<) \longrightarrow (\mathbb{R},\leq,t_{nat}) \times ([0,\omega[,<)$. This obvious consideration already clarifies the relation of the multi-utility approach for representing a preorder with the Richter–Peleg approach for representing a preorder (cf. Herden [18], Peleg [19], and Evren and Ok [7]).

We recall that a *Richter–Peleg representation* (or *utility function*, or else *order-preserving function*) of a preordered set $(X,\precsim)$ is a real-valued function u on X which is *order-preserving* (i.e., u is *increasing* and, for all points $x, y \in X$, $x \prec y$ implies that $u(x) < u(y)$).

Let, finally, $\kappa := \mid X_{|\sim} \mid$ and $\xi := \omega \cdot \kappa = \mid \omega \times \kappa \mid$. Then, an application of Theorem 2 (cf. the last paragraph of Section 1.1) guarantees the existence of a continuous order-embedding

$$\Psi := (\mathbb{R},\leq,t_{nat}) \times [0,\omega[\longrightarrow \left((\{0,1\}^{\omega})^{\kappa},\lesssim,t^{\lesssim}\right) = \left(\{0,1\}^{\xi},\lesssim,t^{\lesssim}\right)$$

and of a continuous order-embedding

$$\Upsilon : (X,\precsim,t) \times [0,\omega[\longrightarrow \left(\{0,1\}^{\xi},\lesssim,t^{\lesssim}\right)$$

such that the following diagram commutes

$$\begin{array}{ccc} \left((\{0,1\}^{\omega})^{\kappa},\lesssim,t^{\lesssim}\right) = \left(\{0,1\}^{\xi},\lesssim,t^{\lesssim}\right) & \overset{\Psi}{\leftarrow} & (\mathbb{R},\leq,t_{nat}) \times ([0,\omega[,<) \\ \uparrow \Upsilon & \nearrow \Phi & \\ (X,\precsim,t) \times ([0,\omega[,<) & & \end{array}.$$

In this way, the Chipman approach, the multi-utility approach, and the Richter–Peleg approach for representing a preorder have been combined (cf. Theorem 4)

2. On an Order-Embedding Theorem of Chipman-Type

In the remainder of this section, we shall consider some preordered topological space $(X,\precsim,t)$, whose topology t is finer than the order topology $t^{\precsim}$, and it is such that the quotient topology $t^{\precsim}_{|\sim}$ that is defined on the set $X_{|\sim}$ of indifference classes of $\precsim$ is Hausdorff. Then, it is the first aim of this section to prove and to comment on the following theorem.

Theorem 1. *The following assertions hold:*
(i) *There exists some cardinal number κ for which there exists an order-embedding $\psi : (X,\precsim) \longrightarrow (\{0,1\}^{\kappa},\leq_{lex})$.*
(ii) *There exists some cardinal number κ and a (complete) preorder $\lesssim$ on $\{0,1\}^{\kappa}$, that is coarser than $\leq_{lex}$, for which there exists a continuous order-embedding $\vartheta : (X,\precsim,t) \longrightarrow (\{0,1\}^{\kappa},\lesssim,t^{\lesssim})$.*

Proof. The validity of Assertion (i) is (at least implicitly) well known. It holds without assuming $(X_{|\sim}, t_{|\sim})$ to be a Hausdorff-space. Nevertheless, we must repeat its proof here in order to prepare the proof of Assertion (ii). Let, therefore, κ be the cardinality $| X_{|\sim} |$ of $X_{|\sim}$. Then, we arbitrarily choose some bijective function $\beta : [0, \kappa[\longrightarrow X_{|\sim}$ in order to define the desired order-embedding $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex})$ by identifying $\psi(x)$, for every $x \in X$, with the tuple $(x_{\alpha})_{\alpha<\kappa} \in \{0,1\}^{\kappa}$ that is defined by setting

$$x_{\alpha} := \begin{cases} 1 & \text{if } \beta(\alpha) \precsim_{|\sim} [x] \\ 0 & \text{otherwise} \end{cases}$$

for all ordinal numbers $\alpha < \kappa$. Since for every point $x \in X$ and every point $y \in X$ the validity of the equivalence $x \precsim y \Leftrightarrow d(x) \subseteq d(y)$ holds, it follows that ψ is an order-embedding.

In order to now verify Assertion (ii), let S be an arbitrary subset of $\{0,1\}^{\kappa}$. Then, a lacuna of S is a non-degenerate (non-trivial) interval of $\{0,1\}^{\kappa}$ that is disjoint from S and has an upper and lower bound in S. A gap of S is a maximal lacuna. Although $\precsim$ is not necessarily complete, it is easy to see that the inclusion $t \supseteq t^{\precsim}$ would imply the continuity of ψ considered as a function $\psi : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$, if $\psi(X)$ would not have any gaps of the form $[u, v[$ or $]u, v]$, i.e., half-closed half-open or half-open half-closed gaps (cf. Beardon [1]). But in order to eliminate these gaps, a difficulty appears. Indeed, let $(x_i)_{i \in I}$ be a net that converges to some point $x \in X$. Then, it may happen that $\lim_{i \in I, \psi(x_i) \leq_{lex} \psi(x)} x_i = x$ or that $\lim_{i \in I, \psi(x_i) \geq_{lex} \psi(x)} x_i = x$.

