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Antiperiodic Solutions for Nonlinear Asymmetric Equations Near Resonance

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Abstract
We investigate the existence of solutions to second order scalar differential equations with asymmetric nonlinearities, subject to antiperiodic boundary conditions. Both resonance and nonresonance cases are examined, with the Landesman–Lazer conditions imposed in the resonant setting. The proofs rely on topological degree theory.

Keywords Antiperiodic solutions · Landesman–Lazer condition · Topological degree theory · Fučík spectrum

Mathematics Subject Classification 34B15

1 Introduction

We are interested in the T-antiperiodic problem associated with the scalar second order equation

$$\ddot{x} + g(t, x) = 0\,, \tag{1}$$

where $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the growth conditions

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$$
\begin{aligned}
\mu_1 \leqslant \liminf_{u \to +\infty} \frac{g(t,x)}{x} \leqslant \limsup_{x \to +\infty} \frac{g(t,x)}{x} \leqslant \mu_2 \,, \\
\nu_1 \leqslant \liminf_{x \to -\infty} \frac{g(t,x)}{x} \leqslant \limsup_{x \to -\infty} \frac{g(t,x)}{x} \leqslant \nu_2 \,,
\end{aligned}
\tag{2}
$$

uniformly in $t \in [0, T]$, for some positive constants μ_1, μ_2, ν_1, and ν_2.

The investigation of (1) under T-antiperiodic boundary condition

$$
(x(0), \dot{x}(0)) = -(x(T), \dot{x}(T))
$$

shares certain similarities with other classical boundary value problems. For instance, let us recall some existence results associated with the T-periodic boundary condition $(x(0), \dot{x}(0)) = (x(T), \dot{x}(T))$, the Neumann and the Dirichlet boundary conditions, $\dot{x}(0) = 0 = \dot{x}(T)$ and $x(0) = 0 = x(T)$, respectively.

In 1969, Lazer and Leach [26] considered (1) with T-periodic boundary conditions. In that paper, $g(t, x) = \lambda x + r(t, x)$, where r is continuous, uniformly bounded and T-periodic in t and $\lambda = \left(\frac{2\pi n}{T}\right)^2$ for some positive integer n. They established that a sufficient condition for the existence of a T-periodic solution is the following: for every non-zero η satisfying $\ddot{\eta} + \lambda\eta = 0$,

$$
\int_{\{\eta<0\}} \limsup_{x \to -\infty} r(t,x)\eta(t)dt + \int_{\{\eta>0\}} \liminf_{x \to +\infty} r(t,x)\eta(t)dt > 0 \,. \tag{3}
$$

The following year, Landesman and Lazer [25] introduced a similar condition for a Dirichlet problem associated with an elliptic operator. Since then, (3) is referred to as *Landesman–Lazer condition*. This condition is crucial for the nonlinearity to be kept sufficiently far from resonance. Their work has served as a foundation for numerous generalizations, see for example [5, 9, 12–15, 19, 32, 34].

Some years later, studying the T-periodic problem, Fučík [18] and Dancer [7, 8] introduced the so-called Fučík spectrum, defined as the set Σ of points $(\mu, \nu) \in \mathbb{R}^2$ such that the asymmetric oscillator

$$
\ddot{x} + \mu x^+ - \nu x^- = 0 \,, \tag{4}
$$

where $x^\pm = \max\{\pm x, 0\}$, has nontrivial T-periodic solutions.

The Fučík spectrum generalizes classical eigenvalue problems. When $\mu = \nu$, the equation reduces to the symmetric case and the spectrum becomes the usual set of eigenvalues. However, in many applications, systems exhibit asymmetry, such as different restoring forces when a displacement is positive or negative. This reflects nonlinear physical phenomena, like piecewise linear springs or membranes. When $(\mu, \nu) \in \Sigma$, the problem is said to be *resonant*. At resonance, standard existence results fail or become more subtle. The interaction of resonance and asymmetry makes the problem delicate and leads to the development of refined analytical tools.

In [10], it was shown that if the function g satisfies (2) and the rectangle $\mathcal{R} = [\mu_1, \mu_2] \times [\nu_1, \nu_2]$ does not intersect the Fučík spectrum Σ, then the equation (1) admits

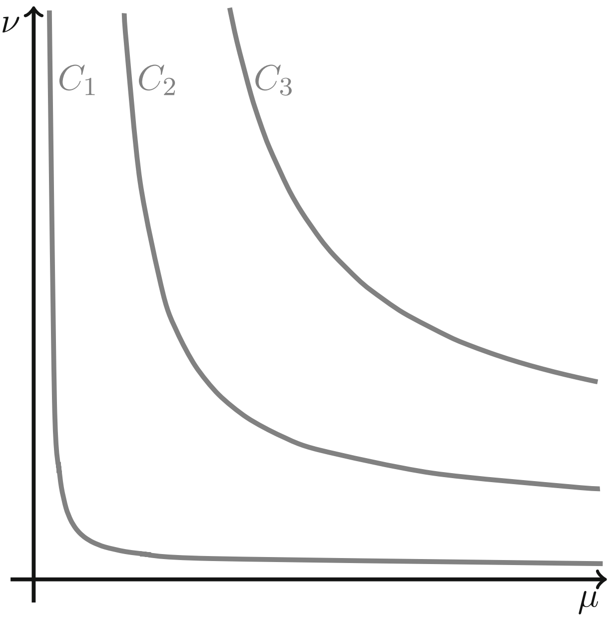

Fig. 1 The Fučík spectrum for the T-periodic problem, where $C_i = \{(\mu, \nu) \in \mathbb{R}^2 : \mu > 0,\ \nu > 0,\ \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{i},\ i = 1, 2, \cdots\}$. It coincides with the Fučík spectrum for the Neumann problem $\dot{x}(0) = 0 = \dot{x}(T/2)$

