

Stochastic Duality and Eigenfunctions

Frank Redig^(✉) and Federico Sau

Delft Institute of Applied Mathematics, Delft University of Technology,
van Mourik Broekmanweg 6, 2628 XE Delft, The Netherlands
f.h.j.redig@tudelft.nl, F.Sau@tudelft.nl

Abstract. We start from the observation that, anytime two Markov generators share an eigenvalue, the function constructed from the product of the two eigenfunctions associated to this common eigenvalue is a duality function. We push further this observation and provide a full characterization of duality relations in terms of spectral decompositions of the generators for finite state space Markov processes. Moreover, we study and revisit some well-known instances of duality, such as Siegmund duality, and extract spectral information from it. Next, we use the same formalism to construct all duality functions for some solvable examples, i.e., processes for which the eigenfunctions of the generator are explicitly known.

1 Introduction

Stochastic duality is a technique to connect two Markov processes via a so-called *duality function*. This connection, interesting in its own right, turns out to be extremely useful when the *dual* process is more tractable than the original process.

Several applications of stochastic duality may be found in the context of interacting particle systems [28] as, for instance, in the study of hydrodynamic limits and fluctuations [9, 10, 23], characterization of extremal measures [28, 33], derivation of Fourier law of transport [3, 24] and correlation inequalities [17]. Other fields rich of applications are population genetics, where the coalescent process arises as a natural dual process (see [11] and references therein) and branching-coalescing processes [13]. Duality and related notions have already been used in the study of spectral gaps and convergence to stationarity by several authors, see e.g. [6, 12, 14, 29, 32].

Part of the research about stochastic duality deals with the problem of *finding* and *characterizing* duality functions relating two given Markov processes. This means that, for a given pair of Markov generators, one wants to find all duality functions or, alternatively, a basis of the linear space of duality functions. See, for instance, in this direction [30] in the context of population genetics, while for particle systems the works [1, 2, 15, 33] for symmetric and [4, 5, 34] for asymmetric processes. For Markov processes, algebraic constructions of duality relations for specific classes of models have also been provided (see e.g. [1, 4, 16, 19, 26]).

In this paper we first show that, viewing a duality relation as a *spectral* relation among the associated Markov generators, duality functions can be obtained from linear combinations of products of eigenfunctions associated to a common eigenvalue. Secondly, we establish this connection with the general aim of characterizing all possible dualities in terms of eigenfunctions and generalized eigenfunctions of the generators involved. To this purpose, our discussion mainly focuses on continuous-time finite-state Markov chains for which no reversibility is assumed but canonical eigendecompositions of Jordan-type of the generators are available.

We emphasize that this connection between duality and eigenfunctions goes both ways: not only eigenfunctions of a shared spectrum give rise to duality functions, but also the existence of duality relations carries information about the spectrum of the generators. Here we can already see a clear distinction between the notion of *self-duality* and *integrability*: knowing certain linear combinations of products of eigenfunctions (self-duality) rather than knowing the eigenfunctions themselves (integrability).

The rest of the paper is organized as follows. In Sect. 2 we provide all preliminary notions of stochastic duality for continuous-time Markov chains. After an introductory study of self-duality and duality in the reversible setting in Sects. 3 and 4, in Sect. 5, via Jordan canonical decompositions, we make precise to which extent spectrum and eigenstructure of generators in duality are shared. In fact, the assumed orthonormality of the eigenfunctions in Sects. 3 and 4 has the only role of simplifying the exposition at a first reading. There, products of orthonormal eigenfunctions are a natural tensor basis w.r.t. which express duality functions; this fact allows a direct description of the linear subspace of duality functions in terms of this tensor basis. In Sect. 5, we show how, by dropping reversibility of the generators and thus orthonormality of the associated eigenfunctions, a tensor basis in terms of product of generalized eigenfunctions is always possible.

We further investigate the connection between eigenfunctions and particular instances of dualities that typically appear in the context of interacting particle systems, see e.g. [15, 33], in Sects. 3 and 4. In Sect. 5.4 we revisit the notion of *intertwining* (see e.g. [21]) in this setting and provide an application to the symmetric exclusion process in Sect. 5.5. In Sect. 6 we provide an alternative way of proving and characterizing Siegmund duality [21, 36] in the finite context.

2 Setting and Notation

Let Ω be a finite state space with cardinality $|\Omega| = n$. We consider an *irreducible* continuous-time Markov process $\{X_t, t \geq 0\}$ on Ω , with generator L given by

$$Lf(x) = \sum_{y \in \Omega} \ell(x, y)(f(y) - f(x)),$$

where $f : \Omega \rightarrow \mathbb{R}$ is a real-valued function and $\ell : \Omega \times \Omega \rightarrow [0, +\infty)$ gives the transition rates. For $x \in \Omega$, we define the exit rate from $x \in \Omega$ as

$$\ell(x) = \sum_{y \in \Omega \setminus \{x\}} \ell(x, y).$$

In the finite context we can identify L with the matrix, still denoted by L , given by

$$L(x, y) = \ell(x, y) \text{ for } x \neq y, \quad L(x, x) = -\ell(x).$$

Given two state spaces $\Omega, \widehat{\Omega}$ of cardinalities $|\Omega| = n, |\widehat{\Omega}| = \widehat{n}$, and two *Markov processes* with generators L, \widehat{L} , we say that they are *dual* with *duality function* $D : \widehat{\Omega} \times \Omega \rightarrow \mathbb{R}$ if, for all $x \in \Omega$ and $\widehat{x} \in \widehat{\Omega}$, we have

$$\widehat{L}_{\text{left}} D(\widehat{x}, x) = L_{\text{right}} D(\widehat{x}, x), \quad (1)$$

where “left”, resp. “right”, refers to action on the left, resp. right, variable. If the laws of the two processes coincide, we speak about *self-duality*. The same notion in terms of matrix multiplication, where D also denotes the matrix with entries $\{D(\widehat{x}, x), \widehat{x} \in \widehat{\Omega}, x \in \Omega\}$, is expressed as

$$\sum_{\widehat{y} \in \widehat{\Omega}} \widehat{L}(\widehat{x}, \widehat{y}) D(\widehat{y}, x) = \sum_{y \in \Omega} L(x, y) D(\widehat{x}, y),$$

or, shortly, as

$$\widehat{L}D = DL^{\top}, \quad (2)$$

where the symbol $^{\top}$ denotes *matrix transposition*, i.e., for a matrix A ,

$$(A^{\top})(x, y) = A(y, x), \quad x, y \in \Omega.$$

More generally, we define two *operators* \widehat{L} and L *dual* with duality function D if relation (1), or equivalently (2) in matrix notation, holds.

3 Self-duality from Eigenfunctions: Reversible Case

As in Sect. 2, let Ω be a finite set of cardinality $|\Omega| = n$, and let L be a generator of an irreducible *reversible* Markov process on Ω w.r.t. the positive measure μ . This measure then satisfies the detailed balance condition

$$\mu(x)L(x, y) = \mu(y)L(y, x), \quad (3)$$

for all $x, y \in \Omega$. This relation can be rewritten as a self-duality with self-duality function the so-called *cheap self-duality function*:

$$D_{\text{cheap}}(x, y) = \frac{\delta_{x,y}}{\mu(y)}. \quad (4)$$

The reversibility of μ implies that L is self-adjoint in $L^2(\mu)$ and, as a consequence, there exists a basis $\{u_1, \dots, u_n\}$ of eigenfunctions of L with $u_1(x) = 1/\sqrt{n}$ corresponding to eigenvalue zero and $\{u_1, \dots, u_n\}$ orthonormal, i.e., $\langle u_i, u_j \rangle_\mu = \delta_{i,j}$ where $\langle \cdot, \cdot \rangle_\mu$ denotes inner product in $L^2(\mu)$. We denote by $\{\lambda_1, \dots, \lambda_n\}$ the corresponding real eigenvalues with

$$0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n.$$

The following proposition then shows how to obtain and characterize self-duality functions in terms of this orthonormal system. The last statement recovers an earlier result from [16].

Proposition 1. (i) For $a_1, a_2, \dots, a_n \in \mathbb{R}$, the function

$$D(x, y) = \sum_{i=1}^n a_i u_i(x) u_i(y) \quad (5)$$

is a self-duality function.

(ii) Every self-duality function has a unique decomposition of the form

$$D(x, y) = \sum_{i,j:\lambda_i=\lambda_j} a_{i,j} u_i(x) u_j(y). \quad (6)$$

(iii) If a function of the form $D(x, y) = f(x)g(y)$ is a non-zero self-duality function, then f and g are eigenfunctions corresponding to the same eigenvalue.

(iv) The $L^2(\mu)$ inner product of self-duality functions produces self-duality functions, i.e., if D and D' are self-duality functions, then

$$\langle D(x, \cdot), D'(x', \cdot) \rangle_\mu = D''(x, x') \quad (7)$$

defines a self-duality function D'' .

Proof. For (i), by definition of eigenfunction $Lu_i = \lambda_i u_i$ with $\lambda_i \in \mathbb{R}$, we obtain

$$\begin{aligned} L_{\text{left}} D(x, y) &= \sum_{i=1}^n a_i Lu_i(x) u_i(y) = \sum_{i=1}^n a_i \lambda_i u_i(x) u_i(y) \\ &= \sum_{i=1}^n a_i u_i(x) \lambda_i u_i(y) = \sum_{i=1}^n a_i u_i(x) Lu_i(y) = L_{\text{right}} D(x, y), \end{aligned}$$

hence (1).

For (ii), start by noticing that every function $D : \Omega \times \Omega \rightarrow \mathbb{R}$ can be written in a unique way as

$$D(x, y) = \sum_{i,j=1}^n a_{i,j} u_i(x) u_j(y),$$

Now using the duality relation (1), it follows that

$$\sum_{i,j} a_{i,j} \lambda_i u_i(x) u_j(y) = \sum_{i,j} a_{i,j} \lambda_j u_i(x) u_j(y),$$

which implies that, for all $i, j = 1, \dots, n$,

$$a_{i,j}\lambda_i = a_{i,j}\lambda_j.$$

For item (iii), first write

$$f(x)g(y) = \sum_{i,j=1}^n a_{i,j}u_i(x)u_j(y).$$

Then we find $a_{ij} = \langle f, u_i \rangle_\mu \langle g, u_j \rangle_\mu =: \alpha_i \beta_j$. From self-duality we conclude, for all $i, j = 1, \dots, n$,

$$\alpha_i \beta_j (\lambda_i - \lambda_j) = 0.$$

Now use that $f(x)g(y)$ is not identically zero to conclude that there exists i with $\alpha_i \neq 0$. Then if $\lambda_j \neq \lambda_i$ we conclude $\beta_j = 0$, which implies that g is an eigenfunction with eigenvalue λ_i . Because g is not identically zero, we can reverse the argument and conclude.