This means that a (crucial) gap to be eliminated in order to guarantee continuity of ψ at x is of the form

$$\left[\sup_{i \in I, \psi(x_i) <_{lex} \psi(x)} \psi(x_i), \psi(x) \right[$$

or

$$\left] \sup_{i \in I, \psi(x_i) \leq_{lex} \psi(x)} \psi(x_i), \psi(x) \right]$$

or

$$\left[\psi(x), \inf_{i \in I, \psi(x_i) \geq_{lex} \psi(x)} \psi(x_i) \right[$$

or

$$\left] \psi(x), \inf_{i \in I, \psi(x_i) >_{lex} \psi(x)} \psi(x_i) \right].$$

Of course, there may exist another net $(y_j)_{j \in J}$ that converges to x. It, thus, follows that a possibility to eliminate these (crucial) gaps only exists if

$$\left[\sup_{i \in I, \psi(x_i) <_{lex} \psi(x)} \psi(x_i), \psi(x) \right[= \left[\sup_{j \in J, \psi(y_j) <_{lex} \psi(x)} \psi(y_j), \psi(x) \right[$$

or

$$\left] \sup_{i \in I, \psi(x_i) \leq_{lex} \psi(x)} \psi(x_i), \psi(x) \right] = \left] \sup_{j \in J, \psi(y_j) \leq_{lex} \psi(x)} \psi(y_j), \psi(x) \right]$$

in case that

$$\lim_{i \in I, \psi(x_i) \leq_{lex} \psi(x)} x_i = \lim_{j \in J, \psi(y_j) \leq_{lex} \psi(x)} y_j = x$$

and that

$$\left[\psi(x), \inf_{i \in I, \psi(x_i) \geq_{lex} \psi(x)} \psi(x_i) \right[= \left[\psi(x), \inf_{j \in J, \psi(y_j) \geq_{lex} \psi(x)} \psi(y_j) \right[,$$

in case that

$$\lim_{i\in I, \psi(x_i)\geq_{lex}\psi(x)} x_i = \lim_{j\in J, \psi(y_j)\geq_{lex}\psi(x)} y_j = x,$$

i.e., the precise form of the (crucial) gaps must be independent of the considered net that converges to x. In order to guarantee the independence of (crucial) gaps from particularly chosen nets that converge to x, the assumption $t^{\precsim}_{|\sim}$ to be a Hausdorff-topology is needed (cf. Example 1). Indeed, if $t^{\precsim}_{|\sim}$ is Hausdorff, independence of (crucial) gaps from particularly considered nets that converge to x follows by distinguishing between four different cases (cf. the above-described possibilities of (crucial) gaps), each of which can be done by applying always the same indirect argument that is based upon condition **SB** and the definition of ψ applied to the equations $\lim_{i\in I} x_i = x = \lim_{j\in J} y_j$. Since this argument is routine and obvious in nature, it may be omitted for the sake of brevity. The independence of (crucial) gaps from particularly chosen nets that converge to x allows us to now proceed by considering the collection $\mathbb{H}$ of all half-closed half-open and all half-open half-closed gaps of $\psi(X)$. In accordance with Beardon [1], we define as follows an equivalence relation $\sim$ on $\{0,1\}^{\kappa}$ with respect to $\mathbb{H}$. If $[r,s[$ and $[s,t[$ are adjacent gaps of $X \setminus \bigcup \mathbb{H}$, then $[r,t]$ is an equivalence class of $\sim$. In addition, if $[u,v[$ and $]w,z]$, respectively, are gaps that do not belong to pairs of adjacent gaps of $X \setminus \bigcup \mathbb{H}$, then $[u,v]$ and $[w,z]$, respectively, define the corresponding equivalence classes of $\sim$. All the other equivalence classes of $\sim$ are defined to be singletons. Since the equivalence classes of $\sim$ are closed intervals of $(\{0,1\}^{\kappa}, \leq_{lex})$, we may define the desired preorder $\lesssim$ on $\{0,1\}^{\kappa}$ that is coarser than $\leq_{lex}$ by setting

$$x \lesssim y \Leftrightarrow [x] = [y] \text{ or} \sup[x] <_{lex} \inf[y]$$

for all $x \in \{0,1\}^{\kappa}$ and all $y \in \{0,1\}^{\kappa}$. Since, for all $x \in \{0,1\}^{\kappa}$ and all $y \in \{0,1\}^{\kappa}$, the inequality $x \leq_{lex} y$ implies that $x \lesssim y$, it follows that $\lesssim$, actually, is coarser than $\leq_{lex}$. Hence, we now consider the (continuous) identity-function $id : (\{0,1\}^{\kappa}, \leq_{lex}, t^{\leq_{lex}}) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$. Furthermore, the definition of $\sim$ implies that the image of the composition

$$\vartheta : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$$

of the functions $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex})$ and $id : (\{0,1\}^{\kappa}, \leq_{lex}, t^{\leq_{lex}}) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$ has, with respect to $\lesssim$, neither half-closed half-open nor half-open half-closed gaps.

Since $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\kappa}, \leq_{lex})$ is an order-embedding, we, therefore, may conclude that $\vartheta : (X, \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$ is a continuous order-embedding. This last observation finishes the proof of the theorem. □

Example 1. Let X be the real unit interval $[0,1]$. Then, we endow X with the order

$$\trianglelefteq := \{(x,x) \mid x \in X\} \cup \{(x,y) \in [0,1[\cup [0,1[\mid x \leq y\}.$$

It follows that $t^{\trianglelefteq} = t_{nat|[0,1]} \cup \{[0,1]\}$, which means that $t^{\trianglelefteq}$ is not Hausdorff. Let $\psi : (X, \trianglelefteq) \longrightarrow (\{0,1\}^{2^{\aleph_0}}, \leq_{lex})$ be the order-embedding that has been described in Theorem 1. Then, we may conclude that the nets (sequences) $\left(\frac{1}{2} - \frac{1}{2n}\right)_{n\geq 1}$ and $\left(\frac{1}{4} - \frac{1}{4n}\right)_{n\geq 1}$ converge, with respect to $t^{\trianglelefteq}$, to 1, but that

$$\left[\sup_{n\geq 1, \psi\left(\frac{1}{2}-\frac{1}{2n}\right)<_{lex}\psi(1)} \psi\left(\frac{1}{2} - \frac{1}{2n}\right), \psi(1)\right[\subsetneqq \left[\sup_{n\geq 1, \psi\left(\frac{1}{4}-\frac{1}{4n}\right)<_{lex}\psi(1)} \psi\left(\frac{1}{4} - \frac{1}{4n}\right), \psi(1)\right[.$$

Let **ORD** be the class of ordinal numbers. As it already has been outlined to some degree in the last paragraph of Section 1.1, we now prove a theorem that underlines the universal character of the class $\{(\{0,1\}^{\lambda}, \leq_{lex}) \mid \lambda \in \textbf{ORD}\}$ of chains.