at least one T-periodic solution. This represents a typical *nonresonance* situation. See also [9, 11, 24] for related results. When the set $\mathcal{R} \cap \Sigma$ consists of only one or both the vertices (μ_1, ν_1) and (μ_2, ν_2) of the rectangle, in order to avoid *resonance*, additional hypotheses are required. For instance, in [12, 13, 15, 32], the *double resonance* case was addressed by imposing Landesman–Lazer-type conditions on both sides, ensuring the existence of a T-periodic solution. Concerning the Neumann and the Dirichlet problems associated with (1), we refer to [19, 29, 32, 34] (Fig. 2).

Most classical studies assume Dirichlet, Neumann, or periodic boundary conditions. Antiperiodic conditions model systems with a “phase shift of half a period”, common in wave propagation or oscillations with inversion symmetry. The geometry of the Fučík spectrum changes with the boundary conditions, so antiperiodicity opens new and less-explored analytical territory. Understanding these problems not only advances mathematical theory but also deepens our insight into asymmetric and resonant systems.

As mentioned above, if compared with the literature available for periodic, Neumann and Dirichlet problems, in the study of the antiperiodic problems the number of references is considerably smaller. For instance, in [6], the existence of antiperiodic solutions for Liénard-type and Duffing-type differential equations with the p-Laplacian operator was established using degree theory. In [20], a resonant second order problem of the form $\ddot{x} = f(t, x, \dot{x})$ satisfying antiperiodic and periodic boundary conditions was analyzed. In [38], the existence of antiperiodic solutions for

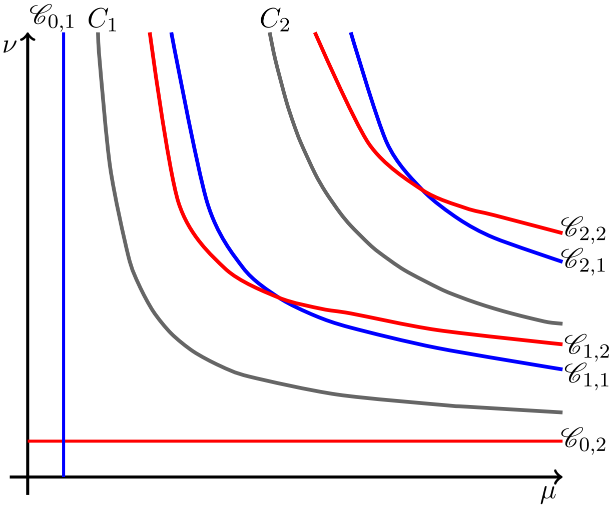

Fig. 2 The Fučík spectrum for the Dirichlet problem. The curves $\mathscr{C}_{k,j}$ are to be defined in Section 2, while the curves C_i are the same as those in Figure 1

a second order ordinary differential equation was explored, studying the interaction of the nonlinearity with the Fučík spectrum. Furthermore, in [21, 22], existence results were proved also for a wider class of boundary value problems where the differential operator could even be non-selfadjoint (see also [30, 31]). In [33], antiperiodic oscillations are obtained for a forced Duffing equation with negative linear stiffness, demonstrating how they develop multiple peaks under increasing forcing strength. For further related studies on second order differential equations we refer to [1, 4, 35, 37]. First order equations have been considered in [16, 17, 28]. For an insight into other types of equations see, e.g., [2, 3, 23, 27, 36].

To the best of our knowledge, the antiperiodic problem associated with asymmetric scalar second order equations under resonance with respect to the Fučík spectrum has not yet been explored. In particular, Landesman–Lazer-type conditions have not been employed in such kind of problems. In the present paper, it is our aim to fill such a gap. The main difficulty in the antiperiodic setting is related to the different structure of the nontrivial solutions of equation (4) when the parameters μ and ν vary. Although the statements of our results in Section 3 look simple, they hide a variety of situations that need to be considered.

The paper is organized as follows. In Sect. 2, we analyze the Fučík spectrum corresponding to an antiperiodic problem and present some key properties. Then, in Sect. 3, we state and prove our main results for the antiperiodic problem under both nonresonance and double resonance situations. The last section is devoted to some remarks, an example of application, and an open problem.

2 Preliminaries

In this section, we discuss about the Fučík spectrum corresponding to the antiperiodic problem, and present some preliminary lemmas.