For (iv), by exchanging the order of summations and using $\langle u_j, u_l \rangle_\mu = \delta_{j,l}$, the l.h.s. of (7) reads

$$\begin{aligned} & \sum_{y \in \Omega} D(x, y) D(x', y) \mu(y) \\ &= \sum_{y \in \Omega} \left(\sum_{i,j:\lambda_i=\lambda_j} a_{i,j} u_i(x) u_j(y) \right) \left(\sum_{k,l:\lambda_k=\lambda_l} a_{k,l} u_k(x') u_l(y) \right) \mu(y) \\ &= \sum_{j=1}^n \left(\sum_{i:\lambda_i=\lambda_j} a_{i,j} u_i(x) \right) \left(\sum_{k:\lambda_k=\lambda_j} a_{k,j} u_k(x') \right). \end{aligned}$$

By noting that, for all $j = 1, \dots, n$, the function $u'_j = \sum_{i:\lambda_i=\lambda_j} a_{i,j} u_i$ is either vanishing or is an eigenfunction of L associated to λ_j , the proof is concluded. \square

In the next propositions we study particular instances of self-duality functions. More precisely, by using Proposition 1, we recover the cheap self-duality function in (4), while in Proposition 3 we characterize *orthogonal* self-duality functions (cf. (11)–(12) below).

Proposition 2 (Cheap self-duality)

(i) For the choice $a_1 = a_2 = \dots = a_n = 1$ in (5), we obtain the cheap self-duality function, i.e.,

$$D_{cheap}(x, y) = \frac{\delta_{x,y}}{\mu(y)} = \sum_{i=1}^n u_i(x) u_i(y). \quad (8)$$

(ii) Conversely, if $\{v_1, \dots, v_n\}$ is a basis of $L^2(\mu)$ and satisfies

$$\sum_{i=1}^n v_i(x)v_i(y) = \frac{\delta_{x,y}}{\mu(y)} \quad (9)$$

for all $x, y \in \Omega$, then $\{v_1, \dots, v_n\}$ is an orthonormal basis of $L^2(\mu)$.

Proof. To show (8), by the positivity of μ , we need to show that, for all $f : \Omega \rightarrow \mathbb{R}$ and $x \in \Omega$,

$$\sum_{y \in \Omega} \sum_{i=1}^n u_i(x)u_i(y)\mu(y)f(y) = f(x).$$

Now note, by interchanging the sum over i with the sum over y , that the l.h.s. equals

$$\sum_{i=1}^n u_i(x)\langle u_i, f \rangle_\mu = f(x),$$

and hence we obtain (i).

For (ii) we need to show that for all $f : \Omega \rightarrow \mathbb{R}$ and $x \in \Omega$

$$f(x) = \sum_{i=1}^n v_i(x)\langle v_i, f \rangle_\mu = \sum_{i=1}^n \sum_{y \in \Omega} v_i(x)v_i(y)f(y)\mu(y). \quad (10)$$

We conclude by interchanging the order of the two summations in the r.h.s. above and using (9), we indeed obtain (10). \square

Remark that the cheap self-duality function is the only, up to multiplicative constants, *diagonal* self-duality, and that it is *orthogonal* in the sense that, for all $x, x' \in \Omega$,

$$\langle D_{\text{cheap}}(x, \cdot), D_{\text{cheap}}(x', \cdot) \rangle_\mu = \delta_{x,x'} \langle D_{\text{cheap}}(x, \cdot), D_{\text{cheap}}(x, \cdot) \rangle_\mu, \quad (11)$$

and similarly, for all $y, y' \in \Omega$,

$$\langle D_{\text{cheap}}(\cdot, y), D_{\text{cheap}}(\cdot, y') \rangle_\mu = \delta_{y,y'} \langle D_{\text{cheap}}(\cdot, y), D_{\text{cheap}}(\cdot, y) \rangle_\mu. \quad (12)$$

The next proposition shows how to find *all* orthogonal self-duality functions.

Proposition 3 (Orthogonal self-duality)

(i) If $\{\tilde{u}_1, \dots, \tilde{u}_n\}$ is an orthonormal system in $L^2(\mu)$ of eigenfunctions of L , corresponding to the same eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, then

$$D(x, y) = \sum_{i=1}^n \tilde{u}_i(x)u_i(y) \quad (13)$$

is an orthogonal self-duality function. More precisely, for all $x, x' \in \Omega$,

$$\langle D(x, \cdot), D(x', \cdot) \rangle_\mu = \frac{\delta_{x,x'}}{\mu(x')}. \quad (14)$$

(ii) The self-duality functions of the form (13) are the only, up to a multiplicative factor, orthogonal self-duality functions.

Proof. For (i), we compute, for all $k = 1, \dots, n$ and $x \in \Omega$, the following quantity

$$\sum_{x' \in \Omega} \langle D(x, \cdot), D(x', \cdot) \rangle_{\mu} \tilde{u}_k(x') \mu(x').$$

By $\langle u_i, u_j \rangle_{\mu} = \langle \tilde{u}_i, \tilde{u}_j \rangle_{\mu} = \delta_{i,j}$, the line above rewrites as follows:

$$\begin{aligned} & \sum_{x' \in \Omega} \sum_{y \in \Omega} \left(\sum_{i=1}^n \tilde{u}_i(x) u_i(y) \right) \left(\sum_{j=1}^n \tilde{u}_j(x') u_j(y) \right) \mu(y) \tilde{u}_k(x') \mu(x') \\ &= \sum_{i=1}^n \sum_{j=1}^n \tilde{u}_i(x) \left(\sum_{y \in \Omega} u_i(y) u_j(y) \mu(y) \right) \left(\sum_{x' \in \Omega} \tilde{u}_j(x') \tilde{u}_k(x') \mu(x') \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \tilde{u}_i(x) \delta_{i,j} \delta_{j,k} = \tilde{u}_k(x). \end{aligned}$$

This together with Proposition 2 concludes the proof of part (i).

For (ii), by starting from a general self-duality function

$$D(x, y) = \sum_{i,j: \lambda_i = \lambda_j} a_{i,j} u_i(x) u_j(y),$$

the l.h.s. of (14) rewrites as

$$\sum_{j=1}^n u'_j(x) u'_j(x'),$$

where $\{u'_1, \dots, u'_n\}$ is defined as

$$u'_j(x) = \sum_{i: \lambda_i = \lambda_j} a_{i,j} u_i(x).$$

By remarking that either $u'_j = 0$ or u'_j is an eigenfunction of L associated to λ_j and applying Proposition 2, we have that

$$\langle u'_i, u'_j \rangle_{\mu} = \delta_{i,j},$$

and that the self-duality function D has the form (13) with $\tilde{u}_i = u'_i$. \square

4 Duality from Eigenfunctions: Reversible Case

Now we consider two generators L, \widehat{L} on the same finite state space Ω with reversible measures $\mu, \widehat{\mu}$ respectively, and orthonormal systems of eigenfunctions $\{u_1, \dots, u_n\}, \{\widehat{u}_1, \dots, \widehat{u}_n\}$ corresponding to the *same* real eigenvalues

$\{\lambda_1, \dots, \lambda_n\}$, i.e., we assume that L and \widehat{L} are self-adjoint in $L^2(\mu)$, resp. in $L^2(\widehat{\mu})$, and that they are iso-spectral.

In what follows we state - without proofs - analogous relations between duality functions and orthonormal systems of eigenfunctions of L and \widehat{L} .

Proposition 4. (i) For $a_1, \dots, a_n \in \mathbb{R}$ the function

$$D(\widehat{x}, x) = \sum_{i=1}^n a_i \widehat{u}_i(\widehat{x}) u_i(x)$$

is a duality function for duality between \widehat{L} and L .

(ii) Every duality function has a unique decomposition of the form

$$D(\widehat{x}, x) = \sum_{i,j:\lambda_i=\lambda_j} a_{ij} \widehat{u}_i(\widehat{x}) u_j(x).$$

(iii) If a function of the form $D(\widehat{x}, x) = f(\widehat{x})g(x)$ is a non-zero duality function, then f and g are eigenfunctions of \widehat{L} , resp. L , corresponding to the same eigenvalue.

(iv) The $L^2(\mu)$ and $L^2(\widehat{\mu})$ inner products of duality functions produce self-duality functions, i.e., if D and D' are duality functions, then

$$\langle D(\widehat{x}, \cdot), D'(\widehat{x}', \cdot) \rangle_{\mu} = \widehat{D}(\widehat{x}, \widehat{x}')$$

defines a self-duality function \widehat{D} for \widehat{L} , and similarly

$$\langle D(\cdot, x), D'(\cdot, x') \rangle_{\widehat{\mu}} = \widetilde{D}(x, x')$$

determines a self-duality function \widetilde{D} for L .

Proposition 5 (Orthogonal duality)

(i) If $\{\tilde{u}_1, \dots, \tilde{u}_n\}$ is an orthonormal system in $L^2(\widehat{\mu})$ of eigenfunctions of \widehat{L} corresponding to the same eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, then

$$D(\widehat{x}, x) = \sum_{i=1}^n \tilde{u}_i(\widehat{x}) u_i(x) \tag{15}$$

is an orthogonal duality function, i.e.,

$$\langle D(\widehat{x}, \cdot), D(\widehat{x}', \cdot) \rangle_{\mu} = \frac{\delta_{\widehat{x}, \widehat{x}'}}{\widehat{\mu}(\widehat{x}')}$$

and

$$\langle D(\cdot, x), D(\cdot, x') \rangle_{\widehat{\mu}} = \frac{\delta_{x, x'}}{\mu(x')}.$$

(ii) These are the only, up to multiplicative constants, orthogonal dualities between \widehat{L} and L .

5 Duality from Eigenfunctions: Non-reversible Case

Working in the *non-reversible* context, i.e., whenever there *does not* exist a probability measure μ on Ω for which the generator L is self-adjoint in $L^2(\mu)$, a spectral decomposition of the generator in terms of real non-positive eigenvalues and orthonormal real eigenfunctions is typically lost. In recent years, the study of the eigendecomposition of non-reversible generators has received an increasing attention [6–8, 32, 37] and duality-related notions have been introduced to relate spectral information of one process, typically a reversible one, to another, typically non-reversible [14, 29].

However, regardless of the spectral eigendecomposition of the generators, in principle interesting dualities can still be constructed from eigenfunctions, either real or complex, and generalized eigenfunctions of the generators involved. The key on which this relation builds up, in the finite context, is the *Jordan canonical decomposition* of the generators. A relation between duality and the Jordan canonical decomposition has already been used in the context of models of population dynamics in [30].