Theorem 2. *Let $\phi : (X, \precsim) \longrightarrow (D, \leq)$ be an order-embedding of $(X, \precsim)$ into the Dedekind-complete chain $(D, \leq)$. Then, the following assertions hold:*

(i) *There exist some ordinal number λ and order-embeddings*

$$\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$$

and

$$\eta :\longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$$

such that the following diagram commutes

$$\begin{array}{ccc} (\{0,1\}^{\lambda}, \leq_{lex}) & \overset{\eta}{\leftarrow} & (D, \leq) \\ \uparrow \psi & \nearrow \phi & \\ (X, \precsim) & & \end{array}.$$

(ii) *In addition to Assertion* (i)*, (total) preorders $\leq \subset \trianglelefteq$ on D and $\leq_{lex} \subset \lesssim$ and order-embeddings*

$$\nu : (\phi(X), \leq) \longrightarrow (D, \trianglelefteq)$$

and

$$\rho : (\psi(X), \leq_{lex}) \longrightarrow (\{0,1\}^{\lambda}, \lesssim)$$

can be chosen in such a way that the compositions

$$\theta := \nu \circ \phi : (X \precsim, t) \longrightarrow (D, \trianglelefteq, t^{\trianglelefteq}),$$

$$\vartheta := \rho \circ \psi : (X \precsim, t) \longrightarrow (\{0,1\}^{\lambda}, \lesssim, t^{\lesssim})$$

and

$$\chi := \rho \circ \eta : (D, \trianglelefteq, t^{\trianglelefteq}) \longrightarrow (\{0,1\}^{\lambda}, \lesssim, t^{\lesssim})$$

are continuous and the following diagram commutes

$$\begin{array}{ccc} (\{0,1\}^{\lambda}, \lesssim, t^{\lesssim}) & \overset{\chi}{\leftarrow} & (D, \trianglelefteq, t^{\trianglelefteq}) \\ \uparrow \vartheta & \nearrow \theta & \\ (X, \precsim, t) & & \end{array}.$$

Proof. Of course, we may assume without loss of generality that $\kappa := | X_{|\sim} | \geq 2$. In order to now verify Assertion (i), we first notice that

$$x \precsim y \Leftrightarrow d(x) \subseteq d(y) \Leftrightarrow i(x) \supseteq i(y),$$

for all points $x \in X$ and $y \in X$. Let λ be the maximum of $| D |$, and $\beta : [0, \kappa[\longrightarrow X_{|\sim}$ an arbitrarily chosen bijective function. Then condition **LR** implies the existence of an order-embedding $\eta : (D, \leq) \longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$ that is defined by identifying $\eta(a)$, for every $a \in D$, with the tuple $\left(\underbrace{1, 1, \ldots, 1, \ldots}_{\lambda\text{-times}} \right) \in \{0,1\}^{\lambda}$, if $a \notin \phi(X)$. Let, therefore, $a \in \phi(X)$. Then, we must at first verify that the subsequent definition is independent of the particular chosen point $x \in X$ such that $\phi(x) = a$. This means that we must prove that the equation $\phi(x) = \phi(y)$ implies that $[x] = [y]$. Indeed, if $\phi(x) = \phi(y)$, then we have that $l(x) = l(y)$ and $r(x) = r(y)$. Hence, it follows that assumptions of condition **LR** are satisfied, which implies that $[x] = [y]$. Let, consequently, some $x \in X$ such that $a = \phi(x)$ be arbitrarily chosen. Then, we may identify $\eta(a)$ with the tuple $(a_{\alpha})_{\alpha < \lambda}$ that is defined by setting

$$a_{\alpha} := \begin{cases} 1 & \text{if } \alpha < \kappa \text{ and } \beta(\alpha) \precsim_{|\sim} [x] \\ 0 & \text{otherwise} \end{cases}$$

for all ordinal numbers $\alpha < \lambda$ (cf. the definition of ψ in the proof of Theorem 1). Of course, the order-embedding $\psi : (X, \precsim) \longrightarrow (\{0,1\}^{\lambda}, \leq_{lex})$ also has to be defined in the same way, i.e., by identifying $\psi(x)$ for every $x \in X$ with the tuple $(x_\alpha)_{\alpha<\lambda}$ that is defined by setting

$$x_\alpha := \begin{cases} 1 & \text{if } \alpha < \kappa \text{ and } \beta(\alpha) \precsim_{|\sim} [x] \\ 0 & \text{otherwise} \end{cases},$$

for every ordinal number $\alpha < \lambda$. The definitions of the order-embeddings η and ψ imply that $\psi = \eta \circ \phi$, which proves Assertion (i).