2.1 The Fučík Spectrum

Consider the asymmetric oscillator under antiperiodic boundary conditions

$$\begin{cases} \ddot{x} + \mu x^{+} - \nu x^{-} = 0\,, \\ x(0) + x(T) = 0\,, \quad \dot{x}(0) + \dot{x}(T) = 0\,. \end{cases} \tag{5}$$

If μ and ν are positive, the solutions of the differential equation in (5) are all periodic, with period

$$\mathcal{T}_{\mu,\nu} = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\,. \tag{6}$$

One particular solution is given by

$$\varphi_{\mu,\nu}(t) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\, t\right) & \text{if } t \in \left[0, \frac{\pi}{\sqrt{\mu}}\right], \\ -\frac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\left(t - \frac{\pi}{\sqrt{\mu}}\right)\right) & \text{if } t \in \left[\frac{\pi}{\sqrt{\mu}}, \mathcal{T}_{\mu,\nu}\right], \end{cases} \tag{7}$$

extended by $\mathcal{T}_{\mu,\nu}$-periodicity to the whole $\mathbb{R}$. All the other solutions are of the form $x(t) = \rho\varphi_{\mu,\nu}(t-\theta)$ with $\rho \geqslant 0$ and $\theta \in \mathbb{R}$.

We define

$$\Sigma = \{(\mu,\nu) \in \mathbb{R}^2 : \text{ (5) has a nontrivial solution}\}\,,$$

that is, the Fučík spectrum of the operator $-\ddot{x}$ under the antiperiodic boundary conditions. Easy computations show that

$$\Sigma = \bigcup_{k\in\mathbb{N}} \mathscr{C}_k\,,$$

where the set $\mathscr{C}_0$ consists of the two lines

$$\mathscr{C}_{0,1} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu = \left(\frac{\pi}{T}\right)^2\right\},$$

$$\mathscr{C}_{0,2} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \nu = \left(\frac{\pi}{T}\right)^2\right\},$$

while, for $k \geqslant 1$, $\mathscr{C}_k = \mathscr{C}_{k,1} \cup \mathscr{C}_{k,2}$, with

$$\mathscr{C}_{k,1} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ (k+1)\frac{\pi}{\sqrt{\mu}} + k\frac{\pi}{\sqrt{\nu}} = T\right\},$$

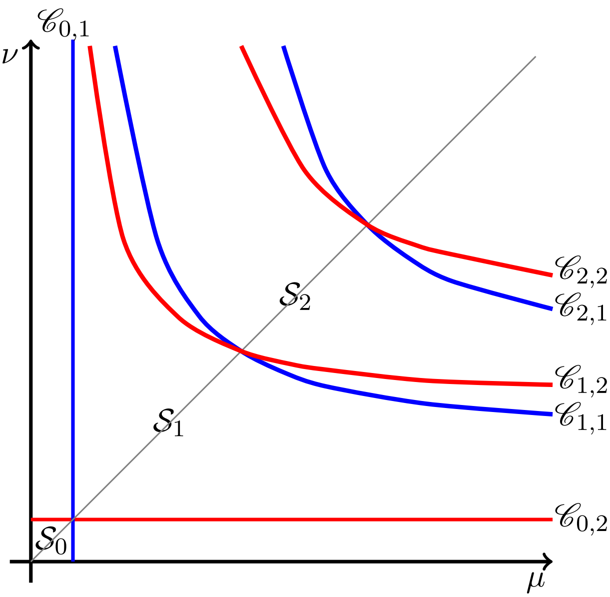

Fig. 3 The Fučík spectrum for the antiperiodic problem and the sets $\mathcal{S}_k$

and

$$\mathscr{C}_{k,2} = \left\{(\mu, \nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ k\frac{\pi}{\sqrt{\mu}} + (k+1)\frac{\pi}{\sqrt{\nu}} = T\right\}.$$

Notice that the curves $\mathscr{C}_{k,1}$ and $\mathscr{C}_{k-1,2}$, for $k \geqslant 1$, share the same horizontal asymptote $\nu = (k\pi/T)^2$.

It can be seen that Σ is a subset of the Fučík spectrum of the operator $-\ddot{x}$ under the corresponding Dirichlet boundary condition $x(0) = 0 = x(T)$.

We use the notations

$$m_{\mu,\nu} = \min\left\{\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\nu}}\right\}, \qquad M_{\mu,\nu} = \max\left\{\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\nu}}\right\},$$

and define the set $\mathcal{S} \subseteq \mathbb{R}^2$ as follows:

$$\mathcal{S} = \bigcup_{k\in\mathbb{N}} \mathcal{S}_k\,, \tag{8}$$

where, as depicted in Fig. 3,

$$\mathcal{S}_k = \left\{(\mu, \nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ (k-1)\mathcal{T}_{\mu,\nu} + M_{\mu,\nu} < T < k\mathcal{T}_{\mu,\nu} + m_{\mu,\nu}\right\}.$$

We now examine the nontrivial solutions of problem (5) in three specific cases.

(i) If $(\mu, \nu) \in \mathscr{C}_{k,1}$ with $\mu \neq \nu$, then the nontrivial solutions of problem (5) are of the type $x(t) = \rho\varphi_{\mu,\nu}(t)$ with $\rho > 0$. In particular, $x(0) = 0 = x(T)$ and $\dot{x}(0) > 0$.