Below, before studying the most general result that exploits the Jordan form of the generators, we treat some special cases reminiscent of the previous sections. In the sequel, for a function $u : \Omega \rightarrow \mathbb{C}$, we denote by $u^* : \Omega \rightarrow \mathbb{C}$ its complex conjugate.

5.1 Duality from Complex Eigenfunctions

A first feature that typically drops as soon as one moves to the non-reversible situation is the appearance of only real eigenvalues. Indeed, given a non-reversible generator L of an irreducible Markov process on Ω , pairs of complex conjugates eigenvalues $\{\lambda, \lambda^*\}$ and eigenfunctions $\{u, u^*\}$ may arise as in the following example.

Example 1. The continuous-time Markov chain on the state space $\Omega = \{1, 2, 3\}$ and described by the generator L , which, viewed as a matrix, reads

$$L = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

represents a basic example of this situation. Indeed, the Markov chain is irreducible, the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3^* = -\frac{3}{2} + i\frac{\sqrt{3}}{2},$$

while the associated eigenfunctions $\{u_1, u_2, u_3\}$ are, for $x \in \{1, 2, 3\}$,

$$u_1(x) = \frac{1}{\sqrt{3}}, \quad u_2(x) = u_3^*(x) = e^{(i\frac{2}{3}\pi)x}.$$

□

Let us, thus, consider two irreducible non-reversible generators L, \widehat{L} on the same state space Ω . We investigate the situation in which there exist $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and functions $u, \widehat{u} : \Omega \rightarrow \mathbb{C}$ such that

$$Lu = \lambda u, \quad \widehat{L}\widehat{u} = \lambda\widehat{u}. \quad (16)$$

Remark that, as L, \widehat{L} are real operators, this implies that

$$Lu^* = \lambda^* u^*, \quad \widehat{L}\widehat{u}^* = \lambda^* \widehat{u}^*. \quad (17)$$

A real duality function arising from a shared pair of complex eigenvalues is obtained in the following proposition.

Proposition 6. *For $a \in \mathbb{R}$, the function*

$$D(\widehat{x}, x) = a\widehat{u}(\widehat{x})u(x) + a\widehat{u}^*(\widehat{x})u^*(x)$$

takes values in \mathbb{R} and is a duality function between \widehat{L} and L .

Proof. It is clear that $D(\widehat{x}, x)$ is in \mathbb{R} . Then, by using (16) and (17), we obtain

$$\begin{aligned} \widehat{L}_{\text{left}} D(\widehat{x}, x) &= a(\widehat{L}\widehat{u})(\widehat{x})u(x) + a(\widehat{L}\widehat{u}^*)(\widehat{x})u^*(x) \\ &= a\lambda\widehat{u}(\widehat{x})u(x) + a\lambda^*\widehat{u}^*(\widehat{x})u^*(x) = a\widehat{u}(\widehat{x})\lambda u(x) + a\widehat{u}^*(\widehat{x})\lambda^* u^*(x) \\ &= a\widehat{u}(\widehat{x})(Lu)(x) + a\widehat{u}^*(\widehat{x})(Lu^*)(x) = L_{\text{right}} D(\widehat{x}, x). \end{aligned}$$

□

5.2 Duality from Generalized Eigenfunctions

A second feature that may be lacking is the existence of a linear independent system of eigenfunctions. However, if L is an irreducible non-reversible generator on the state space Ω with real non-negative eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, there always exists a linearly independent system of so-called *generalized eigenfunctions*, i.e., for each eigenvalue λ_i , there exists a set of linearly independent functions $\{u_i^{(1)}, \dots, u_i^{(m_i)}\}$ such that $m_i \leq n$,

$$Lu_i^{(1)} = \lambda_i u_i^{(1)}$$

and, for $1 < k \leq m_i$,

$$Lu_i^{(k)} = \lambda_i u_i^{(k)} + u_i^{(k-1)}.$$

We refer to $u_i^{(k)}$ as the *k-th order generalized eigenfunction* associated to λ_i . Moreover, if $\lambda_i \neq \lambda_j$, then the set $\{u_i^{(1)}, \dots, u_i^{(m_i)}, u_j^{(1)}, \dots, u_j^{(m_j)}\}$ is linearly independent and any arbitrary function $f : \Omega \rightarrow \mathbb{R}$ can be written as linear combination of functions in $\{u_i^{(k)}, i = 1, \dots, n; k = 1, \dots, m_i\}$.

Example 2. The irreducible generator L on the state space $\Omega = \{1, 2, 3, 4\}$ given by

$$L = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix},$$

represents a basic example of this situation. Indeed, the eigenvalue $\lambda = -1$ has $u^{(1)}$ given by

$$u^{(1)}(x) = \frac{(-1)^x}{2}, \quad x \in \{1, 2, 3, 4\},$$

as eigenfunction and

$$u^{(2)}(x) = \cos\left(\frac{\pi}{2}(x+1)\right), \quad x \in \{1, 2, 3, 4\},$$

as a second order generalized eigenfunction, i.e.,

$$Lu^{(2)} = -u^{(2)} + u^{(1)}.$$

□

In this situation, in case of two generators L, \widehat{L} sharing a real eigenvalue λ with associated generalized eigenfunctions $\{u^{(1)}, \dots, u^{(m)}\}, \{\widehat{u}^{(1)}, \dots, \widehat{u}^{(m)}\}$, the main idea is that a duality function is readily constructed from sums of products of generalized eigenfunctions whose order is, nevertheless, reversed. This connection is the content of the following proposition.

Proposition 7. *The function*

$$D(\widehat{x}, x) = \sum_{k=1}^m \widehat{u}^{(k)}(\widehat{x})u^{(m+1-k)}(x)$$

is a duality function between \widehat{L} and L .

Proof. By using the definition of k -th order generalized eigenfunction, we obtain

$$\begin{aligned} \widehat{L}_{\text{left}}D(\widehat{x}, x) &= \sum_{k=1}^m (\widehat{L}\widehat{u}^{(k)})(\widehat{x})u^{(m+1-k)}(x) \\ &= \sum_{k=1}^m \lambda \widehat{u}^{(k)}(\widehat{x})u^{(m+1-k)}(x) + \sum_{k=2}^m \widehat{u}^{(k-1)}(\widehat{x})u^{(m+1-k)}(x) \\ &= \sum_{k=1}^m \lambda \widehat{u}^{(k)}(\widehat{x})u^{(m+1-k)}(x) + \sum_{k=1}^{m-1} \widehat{u}^{(k)}(\widehat{x})u^{(m-k)}(x) \\ &= \sum_{k=1}^m \widehat{u}^{(k)}(\widehat{x})(Lu^{(m+1-k)})(x) = L_{\text{right}}D(\widehat{x}, x). \end{aligned}$$

□

5.3 Duality and the Jordan Canonical Decomposition: General Case

In this section we provide a general framework that allows us to cover all instances of duality encountered so far in the finite setting. The standard strategy of decomposing generators - viewed as matrices - into their Jordan canonical form builds a bridge between dualities and spectral information of the generators involved. In particular, this linear algebraic approach is useful for the problem of *existence* and *characterization* of duality functions: on one side, the existence of a Jordan canonical decomposition for any generator leads, for instance, to the existence of self-dualities; on the other side, dualities between generators carry information about a common, at least partially, spectral structure of the generators.

Before stating the main result, we introduce some notation. Given a generator L on the state space Ω with cardinality $|\Omega| = n$, L is in *Jordan canonical form* if it can be written as

$$L = UJU^{-1},$$

where $J \in \mathbb{C}^{n \times n}$ is the *unique*, up to permutations, *Jordan matrix* [20, Definition 3.1.1] associated to L and $U \in \mathbb{C}^{n \times n}$ is an invertible matrix. Recall that columns $\{u_1, \dots, u_n\}$ of U consists of (possibly generalized) eigenfunctions of L , while the rows $\{w_1, \dots, w_n\}$ of U^{-1} the (possibly generalized) eigenfunctions of L^\top , chosen in such a way that

$$\langle w_i, u_j \rangle = \sum_{x \in \Omega} w_i(x) u_j^*(x) = \delta_{i,j}.$$

For all Jordan matrices $J \in \mathbb{C}^{n \times n}$ of the form

$$J = \begin{pmatrix} J_{m_1}(\lambda_1) & & \cdots & 0 \\ & J_{m_2}(\lambda_2) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & J_{m_k}(\lambda_k) \end{pmatrix},$$

with $m_1 + \dots + m_k = n$ and *Jordan blocks* $J_m(\lambda)$ of size m associated to eigenvalue $\lambda \in \mathbb{C}$, we define the matrix $B_J \in \mathbb{R}^{n \times n}$ as follows

$$B_J = \begin{pmatrix} H_{m_1} & & \cdots & 0 \\ & H_{m_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & H_{m_k} \end{pmatrix},$$

where, for all $m \in \mathbb{N}$, the matrix $H_m \in \mathbb{R}^{m \times m}$ is defined as

$$H_m = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \ddots \\ \cdot & \cdot & \cdot & \vdots \\ 1 & \cdots & 0 \end{pmatrix},$$

i.e., in such a way that $B_J^\top = B_J^{-1} = B_J$ and $JB_J = B_J J^\top$. Moreover, we say that two matrices $L \in \mathbb{R}^{n \times n}$, $\widehat{L} \in \mathbb{R}^{\widehat{n} \times \widehat{n}}$ are r -similar for some $r = 1, \dots, \min\{n, \widehat{n}\}$ if there exist Jordan canonical forms

$$L = UJU^{-1}, \quad \widehat{L} = \widehat{U}\widehat{J}\widehat{U}^{-1}, \quad (18)$$

matrices $S_r \in \mathbb{R}^{\widehat{n} \times n}$ and $I_r \in \mathbb{R}^{r \times r}$ of the form

$$S_r = \begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad I_r = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

and permutation matrices $\widehat{P} \in \mathbb{R}^{\widehat{n} \times \widehat{n}}$ and $P \in \mathbb{R}^{n \times n}$ such that

$$T_r = \widehat{P}S_rP$$

and

$$\widehat{J}T_r = T_rJ. \quad (19)$$

Of course, if two matrices are r -similar, then they are necessarily r' -similar, for all $r' = 1, \dots, r$ and if $r = n = \widehat{n}$ then we simply say that they are *similar*.

In the following theorem we establish a general connection between duality relations and Jordan canonical forms for generators L, \widehat{L} .

Theorem 1. *The following statements are equivalent:*

- (i) *There exists a duality function $D(\widehat{x}, x)$ of rank r between \widehat{L} and L .*
- (ii) *L and \widehat{L} are r -similar.*

If either condition holds, any duality function is of the form

$$D = \widehat{U}T_rB_JU^\top. \quad (20)$$

In particular if $L = \widehat{L}$, for any $r = 1, \dots, n$, there always exists a self-duality function D of rank r and it must be of the form (20).