In order to prove the validity of Assertion (ii), we use the notation that has been introduced in the proof of Assertion (ii) of Theorem 1. Then, we apply the arguments that have been used in the proof of Assertion (ii) of Theorem 1 in order to verify that the image of the order-embedding
$\rho := id_{\psi(X)} : (\psi(X), \leq_{lex}) \longrightarrow (\{0,1\}^{\lambda}, \precsim)$ neither has half-closed half-open nor half-open half-closed gaps. The proof of Assertion (i) implies that $\eta^{-1} : (\psi(X), \leq_{lex}) \longrightarrow (\phi(X), \leq)$ is an order-isomorphism. Hence, the validity of the following implications, which will be abbreviated by (*), holds:

If the crucial gap I of $\phi(X)$ is the image of some half-closed half-open or some half-open half-closed interval J, then J is a crucial gap of $\phi(X)$. And, conversely:

If the crucial gap J of $\psi(X)$ is the image of some half-closed half-open or some half-open half-closed interval I, then I is a crucial gap of $\psi(X)$.

In particular, we may conclude that the crucial gaps of $\phi(X)$ are independent of particularly chosen converging nets. Hence, we may apply the Beardon construction that has been described in the proof of Assertion (ii) of Theorem 1 in order to define a preorder $\trianglelefteq$ on D that is coarser than $\leq$ in such a way that neither the image of the order-embedding $\nu := id_{\phi(X)} : (\phi(X), \leq) \longrightarrow (D, \trianglelefteq)$ nor the image of the order-embedding $\eta : (\phi X), \trianglelefteq) \longrightarrow (\{0,1\}^{\lambda}, \precsim)$ has any half-closed half-open or half-open half-closed gaps. Hence, the validity of the implications (*) allows us to conclude that the compositions $\theta := \nu \circ \phi : (X, \precsim, t) \longrightarrow (D, \trianglelefteq, t^{\trianglelefteq})$, $\vartheta := \rho \circ \psi : (X, \precsim, t) \longrightarrow (\{0,1\}^{\lambda}, \precsim, t^{\precsim})$ and $\chi := \rho \circ \eta : (D, \trianglelefteq, t^{\trianglelefteq}) \longrightarrow (\{0,1\}^{\lambda}, \precsim)$ are continuous (cf. the proof of Assertion (ii) of Theorem 1). In addition, the validity of Assertion (i) guarantees that $\vartheta = \chi \circ \theta$. So, the proof is complete. □

3. On a Relation of the Chipman Approach with the Continuous Multi-Utility Representation Problem of Preorders

Let (X, t) be an arbitrarily chosen topological space. The problem of determining (characterizing) all preorders $\precsim$ on X, which admit a continuous multi-utility representation, is the focus of this section. We shall further assume that the order topology $t^{\precsim}$ of $\precsim$ is coarser than t. It is well known (cf, for instance, Bosi and Herden [3]) that the assumption $\precsim$ to admit a continuous multi-utility representation implies that $\precsim$ must be closed and, therefore, also semi-closed. Hence, the following lemmas provide a first important step towards a complete solution of the just mentioned characterization problem. As in the proof of Theorem 2 throughout this section, we may assume without loss of generality that $| X_{|\sim} | \geq 2$.

In order to proceed, let us denote for every point $x \in X$ by $\mathbf{N}_{\precsim}(x)$ the set of all points $y \in X \setminus l(x)$ such that $not(x \precsim y)$. This notation allows us to verify the validity of the following lemma.

Lemma 1. *Let $x \in X$ be arbitrarily chosen. Then, $\precsim$ satisfies the following conditions:*

HD: *Let $\precsim$ have a continuous multi-utility representation. Then, $(X_{|\sim}, \precsim_{|\sim})$ is a Hausdorff space.*

OC: *Let $(X_{|\sim}, \precsim_{|\sim})$ be a Hausdorff space and let $\precsim$ be closed. Then, $d(y)$ is open (and closed) for every point $y \in \mathbf{N}_{\precsim}(x)$ that is maximal with respect to $(X, \precsim)$.*

Proof. HD: Let $\precsim$ have a continuous multi-utility representation, and let $x \in X$ and $y \in X$ be arbitrarily chosen points such that $not(x \precsim y)$. Then, there exists a continuous and increasing

function
$f_{xy} : (X, \precsim, t^{\precsim}) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(y) < f_{xy}(x)$. Hence, the desired conclusion follows.

OC: Let $y \in \mathbf{N}_{\precsim}(x)$ be a maximal element of $(X, \precsim)$, which means that $r(y) = \varnothing$. Then, the assumption according to which $t^{\precsim}_{|\sim}$ is Hausdorff implies, with help of condition **SB**, that $l(y) \neq \varnothing$. Hence, we may distinguish between the cases when $(l(y), \precsim)$ has a maximal element, and, respectively $(l(y), \precsim)$ has no maximal element. Let us, therefore, assume at first that $(l(y), \precsim)$ has a maximal element m. Then, the interval $]m, y[$ is empty. This means, in particular, that there exists no net $(m(s))_{s \in S}$ of points $m(s) \in r(m)$ that converges to y. Hence, the set $\mathbf{U}(y) := \{t \in r(m) \mid t \in r(m) \setminus [y]\}$ must be closed, and we may conclude that $[y] = r(m) \setminus \mathbf{U}(y)$ is open and closed. We, thus, proceed by showing that both sets $l(y)$ as well as $d(y)$ are open and closed. In order to verify these properties of $l(y)$ and $d(y)$, respectively, it suffices to prove that $l(y)$ is closed and that $d(y)$ is open. Let, therefore, in a first step, some point $p \in \overline{l(y)}$ be arbitrarily chosen. Then, we have to show that $p \in l(y)$. We, thus, consider some net $(p_o)_{o \in O}$ of points $p_o \in l(y)$ that converges to p. Since $\precsim$ is closed and $p_o \prec y$ for all $o \in O$, it follows that $p \precsim y$, and it remains to verify that the equivalence $p \sim y$ can be excluded. Indeed, if $p \sim y$, then the just proved property that $[y]$ is open (and closed) implies that there exists some index o_y such that $p_o \sim y$ for all points $o \in O$ which are at least as great as o_y. This contradiction implies that $l(y)$ must be closed. For later use, in particular in the proof of Theorem 3, we abbreviate this conclusion by (*). Since $l(y)$ is open and $[y]$ is open, it follows, in a second step, that $d(y) = l(y) \cup [y]$ is open, which completes the discussion of the case $(l(y), \precsim)$ to have a maximal element. We now still must think of the situation $\left[\sup\limits_{q \in l(y)} q\right]$ to coincide with $[y]$. Let, in this situation, $(C, \precsim)$ be some sub-chain of $(l(y), \precsim)$ such that $\left[\sup\limits_{c \in C} c\right] = [y]$ Because of property (*), we may assume without loss of generality that $[y]$ is not open (and closed). We, thus, may arbitrarily choose some point $c \in C$ in order to then consider some net $(y_i)_{i \in I}$ of points $y_i \in r(c)$ which converges to y. Because of the maximality of y with respect to $(X, \precsim)$, it follows that, for every $i \in I$, there exist points $c' \in C$ and $c'' \in C$ such that $c \precsim c' \precsim y_i \precsim c''$. Indeed, otherwise the definition of $t^{\precsim}$ implies that $[y]$ is the meet of two open intervals and, thus, it is open (and closed), which contradicts our assumption according to which $[y]$ is not open (and closed). This argument will be abbreviated by (**M**). But this consideration allows us to conclude that, for every point $l \in l(y)$, the set $r(l) \cap d(y)$ is an open neighborhood of y. Hence, it follows that $d(y) = l(y) \cup (r(l) \cap d(y))$ is open (and closed) for every point $l \in l(y)$, which still was to be shown. □