(ii) If $(\mu, \nu) \in \mathscr{C}_{k,2}$ with $\mu \neq \nu$, then the nontrivial solutions of problem (5) can be written as $x(t) = \rho\varphi_{\mu,\nu}(t + \frac{\pi}{\sqrt{\mu}})$ with $\rho > 0$. In particular, $x(0) = 0 = x(T)$ and $\dot{x}(0) < 0$.

(iii) If $\mathscr{C}_{k,1} \cap \mathscr{C}_{k,2} = \{(\mu, \nu)\}$, i.e., $\mu = \nu = ((2k+1)\pi/T)^2$, it follows that the equation in (5) becomes linear and the nontrivial solutions are given by

$$x(t) = \rho \sin(\sqrt{\mu}(t - \theta)), \quad \text{for any } \rho > 0 \text{ and } \theta \in \mathbb{R}.$$

2.2 Auxiliary Results

In this section, we consider the problem

$$\begin{cases} \ddot{v} + \widehat{\mu}(t)v^+ - \widehat{\nu}(t)v^- = 0, \\ v(0) + v(T) = 0, \quad \dot{v}(0) + \dot{v}(T) = 0, \end{cases} \tag{9}$$

with the following hypothesis.

(H) The functions $\widehat{\mu}, \widehat{\nu} \in L^2(\mathbb{R})$ satisfy

$$\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2, \quad \nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2, \tag{10}$$

for almost every $t \in \mathbb{R}$, all constants being positive.

Let us first recall the definition of "rotation number" of a planar curve around the origin. Assume that $s_1 < s_2$ and let $\phi : [s_1, s_2] \to \mathbb{R}^2$ be a continuous curve such that $\phi(t) \neq (0, 0)$ for every $t \in [s_1, s_2]$. Writing $\phi(t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t))$, where $\rho : \mathbb{R} \to]0, +\infty[$ and $\theta : \mathbb{R} \to \mathbb{R}$ are continuous, we define

$$\mathrm{Rot}(\phi; [s_1, s_2]) = -\frac{\theta(s_2) - \theta(s_1)}{2\pi}.$$

In the following, when dealing with a solution x of (1), by a slight abuse of notation we will write $\mathrm{Rot}(x; [s_1, s_2])$ instead of $\mathrm{Rot}((x, \dot{x}); [s_1, s_2])$.

We first need the following result.

Proposition 2.1 *Assume* **(H)** *and let v be a nontrivial solution of the differential equation in (9).*

1. *If* $\mathrm{Rot}(v; [a, b]) = N$ *for some $a < b$ and $N \in \mathbb{N}$, then*

$$N\mathcal{T}_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1}.$$

2. *If instead* $\mathrm{Rot}(v; [a, b]) = N + \frac{1}{2}$, *then*

$$N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1}.$$

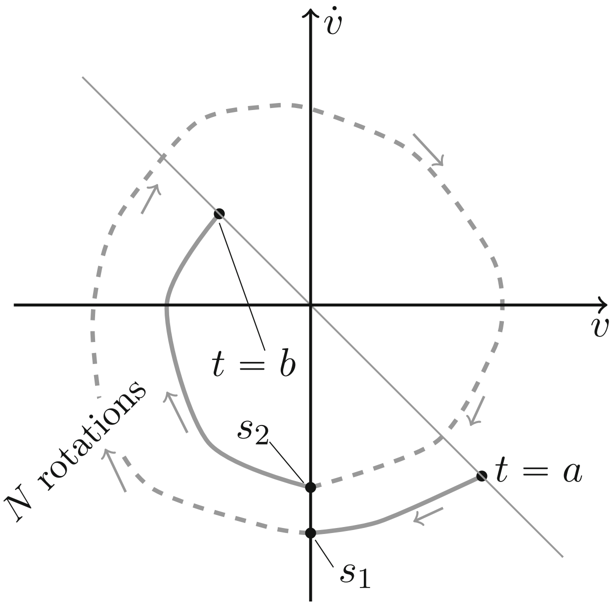

Fig. 4 The rotating solution in the phase plane

Proof The first part of the statement is rather standard (see, e.g., [15]), hence we omit the proof, for briefness. Let us prove the second part. Introducing the polar coordinates

$$(v, \dot{v}) = (\rho \cos \theta, \rho \sin \theta)\,,$$

we see that

$$-\dot{\theta}(t) = \begin{cases} \widehat{\mu}(t) \cos^2 \theta(t) + \sin^2 \theta(t)\,, & \text{if } v(t) \geqslant 0\,, \\ \widehat{\nu}(t) \cos^2 \theta(t) + \sin^2 \theta(t)\,, & \text{if } v(t) \leqslant 0\,. \end{cases} \tag{11}$$

Notice that $-\dot{\theta}(t) > 0$ for almost every t. For definiteness, we assume $\theta(a) \in [-\frac{\pi}{2}, \frac{\pi}{2}[$; the case $\theta(a) \in [\frac{\pi}{2}, \frac{3\pi}{2}[$ can be treated similarly.