Proof. We start with proving that (ii) implies (i). By using the property of r -similarity (19) with Jordan decompositions as in (18), with the choice (20) of the candidate duality function D , we obtain

$$\widehat{L}\widehat{U}T_rB_JU^\top = \widehat{U}\widehat{J}T_rB_JU^\top = \widehat{U}T_rJB_JU^\top = \widehat{U}T_rB_JJ^\top U^\top = \widehat{U}T_rB_JU^\top L^\top,$$

i.e., the duality relation (2) in matrix form.

For the other implication, as the matrices U, \widehat{U} in (18) and B_J are invertible, the following chains of identities are equivalent:

$$\begin{aligned} \widehat{L}D = DL^\top &\iff \widehat{U}\widehat{J}\widehat{U}^{-1}D = D(U^{-1})^\top J^\top U^\top \\ &\iff \widehat{J}\widehat{U}^{-1}D(U^{-1})^\top = \widehat{U}^{-1}D(U^{-1})^\top J^\top \\ &\iff \widehat{J}\widehat{U}^{-1}D(U^{-1})^\top B_J = \widehat{U}^{-1}D(U^{-1})^\top B_JJ. \end{aligned}$$

Moreover, if D has rank r , then $\widehat{U}^{-1}D(U^{-1})^\top B_J$ must have rank r as well. The last relation is of the form

$$\widehat{J}A = AJ,$$

where $A = \widehat{U}^{-1}D(U^{-1})^\top B_J$ is a matrix of rank r . Therefore, we conclude that there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that

$$\widehat{J}S_r = S_r P J P^{-1},$$

i.e., L and \widehat{L} are r -similar according to the Jordan canonical decompositions

$$L = \widetilde{U} \widetilde{J} \widetilde{U}^{-1}, \quad \widehat{L} = \widehat{U} \widehat{J} \widehat{U}^{-1},$$

with $\widetilde{U} = U P^{-1}$ and $\widetilde{J} = P J P^{-1}$. \square

Remark 1. (a) In words, the theorem above states that there exists a rank- r duality matrix if and only if the generators \widehat{L} and L have r eigenvalues (with multiplicities) in common with “compatible” structure of eigenspaces. Additionally, Eq. (20) provides the most general form of the duality function D in terms of matrices U , \widehat{U} . In particular, if J is *diagonal* (i.e., B_J is the identity matrix) *all* duality functions $D(\widehat{x}, x)$ of rank r read as

$$D(\widehat{x}, x) = \sum_{i=1}^r a_i \widehat{u}_i(\widehat{x}) u_i(x),$$

for $a_1, \dots, a_n \in \mathbb{R} \setminus \{0\}$, given $\{u_1, \dots, u_n\}$, $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ are the columns of U , \widehat{U} , invertible matrices in the Jordan decompositions (18) satisfying (19) with $T_r = S_r$. Note the analogy with the duality function described in Propositions 1, 4 and 6. If J is *non-diagonal*, all duality functions D have a similar form up to some index permutations as in Proposition 7.

(b) We note that the *constant duality function* is always a trivial duality function between any two generators L , \widehat{L} on Ω , $\widehat{\Omega}$. Indeed, $\lambda = 0$ is always an eigenvalue for both L and \widehat{L} with associated constant eigenfunctions $u : \Omega \rightarrow \mathbb{R}$, $\widehat{u} : \widehat{\Omega} \rightarrow \mathbb{R}$, i.e., for all $x \in \Omega$ and $\widehat{x} \in \widehat{\Omega}$,

$$u(x) = 1, \quad \widehat{u}(\widehat{x}) = 1,$$

are eigenfunctions for L , \widehat{L} associated to $\lambda = 0$.

(c) Another consequence, as already mentioned in [18], is that in the finite context *self-duality functions always* exist. In fact, a generator L , viewed as a matrix, is always *similar* to itself. Hence, viewing duality relations between generators as similarity relations among matrices allows one to transfer statements about *existence* of Jordan canonical decompositions to statements regarding the *existence* of duality relations, even when neither any explicit formula of the duality functions nor reversible measures for the processes are known. However, Theorem 1 above provides information on how to construct any self-duality matrix. Indeed, given any two Jordan decompositions of L , say

$$LU = UJ, \quad L\widetilde{U} = \widetilde{U}J,$$

the matrix D constructed from U, \tilde{U} and J as in (20), namely

$$D = \tilde{U}B_JU^\top, \quad (21)$$

turns out to be a self-duality function for L and, viceversa, any self-duality matrix D for L is of the form (21).

Typically, to find the eigenvalues and eigenfunctions of the generator associated to a Markov chain is a much more challenging task than establishing duality relations. However, we have seen that the knowledge of the eigenfunctions leads to a full characterization of duality and/or self-duality functions. This is, indeed, the case of the example below, in which we exploit the knowledge of eigenfunctions of two generators to characterize the family of self-duality and duality functions.

Example 3 (One-dimensional symmetric random walks on a finite grid). Let us introduce the symmetric random walk on $\Omega = \{1, \dots, n\}$ reflected on the left and absorbed on the right. We describe the action of the generator L on functions $f : \Omega \rightarrow \mathbb{R}$ as

$$Lf(x) = (f(x+1) - f(x)) + (f(x-1) - f(x)), \quad x \in \Omega \setminus \{1, n\},$$

while for $x \in \{1, n\}$ we have

$$Lf(1) = 2(f(2) - f(1)), \quad Lf(n) = 0.$$

Similarly, we denote by \widehat{L} the generator of the symmetric random walk on Ω reflected on the right and absorbed on the left. Namely,

$$\widehat{L}f(x) = (f(x+1) - f(x)) + (f(x-1) - f(x)), \quad x \in \Omega \setminus \{1, n\},$$

and

$$\widehat{L}f(1) = 0, \quad \widehat{L}f(n) = 2(f(n-1) - f(n)).$$

As an application of Theorem 1, we prove the following dualities: *self-duality* of L , *self-duality* of \widehat{L} and *duality* between L and \widehat{L} . The key is to explicitly find eigenvalues and eigenfunctions of the generators. Indeed, the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of L and \widehat{L} read as follows:

$$\lambda_1 = 0, \quad \lambda_i = 2(\cos(\theta_i) - 1), \quad \theta_i = \frac{i - \frac{1}{2}}{n - 1}\pi, \quad i = 2, \dots, n. \quad (22)$$

The eigenfunctions $\{u_1, \dots, u_n\}$ of L are, for $x \in \Omega$,

$$u_1(x) = \frac{1}{\sqrt{n}}, \quad u_i(x) = \frac{1}{\sqrt{n}} \cos(\theta_i(x-1)), \quad i = 2, \dots, n,$$

while the eigenfunctions $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ of \widehat{L} are, for $x \in \Omega$,

$$\widehat{u}_1(\widehat{x}) = \frac{1}{\sqrt{n}}, \quad \widehat{u}_i(\widehat{x}) = \frac{1}{\sqrt{n}} \sin(\theta_i(\widehat{x}-1)), \quad i = 2, \dots, n.$$

Hence, we conclude the following:

(a) *Self-duality functions for L .* For all values $a_1, \dots, a_n \in \mathbb{R}$, the function

$$D(x, y) = \sum_{i=1}^n a_i u_i(x) u_i(y) = \frac{a_1}{n} + \sum_{i=2}^n \frac{a_i}{n} \cos(\theta_i(x-1)) \cos(\theta_i(y-1)) \quad (23)$$

is a self-duality function for L and all self-duality functions are of this form.

(b) *Self-duality functions for \widehat{L} .* For all $a_1, \dots, a_n \in \mathbb{R}$,

$$\widehat{D}(\widehat{x}, \widehat{y}) = \sum_{i=1}^n a_i \widehat{u}_i(\widehat{x}) \widehat{u}_i(\widehat{y}) = \frac{1}{n} + \sum_{i=2}^n \frac{a_i}{n} \sin(\theta_i(\widehat{x}-1)) \sin(\theta_i(\widehat{y}-1)) \quad (24)$$

is a self-duality function for \widehat{L} and all self-duality functions are of this form.

(c) *Duality functions between L and \widehat{L} .* For all $a_1, \dots, a_n \in \mathbb{R}$,

$$D'(\widehat{x}, x) = \frac{a_1}{n} + \sum_{i=2}^n \frac{a_i}{n} \sin(\theta_i(\widehat{x}-1)) \cos(\theta_i(x-1)), \quad (25)$$

is a duality function between L and \widehat{L} and all duality functions are of this form. \square

We can now provide an analogue of Proposition 2 beyond the reversible context. To fix notation, let L be a generator on Ω , with $|\Omega| = n$. Lacking reversibility, we have seen that complex eigenvalues and generalized eigenfunctions of the generator may arise. However, in the irreducible case, i.e., in case there exists a unique stationary measure $\mu > 0$ for which the adjoint of L in $L^2(\mu)$, say L^\dagger , is itself a generator, a trivial duality relation between L and L^\dagger is available. Indeed, from the adjoint relation

$$\langle L^\dagger f, g \rangle_{L^2(\mu)} = \langle f, Lg \rangle_{L^2(\mu)}, \quad f, g : \Omega \rightarrow \mathbb{R},$$

it follows that the diagonal function $D : \Omega \times \Omega \rightarrow \mathbb{R}$ given by

$$D(x, y) = \frac{\delta_{x,y}}{\mu(y)}, \quad x, y \in \Omega, \quad (26)$$

is a duality function for L^\dagger, L . In analogy with (4), we refer to it as *cheap duality function*, also $D = D_{\text{cheap}}$.

From Theorem 1, the above duality tells us that, beside the fact that the generators L and L^\dagger are indeed similar as matrices, the cheap duality function D_{cheap} in (26) should be represented in terms of functions $\{u_1, \dots, u_n\}$ and $\{\widehat{u}_1, \dots, \widehat{u}_n\}$, which, up to suitably reordering, are indeed the generalized eigenfunctions of L and L^\dagger , respectively.

As a consequence of the following lemma, which we use in the proof of Theorem 5, we obtain that a relation of *bi-orthogonality* w.r.t. μ among the generalized eigenfunctions of L and those of L^\dagger can be derived from the duality w.r.t. D_{cheap} . For the proof, we refer back to the proof of Proposition 2.