As it already has been announced in the introduction, we now characterize those semi-closed preorders $\precsim$ on (X, t) for which $t^{\precsim}_{|\sim}$ is Hausdorff.

Lemma 2. *Let $\precsim$ be a semi-closed preorder. Then, in order for $t^{\precsim}_{|\sim}$ to be Hausdorff, it is necessary and sufficient that $\precsim$ satisfies the conditions* **SB** *and* **LR**.

Proof. As it already has been mentioned in the introduction, the validity of the conditions **SB** and **LR** is necessary in order to guarantee that $t^{\precsim}_{|\sim}$ is Hausdorff. Hence, we may concentrate on the sufficient part of the lemma. In order to verify that the assumption according to which $\precsim$ is semi-closed implies, in combination with validity of the conditions **SB** and **LR**, that $t^{\precsim}_{|\sim}$ is a Hausdorff topology on $X_{|\sim}$ we notice at first that condition **SB** is equivalent to condition **LU**, which states that, for every point $x \in X$, at least one of the sets $l(x)$ or $r(x)$ is not empty. Let now points $x \in X$ and $y \in X$ such that $not(x \precsim y)$ be arbitrarily chosen. Then, the cases $y \prec x$ and $not(y \precsim x)$ are possible. Therefore, we have to distinguish between these possible cases.

Case 1: $y \prec x$. In this situation we distinguish between two more cases.

Case 1.1: There exist points $u \in X$ and $v \in X$ such that the interval $]u, v[$ is empty and $y \precsim u \prec v \precsim x$. In this case, $l(v)$ is an open set that contains y, and $r(u)$ is an open set that contains x. Therefore, the equation $l(v) \cap r(u) = \emptyset$ settles 1.1.

Case 1.2: The closed interval $[y, x]$ does not contain any jump. In this situation there exists some point $z \in X$ such that $y \prec z \prec x$. Hence $l(z)$ and $r(z)$, respectively, are disjoint open sets, which contain y and x, respectively.

Case 2: $not(y \precsim x)$. In this situation condition **LU** implies that the lemma will be proven if the cases $l(x) \neq \emptyset$ or $r(x) \neq \emptyset$ successfully have been handled. Since both cases can be settled by completely analogous arguments, it suffices to concentrate on the case when $l(x)$ is not empty. The inequality $l(x) \neq \emptyset$ implies, with help of condition **LR**, that there exists some $z \in l(x)$ such that $y \notin i(z)$, in case that $r(x) = \emptyset$, or that there exist points $v \in l(x)$ and $z \in r(x)$ such that $y \notin [v, z]$, in case that $r(x) \neq \emptyset$. Since $\precsim$ is semi-closed it, thus, follows that $r(z)$ and $X \setminus i(z)$, respectively, or $]v, z[$ and $X \setminus [v, z]$, respectively, are disjoint open sets which contain the point x and the point y, respectively, which still was to be shown. □

It is well known that a closed preorder $\precsim$ on X is semi-closed. On the other side, however, a semi-closed preorder, in general, is not closed. Indeed, in Bosi and Herden [9], (Theorem 3.2) very restrictive necessary and sufficient conditions for a semi-closed preorder to being closed have been presented. Because of this theorem it is somewhat surprising that the following proposition holds, which surely is worth to be stated separately.

Proposition 1. *Let $t^{\precsim}_{|\sim}$ be Hausdorff, and let $\precsim$ be semi-closed. Then, $\precsim$ is closed.*