Set $\theta_0 = \theta(a)$. By assumption we have $\theta(b) = \theta_0 - (2N+1)\pi$. Then we select $a \leqslant s_1 \leqslant s_2 < b$ such that $\theta(s_1) = -\pi/2$ and $\theta(s_2) = -\pi/2 - 2N\pi$, see Fig. 4. By the first part of the statement, we deduce that

$$N\mathcal{T}_{\mu_2,\nu_2} \leqslant s_2 - s_1 \leqslant N\mathcal{T}_{\mu_1,\nu_1}\,. \tag{12}$$

From (10) and (11), we get

$$\sin^2 \theta(t) + \min\{\mu_1, \nu_1\} \cos^2 \theta(t) \leqslant -\dot{\theta}(t) \leqslant \sin^2 \theta(t) + \max\{\mu_2, \nu_2\} \cos^2 \theta(t)\,,$$

which implies that

$$\frac{-\dot{\theta}(t)}{\sin^2 \theta(t) + \max\{\mu_2, \nu_2\} \cos^2 \theta(t)} \leqslant 1 \leqslant \frac{-\dot{\theta}(t)}{\sin^2 \theta(t) + \min\{\mu_1, \nu_1\} \cos^2 \theta(t)}\,.$$

Upon integration over $[a, s_1]$, we have

$$\int_{-\pi/2}^{\theta_0} \frac{d\theta}{\sin^2\theta + \max\{\mu_2, \nu_2\}\cos^2\theta} \leqslant s_1 - a \leqslant \int_{-\pi/2}^{\theta_0} \frac{d\theta}{\sin^2\theta + \min\{\mu_1, \nu_1\}\cos^2\theta}. \tag{13}$$

Similarly, integrating over $[s_2, b]$,

$$\int_{\theta_0-(2N+1)\pi}^{-\pi/2-2N\pi} \frac{d\theta}{\sin^2\theta + \max\{\mu_2, \nu_2\}\cos^2\theta} \leqslant b - s_2 \leqslant \int_{\theta_0-(2N+1)\pi}^{-\pi/2-2N\pi} \frac{d\theta}{\sin^2\theta + \min\{\mu_1, \nu_1\}\cos^2\theta},$$

which is equivalent to write (since the integrand is π-periodic)

$$\int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin^2\theta + \max\{\mu_2, \nu_2\}\cos^2\theta} \leqslant b - s_2 \leqslant \int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin^2\theta + \min\{\mu_1, \nu_1\}\cos^2\theta}. \tag{14}$$

Summing in (13) and (14), we get

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2\theta + \max\{\mu_2, \nu_2\}\cos^2\theta} \leqslant (s_1 - a) + (b - s_2) \leqslant \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2\theta + \min\{\mu_1, \nu_1\}\cos^2\theta}, \tag{15}$$

and so

$$m_{\mu_2,\nu_2} = \frac{\pi}{\max\{\sqrt{\mu_2}, \sqrt{\nu_2}\}} \leqslant (s_1 - a) + (b - s_2) \leqslant \frac{\pi}{\min\{\sqrt{\mu_1}, \sqrt{\nu_1}\}} = M_{\mu_1,\nu_1}. \tag{16}$$

Adding the estimates in (12) and (16), we have

$$N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1}, \tag{17}$$

which completes the proof. □

We recall the notation $\mathcal{R} = [\mu_1, \mu_2] \times [\nu_1, \nu_2]$.

Lemma 2.2 *Let assumption* **(H)** *hold. If $\mathcal{R} \subseteq \mathcal{S}$, then problem (9) only has the zero solution.*

Proof By contradiction, assume that there is a nontrivial solution $v(t)$ for problem (9). Then $(v(t), \dot{v}(t)) \neq (0,0)$ for every $t \in [0,T]$, and there is an integer $K \geqslant 0$ such that $\mathrm{Rot}(v;[0,T]) = K + \frac{1}{2}$.

Recalling the definition of $\mathcal{S}$ in (8), we discuss two different cases.

Case 1: $\mathcal{R} \subseteq \mathcal{S}_0$.

From Proposition 2.1, we get

$$T < m_{\mu_2,\nu_2} \leqslant m_{\mu_2,\nu_2} + K\mathcal{T}_{\mu_2,\nu_2} \leqslant T\,,$$

which is impossible.

Case 2: $\mathcal{R} \subseteq \mathcal{S}_N$, for $N \geqslant 1$.

If $K \leqslant N-1$, then by Proposition 2.1, we have

$$T \leqslant K\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} \leqslant (N-1)\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} < T\,,$$

which is impossible. Similarly, if $K \geqslant N$, then we have

$$T \geqslant K\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \geqslant N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} > T\,,$$

which is again impossible.

This completes the proof of the lemma. □

We now consider the case when the rectangle $\mathcal{R}$ is contained in the closure of the set $\mathcal{S}$ and touches two curves of the Fučík spectrum.