Proposition 8. *Let L be a generator, μ a positive measure on Ω (not necessarily stationary for L) and let L^\dagger be the adjoint operator of L in $L^2(\mu)$. Let the spans of the generalized eigenfunctions of L and L^\dagger , say $\{u_1, \dots, u_n\}$ and $\{\tilde{u}_1, \dots, \tilde{u}_n\}$, both coincide with $L^2(\mu)$. Then the following statements are equivalent:*

(i) Cheap duality from generalized eigenfunctions. For $x, y \in \Omega$,

$$\sum_{i=1}^n \tilde{u}_i(x) u_i(y) = \frac{\delta_{x,y}}{\mu(y)}.$$

(ii) Bi-orthogonality of generalized eigenfunctions. For all $i, j = 1, \dots, n$,

$$\langle \tilde{u}_i, u_j^* \rangle_\mu = \sum_{x' \in \Omega} \tilde{u}_i(x') u_j(x') \mu(x') = \delta_{i,j}. \quad (27)$$

Two families $\{u_1^*, \dots, u_n^*\}$, $\{\tilde{u}_1, \dots, \tilde{u}_n\}$ satisfying condition (27) are also said to be bi-orthogonal w.r.t. the measure μ .

5.4 Intertwining Relations, Duality and Generalized Eigenfunctions

Symmetries of the generators or, more generally, *intertwining relations* have proved to be useful in producing new duality relations from existing ones, e.g. cheap dualities [4, 33]. Here, we analyze this technique and revisit [33, Theorem 5.1] from the point of view of generalized eigenfunctions.

Theorem 2 (Intertwining relations and duality). *Let L, \tilde{L} and \hat{L} be three generators on Ω , $\tilde{\Omega}$ and $\hat{\Omega}$ respectively. We assume that L and \tilde{L} are intertwined, i.e., there exists a linear operator $\Lambda : L^2(\Omega) \rightarrow L^2(\tilde{\Omega})$ such that, for all $f \in L^2(\Omega)$, we have*

$$\tilde{L}\Lambda f = \Lambda Lf. \quad (28)$$

Moreover, we assume that L and \hat{L} are dual with duality function $D : \hat{\Omega} \times \Omega \rightarrow \mathbb{R}$, i.e.,

$$\hat{L}_{\text{left}} D(\hat{x}, x) = L_{\text{right}} D(\hat{x}, x).$$

Then, the function $\Lambda_{\text{right}} D : \hat{\Omega} \times \tilde{\Omega} \rightarrow \mathbb{R}$ is a duality function for \tilde{L} and \hat{L} , i.e.,

$$\hat{L}_{\text{left}} \Lambda_{\text{right}} D(\hat{x}, \tilde{x}) = \tilde{L}_{\text{right}} \Lambda_{\text{right}} D(\hat{x}, \tilde{x}).$$

Proof. We observe that the *intertwining operator* Λ maps eigenspaces of L to eigenspaces of \tilde{L} . More formally, if there exists a subset $\{u^{(1)}, \dots, u^{(m)}\}$ of $L^2(\Omega)$ such that, for some $\lambda \in \mathbb{C}$,

$$Lu^{(1)} = \lambda u^{(1)}, \quad Lu^{(k)} = \lambda u^{(k)} + u^{(k-1)}, \quad k = 2, \dots, m, \quad (29)$$

then, by (28), the subset $\{\Lambda u^{(1)}, \dots, \Lambda u^{(m)}\}$ in $L^2(\tilde{\Omega})$ satisfy the same identities as in (29) up to replace L by \tilde{L} :

$$\tilde{L}\Lambda u^{(1)} = \lambda \Lambda u^{(1)}, \quad \tilde{L}\Lambda u^{(k)} = \lambda \Lambda u^{(k)} + \Lambda u^{(k-1)}, \quad k = 2, \dots, m. \quad (30)$$

By Theorem 1, the duality function is given by

$$D(\widehat{x}, x) = \sum_{i=1}^n \widehat{u}_i(\widehat{x}) u_i(x),$$

where $\{u_1, \dots, u_n\}, \{\widehat{u}_1, \dots, \widehat{u}_n\}$ are sets of (possibly generalized) eigenfunctions of L, \widehat{L} . Then, by applying the intertwining operator Λ on the right variables, we obtain

$$A_{\text{right}} D(\widehat{x}, \widetilde{x}) = \sum_{i=1}^n \widehat{u}_i(\widehat{x}) (\Lambda u_i)(\widetilde{x}).$$

We conclude from the considerations in (30), (29) and Theorem 1. \square

Typical examples of intertwining relations occur when either Λ is a *symmetry* of a generator, i.e., $\widetilde{L} = L$ in (28) (see e.g. [4]) or when Λ is a *positive contractive operator* such that $\Lambda 1 = 1$, i.e., viewed as a matrix, it is a stochastic matrix from the space $\widetilde{\Omega}$ to Ω (see e.g. [21]). A particular instance, which recovers the so-called *lumpability*, of this last situation is when Λ is a “deterministic” stochastic kernel, i.e., induced by a map from $\widetilde{\Omega}$ to Ω .

5.5 Intertwining of Exclusion Processes

In this section we provide an application of Theorem 2 above. Indeed, after finding suitable intertwining relations between a particular instance of the symmetric simple exclusion process and a generalized symmetric exclusion process, we obtain as in Theorem 2 a large class of self-duality functions for the latter process from self-duality functions of the former. In what follows, we fix $\gamma \in \mathbb{N}$, a finite set V of cardinality $|V| = m$ and a function $p : V \times V \rightarrow \mathbb{R}_+$ such that $p(x, x) = 0$ for all $x \in V$.

The γ -ladder-SEP is the finite-state Markov process on $\widetilde{\Omega} = \{0, 1\}^{V \times \{1, \dots, \gamma\}}$ with generator \widetilde{L} acting on functions $\widetilde{f} : \widetilde{\Omega} \rightarrow \mathbb{R}$ as

$$\begin{aligned} \widetilde{L} \widetilde{f}(\widetilde{\eta}) = & \sum_{x, y \in V} p(x, y) \left[\sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \widetilde{\eta}(x, a) (1 - \widetilde{\eta}(y, b)) (\widetilde{f}(\widetilde{\eta}^{(x, a), (y, b)}) - \widetilde{f}(\widetilde{\eta})) \right. \\ & \left. + \widetilde{\eta}(y, b) (1 - \widetilde{\eta}(x, a)) (\widetilde{f}(\widetilde{\eta}^{(y, b), (x, a)}) - \widetilde{f}(\widetilde{\eta})) \right], \quad \widetilde{\eta} \in \widetilde{\Omega}, \end{aligned}$$

where $\widetilde{\eta}^{(x, a), (y, b)}$ denotes the configuration obtained from $\widetilde{\eta}$ by removing a particle at position (x, a) and placing it at (y, b) . As already mentioned, this process may be considered as a special case of a simple symmetric exclusion process on the set $\widetilde{V}_\gamma = V \times \{1, \dots, \gamma\}$ where $\widetilde{p} : \widetilde{V} \times \widetilde{V} \rightarrow \mathbb{R}_+$ is such that

$$\widetilde{p}((x, a), (y, b)) = p(x, y), \quad (x, a), (y, b) \in \widetilde{V}_\gamma.$$

The SEP(γ) is the finite-state Markov process on $\Omega = \{0, \dots, \gamma\}^V$ with generator L acting on functions $f : \Omega \rightarrow \mathbb{R}$ as

$$Lf(\eta) = \sum_{x,y \in V} p(x,y) [\eta(x)(\gamma - \eta(y))(f(\eta^{x,y}) - f(\eta)) + \eta(y)(\gamma - \eta(x))(f(\eta^{y,x}) - f(\eta))], \quad \eta \in \Omega.$$

It is well known (see e.g. [18]) that L and \tilde{L} are *intertwined* via a deterministic intertwining operator $\Lambda : L^2(\Omega) \rightarrow L^2(\tilde{\Omega})$. The intertwining operator Λ is defined, given the mapping $\pi : \tilde{\Omega} \rightarrow \Omega$ such that

$$\pi(\tilde{\eta}) = (|\tilde{\eta}(1, \cdot)|, \dots, |\tilde{\eta}(n, \cdot)|) \in \Omega, \quad |\tilde{\eta}(x, \cdot)| := \sum_{a=1}^{\gamma} \tilde{\eta}(x, a),$$

as acting on functions $f : \Omega \rightarrow \mathbb{R}$ as

$$\Lambda f(\tilde{\eta}) = f(\pi(\tilde{\eta})), \quad \tilde{\eta} \in \tilde{\Omega}.$$

The intertwining relation then reads, for all $f : \Omega \rightarrow \mathbb{R}$, as

$$\tilde{L}\Lambda f(\tilde{\eta}) = \Lambda Lf(\tilde{\eta}),$$

for $\tilde{\eta} \in \tilde{\Omega}$. Given any self-duality for L with self-duality function $D(\xi, \eta)$, we can build a duality function, namely $D'(\xi, \tilde{\eta}) = \Lambda_{\text{right}} D(\xi, \tilde{\eta})$ for L and \tilde{L} and, furthermore, a self-duality function $D''(\tilde{\xi}, \tilde{\eta}) = \Lambda_{\text{left}} \Lambda_{\text{right}} D(\tilde{\xi}, \tilde{\eta})$ for \tilde{L} .

However, we ask whether there exists an “inverse” intertwining relation, i.e., $\tilde{\Lambda} : L^2(\tilde{\Omega}) \rightarrow L^2(\Omega)$ such that, for $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{R}$,

$$\tilde{\Lambda}\tilde{L}\tilde{f}(\eta) = L\tilde{\Lambda}\tilde{f}(\eta), \quad \eta \in \Omega. \quad (31)$$

In what follows, we say that $\tilde{\eta} \in \tilde{\Omega}$ is *compatible* with $\eta \in \Omega$ or, shortly, $\tilde{\eta} \sim \eta$, if $\pi(\tilde{\eta}) = \eta$.

Proposition 9. *The operator $\tilde{\Lambda} : L^2(\tilde{\Omega}) \rightarrow L^2(\Omega)$ defined as*

$$\tilde{\Lambda}\tilde{f}(\eta) = \left(\prod_{x \in V} \frac{1}{\binom{\gamma}{\eta(x)}} \right) \sum_{\tilde{\eta} \sim \eta} \tilde{f}(\tilde{\eta}), \quad \eta \in \Omega, \quad (32)$$

is the inverse intertwining in (31). Moreover, the intertwining operator above is a stochastic intertwining.