Proof. In order to verify the proposition we must show that, for any two points $x \in X$ and $y \in X$ such that $not(x \precsim y)$, there exist (open) neighborhoods U of x and V of y such that, for every point $u \in U$ and every point $v \in V$, the relation $not(u \precsim v)$ holds. Indeed, having proved the existence of U and V, it follows that $(U \times V) \cap \precsim = \emptyset$, and we are done. An analysis of the proof of Lemma 2 allows us to concentrate on the case that also the relation $not(y \precsim x)$ holds, and that neither $l(x)$ nor $r(x)$ is empty. Let us, therefore, assume in contrast that every open neighborhood $]p, q[$ of x, and every open neighborhood $]r, s[$ of y which is disjoint from $]p, q[$ contains points $h \in]p, q[$ and $k \in]r, s[$, respectively, such that $h \precsim k$. In order to proceed, we set $\mathbf{I}(x) := \{]a, b[\mid x \in]a, b[\subseteq]p, q[\}$ and $\mathbf{I}(y) := \{]c, d[\mid y \in]c, d[\subseteq]r, s[\}$. Since $t^{\precsim}_{|\sim}$ is Hausdorff, we may conclude that $[x] = \bigcap_{]a,b[\in \mathbf{I}(x)}]a, b[$ and $[y] = \bigcap_{]c,d[\in \mathbf{I}(y)}]c, d[$. Then we distinguish between the cases when $[x]$ as well as $[y]$ are open and closed, $[x]$ is open and closed, and $[y]$ is only closed, $[x]$ is only closed, and $[y]$ is open and closed, and $[x]$ as well as $[y]$ are only closed. The case when $[x]$ as well as $[y]$ are open and closed is trivial. Indeed, in this case, we may set $U := [x]$ and $V := [y]$. The remaining three cases can be done by analogous arguments. Hence, we may concentrate, without loss of generality, on the case when $[x]$ as well as $[y]$ are only closed. Let now, in every interval $]a, b[\in \mathbf{I}(x)$ and every interval $]c, d[\in \mathbf{I}(y)$, points $h_a \in]a, b[$ and $k_c \in]c, d[$ such that $h_a \precsim k_c$ be arbitrarily chosen. Then, we may assume, without loss of generality, that $h_a \precsim h_{a'}$ for all intervals $]a', b'[\in \mathbf{I}(x)$ such that $]a', b'[\subseteq]a, b[$, or that $h_a \succsim h_{a'}$ for all intervals $]a', b'[\in \mathbf{I}(x)$ such that $]a', b'[\subseteq]a, b[$, and that $k_c \precsim k_{c'}$ for all intervals $]c', d'[\in \mathbf{I}(y)$ such that $]c', d'[\subseteq]c, d[$, or that $k_c \succsim k_{c'}$ for all intervals $]a', b'[\in \mathbf{I}(y)$ such that $]c', d'[\subseteq]c, d[$. The symmetry of the cases under consideration allows us to concentrate on the case when $h_a \precsim h_{a'}$ for all intervals $]a', b'[\in \mathbf{I}(x)$ such that $]a', b'[\subseteq]a, b[$, and $k_c \precsim k_{c'}$ for all intervals $]c', d'[\in \mathbf{I}(y)$ such that $]c', d'[\subseteq]c, d[$, and on the case $h_a \precsim h_{a'}$ for all intervals $]a', b'[\in \mathbf{I}(x)$ such that $]a', b'[\subseteq]a, b[$, and $k_c \succsim k_{c'}$ for all intervals $]c', d'[\in \mathbf{I}(y)$ such that $]c', d'[\subseteq]c, d[$. Since $\precsim$ is assumed to be semi-closed, it follows, in the first case, that $d(x) \subseteq d(y)$, which means that $x \precsim y$ and, thus, contradicts our assumption that x is not smaller or equivalent to y. The assumptions of the second case imply that $d(x) \subseteq \bigcap_{]c,d[\in \mathbf{I}(y)} d(k_c)$ But, since $\precsim$ is semi-closed, we may conclude that the smallest closed increasing set which contains $\bigcup_{]c,d[\in \mathbf{I}(y)} i(k_c)$ is $i(y)$. It, thus, follows that $\bigcap_{]c,d[\in\mathbf{I}(y)} d(k_c) = d(y)$, which again implies that $x \precsim y$ and, therefore, contradicts the relation $not(x \precsim y)$. This conclusion, finally, proves the validity of the proposition. □

In combination with Lemma 1, Lemma 2, and Proposition 1, the following theorem now presents a complete solution of the characterization problem, which is in focus of this section, and, in this way, (in opinion of the authors) also allows an interesting comparison of the Chipman approach on one side, and the real-valued approach, on the other side, to mathematical utility theory (cf. Theorem 4).

Theorem 3. *Let $(X, \precsim, t)$ be a preordered topological space, the topology t of which is finer than the order topology $t^{\precsim}$. Then, the following assertions are equivalent:*
(i) *$\precsim$ admits a continuous multi-utility representation.*
(ii) *$t^{\precsim}_{|\sim}$ is Hausdorff and $\precsim$ is closed.*
(iii) *$t^{\precsim}_{|\sim}$ is Hausdorff and $\precsim$ is semi-closed.*

Proof. (i) ⇒ (ii): It already has been mentioned above that a preorder $\precsim$ that admits a continuous multi-utility representation must be closed. Therefore, Lemma 1 guarantees the validity of the implication "(i) ⇒ (ii)".

(ii) ⇒ (iii): Since a closed preorder $\precsim$ is semi-closed nothing has to be proved.

(iii) ⇒ (ii): See Proposition 1.

(ii) ⇒ (i): Let Assertion (ii) be valid, and let points $x \in X$ and $y \in X$ such that $y \in \mathbf{N}_{\precsim}(x)$ be arbitrarily chosen. Then, we must prove that there exists some continuous and increasing function $f_{xy} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(y) < f_{xy}(x)$. In order to verify the existence of f_{xy}, we distinguish between the cases when y is contained in $X \setminus l(x)$, and y is contained in $l(x)$.