Lemma 2.3 *Let μ_1, μ_2, ν_1, and ν_2 satisfy*

$$N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T = (N+1)\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2}\,, \tag{18}$$

for a certain integer $N \geqslant 0$. Suppose that problem (9) admits a nontrivial solution v, with

$$\begin{aligned} &\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2\,, \quad \text{a.e. on } I_+ = \{t \in [0,T] : v(t) > 0\}\,, \\ &\nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2\,, \quad \text{a.e. on } I_- = \{t \in [0,T] : v(t) < 0\}\,. \end{aligned} \tag{19}$$

Then, either

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

or

$$\widehat{\mu}(t) = \mu_2 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_2 \text{ a.e. on } I_-\,.$$

Proof Since v is a nontrivial solution, it satisfies $v^2(t) + \dot{v}^2(t) \neq 0$ for all $t \in [0,T]$, and $\mathrm{Rot}(v;[0,T]) = K + \frac{1}{2}$, for some integer $K \geqslant 0$. We claim that either $K = N$ or $K = N+1$. Indeed, if $K < N$, then by using Proposition 2.1 and (18), we have

$$T \leqslant K\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} < N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T\,,$$

which is impossible. Similarly, if $K > N+1$, we have

$$\begin{aligned} T &\geqslant K\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \\ &> (N+1)\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} = T\,, \end{aligned}$$

which is again impossible.

Let us analyze the alternative $K = N$, as the other situation follows similarly. We need to consider the following three cases.
Case 1. $\mu_1 < \nu_1$, i.e.,

$$T = N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\mu_1}} > N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\nu_1}}\,. \tag{20}$$

Claim $v(0) = 0$ and $\dot{v}(0) > 0$.

Proof of the Claim. Assume by contradiction that $v(0) \neq 0$ or $v(0) = 0$ with $\dot{v}(0) < 0$.
We first discuss the case when $v(0) > 0$ or $v(0) = 0$ with $\dot{v}(0) < 0$. In this case, there exist $0 \leqslant t_0 < t_1 < \cdots < t_{2N} < T$ such that

$$v(t_i) = 0, \quad \dot{v}(t_{2i}) < 0\,, \quad \text{and} \quad \dot{v}(t_{2i+1}) > 0\,. \tag{21}$$

Introducing polar coordinates, recalling that $-\dot{\theta}(t) > 0$ for every t, we deduce that $\mathrm{Rot}(v; [t_i, t_{i+1}]) = 1/2$ for every i. We set $\theta(0) = \theta_0 \in [-\pi/2, \pi/2[\,$. □

From Proposition 2.1, we have

$$t_{2N} - t_0 \leqslant N\mathcal{T}_{\mu_1,\nu_1}.$$

Combining this fact with the equality in (20) and recalling (11), we get

$$\begin{aligned} &\frac{\pi}{\sqrt{\mu_1}} \leqslant (t_0 - 0) + (T - t_{2N}) \\ &= \int_0^{t_0} \frac{-\dot{\theta}(t)dt}{\widehat{\mu}(t)\cos^2\theta(t) + \sin^2\theta(t)} + \int_{t_{2N}}^{T} \frac{-\dot{\theta}(t)dt}{\widehat{\nu}(t)\cos^2\theta(t) + \sin^2\theta(t)}\,, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{\theta_0-\pi}^{\theta_0} \frac{d\theta}{\mu_1\cos^2\theta + \sin^2\theta} \\ &\leqslant \int_{-\frac{\pi}{2}}^{\theta_0} \frac{d\theta}{\widetilde{\mu}(\theta)\cos^2\theta + \sin^2\theta} + \int_{\theta_0-(2N+1)\pi}^{-\frac{\pi}{2}-2N\pi} \frac{d\theta}{\widetilde{\nu}(\theta)\cos^2\theta + \sin^2\theta} \\ &= \int_{-\frac{\pi}{2}}^{\theta_0} \frac{d\theta}{\widetilde{\mu}(\theta)\cos^2\theta + \sin^2\theta} + \int_{\theta_0-\pi}^{-\frac{\pi}{2}} \frac{d\theta}{\widetilde{\nu}(\theta)\cos^2\theta + \sin^2\theta}\,, \end{aligned}$$

where $\widetilde{\mu}(\theta(t)) = \widehat{\mu}(t)$, and $\widetilde{\nu}(\theta(t)) = \widehat{\nu}(t)$. After splitting the left hand side integral, we thus obtain

$$\int_{\theta_0-\pi}^{-\frac{\pi}{2}} \underbrace{\left(\frac{1}{\mu_1\cos^2\theta+\sin^2\theta}-\frac{1}{\widetilde{\nu}(\theta)\cos^2\theta+\sin^2\theta}\right)}_{>0,\ \text{since}\ \widetilde{\nu}\geqslant\nu_1>\mu_1.} d\theta$$
$$\leqslant \int_{-\frac{\pi}{2}}^{\theta_0} \underbrace{\left(\frac{1}{\widetilde{\mu}(\theta)\cos^2\theta+\sin^2\theta}-\frac{1}{\mu_1\cos^2\theta+\sin^2\theta}\right)}_{\leqslant 0,\ \text{since}\ \widetilde{\mu}\geqslant\mu_1.} d\theta\,. \tag{22}$$

The left hand side integral is positive due to the fact that $\theta_0 < \frac{\pi}{2}$, and we get a contradiction.

The case $v(0) < 0$ can be treated similarly, thus completing the proof of the Claim.

Since $v(0) = 0$ and $\dot{v}(0) > 0$, we can select $0 = t_1 < \cdots < t_{2N+2} = T$ satisfying (21).