Proof. Without loss of generality, we consider $V = \{x, y\}$. By expanding the l.h.s. of (31) with $\tilde{\Lambda}$ as in (32), we obtain four terms:

$$\ell_1 = -\frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \sum_{\tilde{\eta} \sim \eta} \sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(x, a)(1 - \tilde{\eta}(y, b))\tilde{f}(\tilde{\eta})$$

$$\begin{aligned}
\ell_2 &= \frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \sum_{\tilde{\eta} \sim \eta} \sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(x, a) (1 - \tilde{\eta}(y, b)) \tilde{f}(\tilde{\eta}^{(x,a),(y,b)}) \\
\ell_3 &= -\frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \sum_{\tilde{\eta} \sim \eta} \sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(y, b) (1 - \tilde{\eta}(x, a)) \tilde{f}(\tilde{\eta}) \\
\ell_4 &= \frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \sum_{\tilde{\eta} \sim \eta} \sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(y, b) (1 - \tilde{\eta}(x, a)) \tilde{f}(\tilde{\eta}^{(y,b),(x,a)}).
\end{aligned}$$

By doing the same thing with the r.h.s., we obtain:

$$\begin{aligned}
r_1 &= -\frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \eta(x) (\gamma - \eta(y)) \sum_{\tilde{\eta} \sim \eta} \tilde{f}(\tilde{\eta}) \\
r_2 &= \frac{1}{\binom{\gamma}{\eta(x)-1}} \frac{1}{\binom{\gamma}{\eta(y)+1}} \eta(x) (\gamma - \eta(y)) \sum_{\tilde{\eta} \sim \eta^{x,y}} \tilde{f}(\tilde{\eta}) \\
r_3 &= -\frac{1}{\binom{\gamma}{\eta(x)}} \frac{1}{\binom{\gamma}{\eta(y)}} \eta(y) (\gamma - \eta(x)) \sum_{\tilde{\eta} \sim \eta} \tilde{f}(\tilde{\eta}) \\
r_4 &= \frac{1}{\binom{\gamma}{\eta(x)+1}} \frac{1}{\binom{\gamma}{\eta(y)-1}} \eta(y) (\gamma - \eta(x)) \sum_{\tilde{\eta} \sim \eta^{y,x}} \tilde{f}(\tilde{\eta}).
\end{aligned}$$

Note that $\ell_1 = r_1$ because, for all $\tilde{\eta} \sim \eta$,

$$\sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(x, a) (1 - \tilde{\eta}(y, b)) = \eta(x) (\gamma - \eta(y)),$$

and similarly for $\ell_3 = r_3$. For $\ell_2 = r_2$ it is enough to verify that, for each $\tilde{\eta}_* \sim \eta^{x,y}$,

$$\sum_{\tilde{\eta} \sim \eta} \sum_{a=1}^{\gamma} \sum_{b=1}^{\gamma} \tilde{\eta}(x, a) (1 - \tilde{\eta}(y, b)) \mathbb{1}\{\tilde{\eta}^{(x,a),(y,b)} = \tilde{\eta}_*\} = (\eta(y) + 1) (\gamma - \eta(x) + 1).$$

This last identity indeed holds, as the configurations $\tilde{\eta} \sim \eta$ can be obtained from $\tilde{\eta}_*$ by picking one of the $\eta(y) + 1$ particles on $y \in V$ and putting it back on one of the $\gamma - \eta(x) + 1$ holes of $x \in V$. Analogously for $\ell_4 = r_4$. \square

As a consequence of this proposition, by starting from self-duality of the γ -ladder-SEP, we can produce duality functions for \tilde{L} and L and self-duality functions for L . We use the following result of [35, Theorem 2.8] to obtain a large class of “factorized” self-duality functions for \tilde{L} .

Theorem 3 ([35]). *The simple symmetric exclusion process $\{\tilde{\eta}_t, t \geq 0\}$ on the vertex set $V \times \{1, \dots, \gamma\}$ is self-dual w.r.t. the duality function*

$$\tilde{D}(\tilde{\xi}, \tilde{\eta}) = \prod_{(x,a) \in V \times \{1, \dots, \gamma\}} (\alpha + \beta \tilde{\eta}(x, a))^{\epsilon + \delta \tilde{\xi}(x,a)}, \quad \tilde{\xi}, \tilde{\eta} \in \tilde{\Omega}, \quad (33)$$

for all α, β, ϵ and $\delta \in \mathbb{R}$.

Now, we apply the intertwining operator \tilde{A} first on the right and then on the left variables of D above.

Theorem 4. *All self-duality functions for SEP(γ) derived from self-duality functions of γ -ladder-SEP as in (33) are all in factorized form, i.e.,*

$$D(\xi, \eta) = \tilde{A}_{\text{left}} \tilde{A}_{\text{right}} \tilde{D}(\xi, \eta) = \prod_{x \in V} d_x^{\alpha, \beta, \epsilon, \delta}(\xi(x), \eta(x)).$$

Moreover, the single-site self-duality functions $d_x^{\alpha, \beta, \epsilon, \delta}(k, n)$, for $k, n \in \{0, \dots, \gamma\}$, are in one of the following forms: either the classical polynomials

$$d_x^{0, \beta, 0, \delta}(k, n) = (\beta^\delta)^k \frac{(\gamma - k)!}{\gamma!} \frac{n!}{(n - k)!} \mathbb{1}\{n \geq k\},$$

the orthogonal polynomials

$$d_x^{\alpha, \beta, \epsilon, \delta}(k, n) = (-1)^{\delta k} \alpha^{\epsilon \gamma - \epsilon n + \delta k} (\alpha + \beta)^{\epsilon n} {}_2F_1 \left[\begin{matrix} -k - n \\ -\gamma \end{matrix}; 1 - \left(1 + \frac{\beta}{\alpha}\right)^\delta \right],$$

or other degenerate functions:

$$\begin{aligned} d_x^{\alpha, \beta, \epsilon, 0}(k, n) &= (\alpha + \beta)^{\epsilon n} \alpha^{\epsilon(\gamma - n)} \\ d_x^{0, \beta, \epsilon, \delta}(k, n) &= \beta^{\epsilon \gamma + \delta k} \mathbb{1}\{n = \gamma\} \\ d_x^{\alpha, 0, \epsilon, \delta}(k, n) &= \alpha^{\epsilon \gamma + \delta k} \\ d_x^{\alpha, -\alpha, \epsilon, \delta}(k, n) &= \alpha^{\epsilon \gamma + \delta k} \mathbb{1}\{n = 0\}. \end{aligned}$$

Proof. First thing to note is that the factorized structure of D is preserved under \tilde{A} . Indeed, if we use the notation

$$\mathbf{d}(k, n) = (\alpha + \beta n)^{\epsilon + \delta k},$$

then

$$\tilde{A}_{\text{right}} D(\tilde{\xi}, \eta) = \prod_{x \in V} \left(\frac{1}{\binom{\gamma}{\eta(x)}} \sum_{\tilde{\eta}(x, \cdot) \sim \eta(x)} \prod_{a=1}^{\gamma} \mathbf{d}(\tilde{\xi}(x, a), \tilde{\eta}(x, a)) \right).$$

Hence we compute only what is inside the parenthesis (which will see does depend on $\tilde{\xi}(x, \cdot)$ only through $|\tilde{\xi}(x, \cdot)|$):

$$\begin{aligned} & d_x^{\alpha, \beta, \epsilon, \delta}(\xi(x), \eta(x)) \\ &= (\alpha + \beta)^{\epsilon \eta(x)} \alpha^{\epsilon(\gamma - \eta(x))} \frac{1}{\binom{\gamma}{\eta(x)}} \sum_{\tilde{\eta}(x, \cdot) \sim \eta(x)} \prod_{a=1}^{\gamma} (\alpha + \beta \tilde{\eta}(x, a))^{\delta \tilde{\xi}(x, a)}. \quad (34) \end{aligned}$$

The last summation

$$\frac{1}{\binom{\gamma}{\eta(x)}} \sum_{\tilde{\eta}(x, \cdot) \sim \eta(x)} \prod_{a=1}^{\gamma} (\alpha + \beta \tilde{\eta}(x, a))^{\delta \tilde{\xi}(x, a)}$$

clearly does not depend on $\tilde{\xi}(x, \cdot)$ but only on $\xi(x) = |\tilde{\xi}(x, \cdot)|$ and equals

$$\frac{1}{\binom{\gamma}{\eta(x)}} \sum_{\ell=0}^{\xi(x)} \binom{\xi(x)}{\xi(x) - \ell} \binom{\gamma - \xi(x)}{\eta(x) - (\xi(x) - \ell)} (\alpha + \beta)^{\delta(\xi(x) - \ell)} \alpha^{\delta \ell}. \quad (35)$$

If $\delta = 0$, this last expression in (35) by Chu-Vandermonde identity equals 1, hence

$$d_x^{\alpha, \beta, \epsilon, 0}(\xi(x), \eta(x)) = (\alpha + \beta)^{\epsilon \eta(x)} \alpha^{\epsilon(\gamma - \eta(x))}.$$

If $\delta \neq 0$ and $\alpha = 0$, expression (35) rewrites as

$$\begin{aligned} \frac{1}{\binom{\gamma}{\eta(x)}} \binom{\gamma - \xi(x)}{\eta(x) - \xi(x)} \beta^{\delta \xi(x)} \mathbb{1}\{\eta(x) \geq \xi(x)\} \\ = (\beta^\delta)^{\xi(x)} \frac{(\gamma - \xi(x))!}{\gamma!} \frac{\eta(x)!}{(\eta(x) - \xi(x))!} \mathbb{1}\{\eta(x) \geq \xi(x)\}, \end{aligned}$$

and hence, for $\epsilon = 0$, (34) becomes

$$d_x^{0, \beta, 0, \delta}(\xi(x), \eta(x)) = (\beta^\delta)^{\xi(x)} \frac{(\gamma - \xi(x))!}{\gamma!} \frac{\eta(x)!}{(\eta(x) - \xi(x))!} \mathbb{1}\{\eta(x) \geq \xi(x)\},$$

i.e., the classical single-site self-duality functions, while, for $\epsilon \neq 0$,

$$d_x^{0, \beta, \epsilon, \delta}(\xi(x), \eta(x)) = \beta^{\epsilon \gamma + \delta \xi(x)} \mathbb{1}\{\eta(x) = \gamma\}.$$

If $\delta \neq 0$ and $\alpha \neq 0$ and $\beta = 0$, then again we get some trivial:

$$d_x^{\alpha, 0, \epsilon, \delta}(\xi(x), \eta(x)) = \alpha^{\epsilon \gamma + \delta \xi(x)}.$$