Case 1: $y \in X \setminus l(x)$. In this case, we must distinguish between the situations when y is a maximal element of $(X, \precsim)$, and y is not a maximal element of$(X, \precsim)$. In the first situation, we may apply property **OC** in order to set

$$f_{xy}(z) := \begin{cases} 0 & \text{if } z \in d(y) \\ 1 & \text{otherwise} \end{cases}$$

for all $z \in X$. In the second situation, i.e., when y is not a maximal element of $(X, \precsim)$, there exists some point $u \in X$ such that $y \prec u$. Of course, u may be a maximal element of $(X, \precsim)$. In this case, however, we may apply the argument which has been applied in the first situation. Hence, we may assume, without loss of generality, that there exist points $u \in X$ and $v \in X$ such that $y \prec u \prec v$. We proceed by assuming, at first, that v and, thus, also u is contained in $\mathbf{N}_{\precsim}(x)$. We abbreviate this assumption by (**). In addition, we assume that both (open) intervals $]y, u[$ and $]u, v[$ are empty. These assumptions imply that there does not exist any open interval $]r, t[$ of $(X, \precsim)$ which contains u and it is completely contained in $]y, v[$. Hence, there cannot exist any net $(u_i)_{i \in I}$ of points $u_i \in]y, v[$ which converges to u, which implies that the set $\mathbf{U}(u)$ of all points $s \in]y, v[$, the indifference classes $[s]$ of which are different from $[u]$, must be closed. This means that we now may apply the conclusion that in the proof of Lemma 1 has been abbreviated by (*), in order to conclude that $l(u)$ is open and closed. Since both points u and v are contained in $\mathbf{N}_{\precsim}(x)$, these considerations, finally, allow us to define the desired continuous and increasing function $f_{xy} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ by setting

$$f_{xy}(z) := \begin{cases} 0 & \text{if } z \in l(u) \\ 1 & \text{otherwise} \end{cases}$$

for all $z \in X$. In addition, the above considerations imply that, for the moment, we may assume without loss of generality that there exist no points $h \in]y, v[$, $k \in]y, v[$ and $q \in]y, v[$ such that $h \prec k \prec q$, and that both intervals $]h, k[$ and $]k, q]$ are empty. With the help of this assumption, we now verify that the preordered set $(]y, v[, \precsim)$ is not scattered. Indeed, the assumption implies, by complete induction, each induction step of which may be settled by some straightforward indirect argument, that $]y,v[,\precsim)$ contains some order-dense subset, i.e., a subset which does not contain any jumps, or that the set $\mathbf{I}(]y,v[) := \{[c,g] \subseteq]y,v[\mid]c,g[= \varnothing\}$, which is ordered by setting $[c,g] \trianglelefteq [c',g'] \Leftrightarrow [c,g] = [c',g']$ or $g \prec c'$ for all intervals $[c,g] \in \mathbf{I}(]y,v[)$ and $[c',g'] \in \mathbf{I}(]y,v[)$, is order-dense. Let $[0,1]$ be the real unit interval. Then, our considerations allow us to conclude that, in any case, there exists an order-embedding $i : \mathbb{Q} \cap [0,1], \leq) \longrightarrow (]y,v[,\precsim)$. We, thus, proceed by showing that, for all rationals $q \in \mathbb{Q} \cap [0,1]$ and $q' \in \mathbb{Q} \cap [0,1]$ such that $q < q'$, the inclusion $\overline{l(i(q))} \subseteq l(i(q'))$ holds. Since $\precsim$ is closed and, therefore, also semi-closed, it follows that $\overline{l(i(q))} \subseteq d(i(q))$. The validity of the strong inequality $i(q) \prec i(q')$, thus, implies the desired inclusions $\overline{l(i(q))} \subseteq d(i(q)) \subseteq l(i(q'))$. These considerations imply that the assumptions of Peleg's Theorem (cf. Peleg [10]) are satisfied or, equivalently, that the family $\{l(i(q))\}_{q \in \mathbb{Q} \cap [0,1]}$ is a (decreasing) separable system in the sense of Herden [8]. Peleg's theorem or Theorem 4.1 in Herden [18], therefore, implies the existence of some continuous and increasing function $f_{xy} : (X,\precsim,t) \longrightarrow (\mathbb{R},\leq,t_{nat})$ such that $f_{xy}(y) = 0$ and $f_{xy}(x) = 1$. Let us abbreviate these arguments by (***). In order to finish the first case, we still must consider the situation when v is not contained in $\mathbf{N}_{\precsim}(x)$, i.e., the situation when v is contained in $i(x)$. Of course, it is possible that $r(y) = i(x)$, which means that also $u \in i(x)$. In this situation, however, $i(x)$ is an open and closed subset of X. Hence, we may define the desired continuous and increasing function $f_{xy} : (X,\precsim,t) \longrightarrow (\mathbb{R},\leq,t_{nat})$ by setting

$$f_{xy}(z) := \begin{cases} 0 & \text{if } z \in X \setminus i(x) \\ 1 & \text{otherwise} \end{cases}$$

for all $z \in X$. These considerations now allow us to assume that $y \prec u \prec v$, but $u \in \mathbf{N}_{\precsim}(x)$ and $v \in i(x)$. Since it already has been shown that we may assume without loss of generality that at least one of the (open) intervals $]y,u[$ or $]u,v[$ is not empty, we first briefly discuss the case when $]y,u[$ is not empty. In this case, however, we may apply the arguments that have been summarized by (***), in order to guarantee the existence of some continuous and increasing function $f_{xy} : (X,\precsim,t) \longrightarrow ([0,1],\leq,t_{nat})$ such that $f_{xy}(y) = 0$ and $f_{xy}(x) = 1$. Hence, we now may assume that the (open) interval $]u,v[$ is not empty. In this situation, reiteration of the just used argument in combination with an analysis of the arguments that have been summarized by (***), imply that there exists some net $(w_k)_{k \in K}$ of points $w_k \in]u,v[$ which converges to u. We now proceed by applying an indirect argument. This means precisely that we assume that each point w_k is contained in $i(x)$, for all indexes $k \in K$. Since $\precsim$ is a closed preorder, this assumption allows us to conclude, however, that $u \in i(x)$, which contradicts our assumption that u is an element of $\mathbf{N}_{\precsim}(x)$. This contradiction guarantees the existence of some point $w \in]u,v[$ such that $w \in \mathbf{N}_{\precsim}(x)$. Since $y \prec u \prec w$, now the same situation is given, as the one which has been described in (**). This reduction to assumption (**), finally, settles the first case.