We now introduce the following modified polar coordinates

$$v = \begin{cases} \dfrac{1}{\sqrt{\mu_1}}\, r\cos\theta & \text{if } v \geqslant 0\,, \\ \dfrac{1}{\sqrt{\nu_1}}\, r\cos\theta & \text{if } v \leqslant 0\,, \end{cases} \qquad \dot{v} = r\sin\theta\,,$$

and observe that

$$\dot{\theta} = \begin{cases} \sqrt{\mu_1}\,\dfrac{\ddot{v}v-\dot{v}^2}{\mu_1 v^2+\dot{v}^2} & \text{if } v > 0\,, \\ \sqrt{\nu_1}\,\dfrac{\ddot{v}v-\dot{v}^2}{\nu_1 v^2+\dot{v}^2} & \text{if } v < 0\,. \end{cases}$$

Then, for $i = 1, \ldots, N+1$, we have

$$\begin{aligned} \frac{\pi}{\sqrt{\mu_1}} &= \int_{t_{2i-1}}^{t_{2i}} \frac{\widehat{\mu}(t)v^2+\dot{v}^2}{\mu_1 v^2+\dot{v}^2}dt \\ &= \int_{t_{2i-1}}^{t_{2i}} \frac{(\widehat{\mu}(t)-\mu_1)v^2}{\mu_1 v^2+\dot{v}^2}dt + \int_{t_{2i-1}}^{t_{2i}} \frac{\mu_1 v^2+\dot{v}^2}{\mu_1 v^2+\dot{v}^2}dt \\ &= \int_{t_{2i-1}}^{t_{2i}} \frac{(\widehat{\mu}(t)-\mu_1)v^2}{\mu_1 v^2+\dot{v}^2}dt + (t_{2i}-t_{2i-1})\,. \end{aligned} \tag{23}$$

Similarly, for $j = 2, \ldots, N+1$, we have

$$\begin{aligned}\frac{\pi}{\sqrt{\nu_1}} &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{\widehat{\nu}(t)v^2 + \dot{v}^2}{\nu_1 v^2 + \dot{v}^2}dt \\ &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2}dt + \int_{t_{2j-2}}^{t_{2j-1}} \frac{\nu_1 v^2 + \dot{v}^2}{\nu_1 v^2 + \dot{v}^2}dt \\ &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2}dt + (t_{2j-1} - t_{2j-2})\,. \end{aligned} \tag{24}$$

Summing (23) for $i = 1, \dots, N+1$ and (24) for $j = 2, \dots, N+1$, by (20) we obtain

$$T = \int_{I_+} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2}dt + \int_{I_-} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2}dt + T\,,$$

which implies that

$$\int_{I_+} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2}dt + \int_{I_-} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2}dt = 0\,.$$

Recalling (19), we get

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

thus completing the proof in Case 1.
Case 2. $\mu_1 > \nu_1$, i.e.,

$$T = N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\nu_1}} > N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\mu_1}}\,. \tag{25}$$

A similar computation as in Case 1 shows that $v(0) = 0$ and $\dot{v}(0) < 0$. Then one can find $0 = t_0 < \cdots < t_{2N+1} = T$ satisfying (21) and obtain estimates as in (23) and (24), so that the conclusion follows similarly.
Case 3. $\mu_1 = \nu_1$, i.e.,

$$T = \frac{(2N+1)\pi}{\sqrt{\mu_1}}\,. \tag{26}$$

In this case,

$$\begin{aligned}\frac{(2N+1)\pi}{\sqrt{\mu_1}} &= \int_0^T \frac{\widehat{\mu}(t)(v^+)^2 + \widehat{\nu}(t)(v^-)^2 + \dot{v}^2}{\mu_1 v^2 + \dot{v}^2}dt \\ &= \int_0^T \frac{(\widehat{\mu}(t) - \mu_1)(v^+)^2 + (\widehat{\nu}(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot{v}^2}dt + \int_0^T \frac{\mu_1 v^2 + \dot{v}^2}{\mu_1 v^2 + \dot{v}^2}dt \\ &= \int_0^T \frac{(\widehat{\mu}(t) - \mu_1)(v^+)^2 + (\widehat{\nu}(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot{v}^2}dt + T\,,\end{aligned}$$

which implies that

$$\int_0^T \frac{(\widehat{\mu}(t) - \mu_1)(v^+)^2 + (\widehat{\nu}(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot{v}^2} dt = 0\,,$$

leading to the same conclusion as in the previous cases. □

3 Main Results

We will need the following assumption.

(G) The function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ has the form

$$g(t, x) = \gamma_+(t, x)x^+ - \gamma_-(t, x)x^- + h(t, x)\,,$$

where γ_+, γ_- and h are continuous functions such that

$$\mu_1 \leqslant \gamma_+(t, x) \leqslant \mu_2\,, \quad \nu_1 \leqslant \gamma_-(t, x) \leqslant \nu_2\,,$$

for every $t \in [0, T]$ and $x \in \mathbb{R}$, the above constants $\mu_1, \mu_2, \nu_1, \nu_2$ all being positive, and h is uniformly bounded.

As above, we denote the rectangle $[\mu_1, \mu_2] \times [\nu_1, \nu_2]$ by $\mathcal{R}$.