The most interesting case is when $\delta \neq 0$, $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq -\beta$. In this case the quantity in (35) equals

$$(\alpha + \beta)^{\delta \xi(x)} \frac{1}{\binom{\gamma}{\eta(x)}} \sum_{\ell=0}^{\xi(x)} \binom{\xi(x)}{\xi(x) - \ell} \binom{\gamma - \xi(x)}{\eta(x) - (\xi(x) - \ell)} \left(\frac{\alpha}{\alpha + \beta} \right)^{\delta \ell},$$

which rewrites, by using two known relations in [31, p. 51], as

$$(-\alpha)^{\delta \xi(x)} {}_2F_1 \left[\begin{matrix} -\xi(x) - \eta(x) \\ -\gamma \end{matrix}; 1 - \left(1 + \frac{\beta}{\alpha}\right)^\delta \right],$$

leading to

$$\begin{aligned} d_x^{\alpha, \beta, \epsilon, \delta}(\xi(x), \eta(x)) \\ = (-1)^{\delta \xi(x)} \alpha^{\epsilon \gamma - \epsilon \eta(x) + \delta \xi(x)} (\alpha + \beta)^{\epsilon \eta(x)} {}_2F_1 \left[\begin{matrix} -\xi(x) - \eta(x) \\ -\gamma \end{matrix}; 1 - \left(1 + \frac{\beta}{\alpha}\right)^\delta \right], \end{aligned}$$

i.e., we recover the *orthogonal polynomial single-site self-duality functions* for the SEP(γ), namely families of Kravchuk polynomials. If $\alpha = -\beta$, then we have

$$d_x^{\alpha, -\alpha, \epsilon, \delta}(\xi(x), \eta(x)) = \alpha^{\epsilon\gamma + \delta\xi(x)} \mathbb{1}\{\eta(x) = 0\}.$$

□

6 Siegmund Duality

This connection between duality functions and eigenfunctions enables us to recover another special instance of duality, the so-called *Siegmund duality*. Siegmund duality, which arises in the context of totally ordered state spaces $\Omega = \Omega$, was first established by Siegmund [36] for pairs of absorbed/reflected-at-0 processes on the positive real line and on the positive integers. Further applications and generalizations of Siegmund dualities were studied by many authors, see for instance [25, 27, 28].

What we focus here on is a finite-context characterization of Siegmund duality already obtained via an intertwining relation in [21]. However, by using a representation of duality in terms of generalized eigenfunctions of the generators, the characterization result of Siegmund duality that we obtain, besides simplifying the proof of an analogous result in [36, Theorem 3], adds spectral information to the proof in [21].

Moreover, as Siegmund duality can be seen as a full-rank duality between two processes, cf. Theorem 1, a spectral approach provides a strategy to find other duality relations in the presence of Siegmund duality.

6.1 Characterization of Siegmund Duality

On the totally ordered state space $\Omega = \{1, \dots, n\}$, two generators L, \widehat{L} are said to be *Siegmund dual* if

$$\widehat{L}_{\text{left}} D_S(x, y) = L_{\text{right}} D_S(x, y), \quad (36)$$

with duality function $D_S : \Omega \times \Omega \rightarrow [0, 1]$ given by

$$D_S(x, y) = \mathbb{1}\{x \geq y\}. \quad (37)$$

Note that the duality relation (36) with duality function D_S (37) reads out

$$\sum_{x'=y}^n \widehat{L}(x, x') = \sum_{y'=1}^x L(y, y'). \quad (38)$$

From (38), a *necessary* relation between two Siegmund dual generators L and \widehat{L} reads as follows:

$$L(y, x) = \sum_{x'=y}^n \widehat{L}(x, x') - \widehat{L}(x-1, x'), \quad x, y \in \Omega, \quad (39)$$

with the convention $\widehat{L}(0, \cdot) = 0$. As (39) implies (38), this condition is indeed also *sufficient*.

Remark 2 (Sub-generators and monotonicity). If we require that only \widehat{L} is a generator, the operator L as defined in (39) is not necessarily a generator. However, the following implications hold:

- (a) If \widehat{L} is a generator and $L(y, x) \geq 0$ for all $x \neq y$, then L is a *sub-generator* on Ω , i.e.,

$$L(y, x) \geq 0, \quad x \neq y \quad \text{and} \quad \sum_{x=1}^n L(y, x) \leq 0, \quad y \in \Omega. \quad (40)$$

The proof goes as follows:

$$\begin{aligned} \sum_{x=1}^n L(y, x) &= \sum_{x'=y}^n \sum_{x=1}^n \widehat{L}(x, x') - \widehat{L}(x-1, x') = \sum_{x'=y}^n \widehat{L}(n, x') \\ &\leq \sum_{x'=1}^n \widehat{L}(n, x') = 0, \end{aligned}$$

where we used (39) in the first equality and the last inequality is a consequence of \widehat{L} being a generator.

- (b) Note that, by [22, Theorem 2.1],

$$\sum_{x'=y}^n \widehat{L}(x, x') - \widehat{L}(x-1, x') \geq 0, \quad x \neq y, \quad (41)$$

is equivalent to require that the continuous-time Markov chain with generator \widehat{L} is *monotone* (see [28]).

As a consequence, L is a sub-generator if and only if \widehat{L} is associated to a monotone process on Ω .

In the following theorem, we study the relation between eigenfunctions of Siegmund dual (sub-)generators and how the Siegmund duality function D_S in (37) is constructed from the eigenfunctions.

Theorem 5. (i) Let L and \widehat{L} be Siegmund dual (sub-)generators in the sense of (36). If \widehat{w} is a k -th order generalized eigenfunction of \widehat{L}^\top associated to eigenvalue λ , then

$$u(x) = \sum_{y=x}^n \widehat{w}(y), \quad x \in \Omega, \quad (42)$$

is a k -th order generalized eigenfunction of L associated to the eigenvalue λ .

- (ii) In the same context as in item (i), given a set $\{\widehat{w}_1, \dots, \widehat{w}_n\}$ of (generalized) eigenfunctions of \widehat{L}^\top whose span coincides with $L^2(\Omega)$, if $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ are (generalized) eigenfunctions of \widehat{L} such that

$$\langle \widehat{w}_i, \widehat{u}_j^* \rangle = \sum_{x=1}^n \widehat{w}_i(x) \widehat{u}_j(x) = \delta_{i,j}, \quad (43)$$

and $\{u_1, \dots, u_n\}$ are defined in terms of $\{\widehat{w}_1, \dots, \widehat{w}_n\}$ as in (42), then the function

$$D(x, y) = \sum_{i=1}^n \widehat{u}_i(x) u_i(y), \quad x, y \in \Omega,$$

is the Siegmund duality function D_S .

(iii) Let L and \widehat{L} be (sub-)generators on Ω . If for any k -th order generalized eigenfunction \widehat{w} of \widehat{L}^\top associated to eigenvalue λ , u as defined in (42) is a k -th order generalized eigenfunction of L associated to the same eigenvalue λ , then L and \widehat{L} are Siegmund dual and D_S is obtained as in item (ii).

Proof. Let \widehat{w} and u be as in item (i). Then,

$$\begin{aligned} \sum_{x=1}^n L(y, x) u(x) &= \sum_{x=1}^n \left(\sum_{x'=y}^n \widehat{L}(x, x') - \widehat{L}(x-1, x') \right) u(x) \\ &= \sum_{x'=y}^n \sum_{x=1}^n \left(\widehat{L}^\top(x', x) u(x) - \widehat{L}^\top(x', x-1) u(x) \right), \end{aligned}$$

which, by noting that $\widehat{w}(n) = u(n)$, reads as

$$\sum_{x'=y}^n \sum_{x=1}^n \widehat{L}^\top(x', x) \widehat{w}(x) = \sum_{x'=y}^n \lambda \widehat{w}(x') = \lambda \sum_{x'=y}^n \widehat{w}(x') = \lambda u(y),$$

thus, u is eigenfunction with eigenvalue λ . For the generalized eigenfunctions, the proof follows the same line.

For item (ii) and (iii), from the sets $\{\widehat{w}_1, \dots, \widehat{w}_n\}$ and $\{u_1, \dots, u_n\}$ of generalized eigenfunctions of \widehat{L}^\top and L related as in (42), by Theorem 1 the function

$$D(x, y) = \sum_{i=1}^n \widehat{u}_i(x) u_i(y) = \sum_{i=1}^n \widehat{u}_i(x) \sum_{x'=y}^n \widehat{w}_i(x') = \sum_{x'=y}^n \sum_{i=1}^n \widehat{u}_i(x) \widehat{w}_i(x') \quad (44)$$

is a full-rank duality for L and \widehat{L} . By Proposition 8 and condition (43), by passing to the conjugates, we obtain

$$\sum_{i=1}^n \widehat{u}_i(x) \widehat{w}_i(x') = \delta_{x, x'},$$

and hence the function $D(x, y)$ in (44) writes as

$$D(x, y) = \sum_{x'=y}^n \delta_{x, x'} = \mathbb{1}\{x \geq y\} = D_S(x, y).$$

□

In this final example, by using item (iii) of Theorem 5, we show how to obtain Siegmund duality from the knowledge of eigenvalues and eigenfunctions of (sub-)generators. The example we consider here concerns two symmetric random walks on $\Omega = \{1, \dots, n\}$.

Example 4 (Blocked vs absorbed random walks on a finite grid). The first symmetric nearest-neighbor random walk is *blocked at the boundaries*, namely the generator \widehat{L} is described, for $f : \Omega \rightarrow \mathbb{R}$, as

$$\widehat{L}f(x) = (f(x+1) - f(x)) + (f(x-1) - f(x)), \quad x \in \Omega \setminus \{1, n\},$$

and, on the boundaries,

$$\widehat{L}f(1) = f(2) - f(1), \quad \widehat{L}f(n) = f(n-1) - f(n).$$

The second random walk is *absorbed at the boundaries*, i.e., it is a sub-Markov process on $\Omega = \{1, \dots, n\}$ with sub-generator L which acts on functions $f : \Omega \rightarrow \mathbb{R}$ as

$$Lf(x) = (f(x+1) - f(x)) + (f(x-1) - f(x)), \quad x \in \Omega \setminus \{1, n\},$$

and

$$Lf(1) = 0, \quad Lf(n) = f(n-1) - 2f(n),$$

i.e. $x = 1$ is an absorbing point, while at $x = n$ the random walk either jumps to the left at rate 1 or “exits the system” at rate 1.