Case 2: In this case we distinguish between the sub-case when $i(x)$ coincides with $r(y)$, and the sub-case when $i(x)$ is properly contained in $r(y)$. But since the sub-case when $r(y)$ coincides with $i(x)$ already has been discussed in the second part of the first case, we only have to consider that strong inclusion $i(x) \subsetneqq r(y)$. Here, also the arguments which have been used in discussing the first case apply. Indeed, let in this sub-case some point $c \in r(y) \setminus i(x)$ be arbitrarily chosen. Then, we have that $c \in l(x)$ or $c \in X \setminus l(x)$. If $c \in l(x)$, then the situation $y \prec c \prec x$ is given, and we may apply the above arguments, concerning the situation when $y \prec u \prec v$. If $c \in X \setminus l(x)$, then the inclusion $c \in r(y) \setminus i(x)$ implies that $c \in \mathbf{N}_{\precsim}(x)$, and we may apply the above arguments concerning the last part of the first case, in order to also handle this situation. This last conclusion completes the proof of the theorem. □

In the second section of this paper, the universal character of the Chipman approach to mathematical utility has been demonstrated (cf. Theorem 1 and Theorem 2). Concentrating on continuous multi-utility representation in Theorem 3, however, it could be shown that the Chipman approach, at least formally, is not as far away from the real-valued approach, as it seems at first sight. Therefore, we now still discuss the relations between the Chipman approach, the continuous multi-utility approach, and the Richter–Peleg approach to mathematical utility theory in more detail (cf. Subsection 1.2 of the introduction). The relation between the Chipman approach and the continuous multi-utility approach can be described by combining Theorems 1 and 2 in order to state the following theorem.

Theorem 4. *Let $\precsim$ be a semi-closed preorder on a topological space (X, t), and let us assume, in addition, that $t^{\precsim}_{|\sim}$ is Hausdorff, and that the order topology $t^{\precsim}$ is coarser than t. Then, the following assertions hold:*
(i) *There exists some cardinal number κ and a preorder $\lesssim$ on $\{0,1\}^{\kappa}$, which is coarser than $\leq_{lex}$, for which there exists a continuous order-embedding $\vartheta : (X \precsim, t) \longrightarrow (\{0,1\}^{\kappa}, \lesssim, t^{\lesssim})$;*
(ii) *$\precsim$ admits a continuous multi-utility representation.*

Proof. Let now ω be the cardinality of the set $\mathbf{N}(\precsim)$ of all pairs $([x],[y])) \in X_{|\sim} \times X_{|\sim}$ such that $not(x \precsim y)$. Then, we consider the direct sum $(\mathbb{R}, \leq, t_{nat}) \times ([0, \omega[, <)$ of ordered topological spaces $(\mathbb{R}, \leq, t_{nat})$, as well as the direct sum $(X, \precsim, t) \times ([0, \omega[, <)$ of preordered topological spaces $(X, \precsim, t)$. As it already has been shown in Subsection 1.2 of the introduction, it follows that there exists a continuous order-preserving function $\Phi : (X, \precsim, t) \times ([0, \omega[, <) \longrightarrow (\mathbb{R}, \leq, t_{nat}) \times ([0, \omega[, <)$. This consideration already clarifies the relation of the continuous multi-utility approach for representing a preorder with the Richter–Peleg approach for representing a preorder (cf. Evren and Ok [7]). Define, finally, $\kappa :=| X_{|\sim} |$ and $\xi := \omega \cdot \kappa =| \omega \times \kappa |$. Then, Theorem 2 implies the existence of a continuous order-embedding

$$\Psi : (\mathbb{R}, \leq, t_{nat}) \times ([0, \omega[, <) \longrightarrow ((\{0,1\}^{\omega})^{\kappa}, \lesssim, t^{\lesssim}) = (\{0,1\}^{\xi}, \lesssim, t^{\lesssim})$$

and of a continuous order-embedding

$$\Upsilon : (X, \precsim, t) \times ([0, \omega[, <) \longrightarrow (\{0,1\}^{\xi}, \lesssim, t^{\lesssim}),$$

such that the following diagram commutes

$$\begin{array}{ccc} ((\{0,1\}^{\omega})^{\kappa}, \lesssim, t^{\lesssim}) = (\{0,1\}^{\xi}, \lesssim, t^{\lesssim}) & \overset{\Psi}{\leftarrow} & (\mathbb{R}, \leq, t_{nat}) \times ([0, \omega[, <) \\ \uparrow \Upsilon & \nearrow \Phi & \\ (X, \precsim, t) \times ([0, \omega[, <) & & \end{array} .$$

□

This last theorem completely clarifies the relations between the Chipman-approach, the continuous multi-utility approach and the Richter–Peleg approach to mathematical utility theory (cf. Subsection 1.2 of the introduction).

4. Conclusions

The Chipman approach to mathematical utility theory, on the one hand, and the continuous multi-utility approach, on the other hand, chiefly are in the focus of this paper. Indeed, it has been shown, in the second and in third sections of this paper, that both approaches to mathematical utility theory are not as far away from each other as they seem at first sight. In Subsection 1.2 of the introduction and in Theorem 4, the (formal) relations of these approaches to mathematical utility theory still have been combined with the usual Richter–Peleg approach to mathematical utility theory, in order to describe and visualize in this way the intimate relations which exist between these generally used approaches. The so-called *Richter–Peleg multi-utility representations*, which take place when all the functions in multi-utility representations are order-preserving for the given preorder, will be studied in a future paper within the perspective of the present paper.

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