To explicitly obtain eigenfunctions and eigenvalues in this setting we use the following *ansatz*:

$$f_{a,b,c,\theta}(x) = a \cos(\theta x + c) + b \sin(\theta x + c), \quad x \in \Omega,$$

where a, b, c and $\theta \in \mathbb{R}$ are the parameters to be determined. Regarding the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, in both cases we have

$$\lambda_1 = 0, \quad \lambda_i = 2(\cos(\theta_i) - 1), \quad \theta_i = \frac{i-1}{n}\pi, \quad i = 2, \dots, n.$$

Hence, all eigenvalues are distinct. The eigenfunctions $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ of \widehat{L} are, for $x \in \{1, \dots, n\}$ and $i = 2, \dots, n$,

$$\widehat{u}_1(x) = \frac{1}{\sqrt{n}},$$

and

$$\widehat{u}_i(x) = \frac{1}{\sqrt{n(1 - \cos(\theta_i))}} (-\sin(\theta_i) \cos(\theta_i(x-1)) + (1 - \cos(\theta_i)) \sin(\theta_i(x-1))).$$

The eigenfunctions $\{u_1, \dots, u_n\}$ of L are given, for $x \in \{1, \dots, n\}$ and $i = 2, \dots, n$, by

$$u_1(x) = \frac{n+1-x}{\sqrt{n}}, \quad u_i(x) = \frac{1}{\sqrt{n(1 - \cos(\theta_i))}} \sin(\theta_i(x-1)).$$

Hence, we note that:

(a) By Theorem 1, L and \widehat{L} are dual and any duality function is of the form

$$D(x, y) = \sum_{i=1}^n a_i \widehat{u}_i(x) u_i(y), \quad (45)$$

for $a_1, \dots, a_n \in \mathbb{R}$.

(b) By denoting by ν the counting measure on $\Omega = \{1, \dots, n\}$, the generator \widehat{L} is self-adjoint in $L^2(\nu)$ and is, as a matrix, symmetric, i.e., $\widehat{L}^\top = \widehat{L}$. As a consequence, $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ are eigenfunctions of both \widehat{L} and \widehat{L}^\top .

(c) For all $i = 1, \dots, n$,

$$u_i(x) = \sum_{y=x}^n \widehat{u}_i(y), \quad x \in \Omega,$$

i.e., the eigenfunctions $\{u_1, \dots, u_n\}$ are related to $\{\widehat{u}_1, \dots, \widehat{u}_n\}$ as in (42).

(d) The eigenfunctions $\widehat{u}_1, \dots, \widehat{u}_n$ are normalized in $L^2(\nu)$, i.e., for all $i, j = 1, \dots, n$,

$$\langle \widehat{u}_i, \widehat{u}_j \rangle_{L^2(\nu)} = \delta_{i,j}.$$

As a consequence, by Theorem 5, for the choice $a_1 = \dots = a_n = 1$, the duality function $D(x, y)$ in (45) is the Siegmund duality function $D_S(x, y)$ in (37), namely, for all $x, y \in \Omega$,

$$\begin{aligned} & \frac{n+1-y}{n} \\ & + \sum_{i=2}^n \frac{\sin(\theta_i(y-1))}{n(1-\cos(\theta_i))} (-\sin(\theta_i) \cos(\theta_i(x-1)) + (1-\cos(\theta_i)) \sin(\theta_i(x-1))) \\ & = \mathbb{1}\{x \geq y\}. \end{aligned}$$

As a final remark, we note that, by adding the cemetery state $\Delta = \{n+1\}$ accessible at rate 1 only from the state $\{n\}$, the absorbed sub-Markov random walk associated to L becomes a proper Markov process with $\{1\}$ and $\{n+1\}$ as absorbing states. If we denote by L^{ext} the generator on the extended space $\Omega \cup \Delta$, it follows that the eigenvalues of L^{ext} remain unchanged, while the new eigenfunctions $\{u_1^{\text{ext}}, \dots, u_n^{\text{ext}}, u_{n+1}^{\text{ext}}\}$ are such that

$$u_{n+1}^{\text{ext}}(x) = 1, \quad x \in \Omega \cup \Delta,$$

and, for all $i = 1, \dots, n$,

$$u_i^{\text{ext}}(n+1) = 0, \quad u_i^{\text{ext}}(x) = u_i(x), \quad x \in \Omega.$$

Hence, the function

$$D_S^{\text{ext}}(x, y) = \sum_{i=1}^n \widehat{u}_i(x) u_i^{\text{ext}}(y), \quad x \in \Omega, \quad y \in \Omega \cup \Delta,$$

equals $\mathbb{1}\{x \geq y\}$. □

Acknowledgments. The authors thank *Institut Henri Poincaré*, where part of this work was done, for very kind hospitality. F.S. acknowledges NWO for financial support via the TOP1 grant 613.001.552. The same author is indebted to G. Carinci for fruitful discussions.

References

1. Borodin, A., Corwin, I., Gorin, V.: Stochastic six-vertex model. *Duke Math. J.* **165**, 563–624 (2016)
2. Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Dualities in population genetics: a fresh look with new dualities. *Stoch. Process. Appl.* **125**, 941–969 (2015)
3. Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Duality for stochastic models of transport. *J. Stat. Phys.* **152**, 657–697 (2013)
4. Carinci, G., Giardinà, C., Redig, F., Sasamoto, T.: A generalized asymmetric exclusion process with $U_q(\mathfrak{sl}_2)$ stochastic duality. *Probab. Theory Relat. Fields* **166**, 887–933 (2016)
5. Carinci, G., Giardinà, C., Redig, F., Sasamoto, T.: Asymmetric stochastic transport models with $\mathcal{U}_q(\mathfrak{su}(1, 1))$ symmetry. *J. Stat. Phys.* **163**, 239–279 (2016)
6. Choi, M.C.H., Patie, P.: A sufficient condition for continuous-time finite skip-free Markov chains to have real eigenvalues. In: Bélair, J., et al. (eds.) *Mathematical and Computational Approaches in Advancing Modern Science and Engineering*, pp. 529–536. Springer, Cham (2016)
7. Choi, M., Patie, P.: Skip-free Markov chains. Preprint, Research gate (2016)
8. Conrad, N.D., Weber, M., Schütte, C.: Finding dominant structures of nonreversible Markov processes. *Multiscale Model. Simul.* **14**, 1319–1340 (2016)
9. Corwin, I., Shen, H., Tsai, L.-C.: ASEP(q, j) converges to the KPZ equation. *Annales de l’Institut Henri Poincaré Probabilités et Statistiques* **54**, 995–1012 (2018)
10. De Masi, A., Presutti, E.: *Mathematical Methods for Hydrodynamic Limits. Lecture Notes in Mathematics*, vol. 1501. Springer, Heidelberg (1991)
11. Depperschmidt, A., Greven, A., Pfaffelhuber, P.: Tree-valued Fleming-Viot dynamics with mutation and selection. *Ann. Appl. Probab.* **22**, 2560–2615 (2012)
12. Diaconis, P., Fill, J.A.: Strong stationary times via a new form of duality. *Ann. Probab.* **18**, 1483–1522 (1990)
13. Etheridge, A., Freeman, N., Penington, S.: Branching Brownian motion, mean curvature flow and the motion of hybrid zones. *Electron. J. Probab.* **22**, 1–40 (2017)
14. Fill, J.A.: On hitting times and fastest strong stationary times for skip-free and more general chains. *J. Theor. Probab.* **22**, 587–600 (2009)
15. Franceschini, C., Giardinà, C.: Stochastic Duality and Orthogonal Polynomials. Preprint, [arXiv:1701.09115](https://arxiv.org/abs/1701.09115) (2016)
16. Franceschini, C., Giardinà, C., Groenevelt, W.: Self-duality of Markov processes and intertwining functions. *Math. Phys. Anal. Geom.* **21**, 29–49 (2018)
17. Giardinà, C., Kurchan, J., Redig, F.: Duality and exact correlations for a model of heat conduction. *J. Math. Phys.* **48**, 033301 (2007)
18. Giardinà, C., Kurchan, J., Redig, F., Vafayi, K.: Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* **135**, 25–55 (2009)
19. Groenevelt, W.: Orthogonal stochastic duality functions from Lie algebra representations. *J. Stat. Phys.* **174**, 97–119 (2019)

20. Horn, R.A., Johnson, C.R.: *Matrix Analysis*. Cambridge University Press, Cambridge (2012)
21. Huillet, T., Martinez, S.: Duality and intertwining for discrete Markov kernels: relations and examples. *Adv. Appl. Prob.* **43**, 437–460 (2011)
22. Keilson, J., Kester, A.: Monotone matrices and monotone Markov processes. *Stoch. Process. Appl.* **5**, 231–241 (1977)
23. Kipnis, C., Landim, C.: *Scaling Limits of Interacting Particle Systems*, vol. 320. Springer, Heidelberg (1999)
24. Kipnis, C., Marchioro, C., Presutti, E.: Heat flow in an exactly solvable model. *J. Stat. Phys.* **27**, 65–74 (1982)
25. Kolokol'tsov, V.N.: Stochastic monotonicity and duality for one-dimensional Markov processes. *Math. Notes* **89**, 652–660 (2011)
26. Kuan, J.: An algebraic construction of duality functions for the stochastic $\mathcal{U}_q(A_n^{(1)})$ vertex model and its degenerations. *Commun. Math. Phys.* **359**, 121–187 (2018)
27. Lee, R.X.: The existence and characterisation of duality of Markov processes in the Euclidean space. Ph.D. thesis, University of Warwick (2013)
28. Liggett, T.M.: *Interacting Particle Systems*. Springer, Heidelberg (2005)
29. Miclo, L.: On the Markovian similarity. In: Donati-Martin, C., Lejay, A., Rouault, A. (eds.) *Séminaire de Probabilités XLIX*, pp. 375–403. Springer, Cham (2018)
30. Möhle, M.: The concept of duality and applications to Markov processes arising in neutral population genetics models. *Bernoulli* **5**, 761–777 (1999)
31. Nikiforov, A.F., Suslov, S.K., Uvarov, V.B.: *Classical Orthogonal Polynomials of a Discrete Variable*. Springer, Heidelberg (1991)
32. Patie, P., Savov, M., Zhao, Y.: Intertwining, Excursion Theory and Krein Theory of Strings for Non-self-adjoint Markov Semigroups. Preprint, [arXiv:1706.08995](https://arxiv.org/abs/1706.08995) (2017)
33. Redig, F., Sau, F.: Factorized duality, stationary product measures and generating functions. *J. Stat. Phys.* **172**, 980–1008 (2018)
34. Schütz, G.M.: Duality relations for asymmetric exclusion processes. *J. Stat. Phys.* **86**, 1265–1287 (1997)
35. Schütz, G.M.: Fluctuations in stochastic interacting particle systems. In: Giacomin, G., Olla, S., Saada, E., Spohn, H. (eds.) *Stochastic Dynamics out of Equilibrium*. PROMS, vol. 282, pp. 67–134. Springer, Cham (2019). <https://indico.math.cnrs.fr/event/852/material/1/0.pdf>
36. Siegmund, D.: The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. *Ann. Probab.* **4**(6), 914–924 (1976)
37. Weber, M.: Eigenvalues of non-reversible Markov chains - a case study. Technical report 17-13, ZIB, Takustr. 7, 14195 Berlin (2017)