Theorem 3.1 (Nonresonance) *If assumption* **(G)** *holds with* $\mathcal{R} \subseteq \mathcal{S}$, *then problem (1) has a solution.*

Proof Let $\mathcal{R} \subseteq \mathcal{S}_k$, for some $k \in \mathbb{N}$, and set

$$\overline{\mu} = \frac{\mu_1 + \mu_2}{2}\,, \quad \overline{\nu} = \frac{\nu_1 + \nu_2}{2}\,.$$

Consider the following family of problems

$$\begin{cases} \ddot{x} + (1-\sigma)[\overline{\mu}x^+ - \overline{\nu}x^-] + \sigma g(t, x) = 0\,, \\ x(0) + x(T) = 0\,, \quad \dot{x}(0) + \dot{x}(T) = 0\,, \end{cases} \tag{27}$$

with $\sigma \in [0, 1]$. We aim to show that there is a $r > 0$ such that, for every solution x of (27), one has $\|x\|_\infty \leq r$.

Assume by contradiction that for every positive integer n there exist $\sigma_n \in [0, 1]$ and a solution x_n of (27), with $\sigma = \sigma_n$, such that $\|x_n\|_\infty > n$. Passing to a subsequence we can assume that $(\sigma_n)_n$ converges to some $\overline{\sigma} \in [0, 1]$. Set $v_n = \frac{x_n}{||x_n||_\infty}$. Then,

$$\begin{cases} \ddot{v}_n + \widehat{\mu}_n(t)v_n^+ - \widehat{\nu}_n(t)v_n^- + \sigma_n \frac{h(t, ||x_n||_\infty v_n)}{||x_n||_\infty} = 0\,, \\ v_n(0) + v_n(T) = 0\,, \quad \dot{v}_n(0) + \dot{v}_n(T) = 0\,, \end{cases} \tag{28}$$

where

$$\widehat{\mu}_n(t) = (1 - \sigma_n)\overline{\mu} + \sigma_n \gamma_+(t, ||x_n||_\infty v_n(t)),$$
$$\widehat{\nu}_n(t) = (1 - \sigma_n)\overline{\nu} + \sigma_n \gamma_-(t, ||x_n||_\infty v_n(t)).$$

Notice that $\mu_1 \leqslant \widehat{\mu}_n(t) \leqslant \mu_2$ and $\nu_1 \leqslant \widehat{\nu}_n(t) \leqslant \nu_2$.

From the differential equation in (28) and the properties of $\widehat{\mu}_n$, $\widehat{\nu}_n$ and h, the sequence $(v_n)_n$ is bounded in $H^2(0, T)$, therefore there exists a function v such that, up to a subsequence, $v_n \to v$ in $C^1([0, T])$ and weakly in $H^2(0, T)$. In particular $||v||_\infty = 1$. Since the sequences $(\widehat{\mu}_n)_n$ and $(\widehat{\nu}_n)_n$ are bounded, we can suppose that, up to a subsequence, they converge weakly in $L^2(0, T)$ to some functions $\widehat{\mu}$ and $\widehat{\nu}$, respectively, with $\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2$ and $\nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2$, almost everywhere on $[0, T]$. Passing to the weak limit in (28), by a standard argument (see, e.g., the proof of [10, Theorem 3.1]), the function v solves

$$\begin{cases} \ddot{v} + \widehat{\mu}(t)v^+ - \widehat{\nu}(t)v^- = 0, \\ v(0) + v(T) = 0, \quad \dot{v}(0) + \dot{v}(T) = 0, \end{cases} \tag{29}$$

for almost every $t \in [0, T]$, which is a contradiction with Lemma 2.2.

To conclude the proof of the theorem, recalling the notation (6), we define $\zeta = \mathcal{T}_{\overline{\mu},\overline{\nu}}$ and consider the curve

$$\mathscr{C} = \left\{ (\mu, \nu) : \mu > 0, \nu > 0, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \zeta \right\},$$

which is contained in $\mathcal{S}_k$ for some k. This curve connects the point $(\overline{\mu}, \overline{\nu})$ with $((\frac{2\pi}{\zeta})^2, (\frac{2\pi}{\zeta})^2)$. We parametrize this part of the curve by a continuous map $\xi : [0, 1] \to \mathbb{R}^2$ as

$$\xi(\sigma) = (\overline{\mu}(\sigma), \overline{\nu}(\sigma)),$$

with $\xi(0) = (\overline{\mu}, \overline{\nu})$, $\xi(1) = ((\frac{2\pi}{\zeta})^2, (\frac{2\pi}{\zeta})^2)$.

By Lemma 2.2, for every $\sigma \in [0, 1]$, the problem

$$\begin{cases} \ddot{v} + \overline{\mu}(\sigma)v^+ - \overline{\nu}(\sigma)v^- = 0, \\ v(0) + v(T) = 0, \quad \dot{v}(0) + \dot{v}(T) = 0 \end{cases} \tag{30}$$

only has the zero solution.

Since the differential operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq L^2(0, T) \to L^2(0, T)$ defined by $\mathcal{L}x = -\ddot{x}$ on the domain

$$\mathcal{D}(\mathcal{L}) = \{x \in H^2(0, T) : x(0) + x(T) = 0, \dot{x}(0) + \dot{x}(T) = 0\}$$

is selfadjoint and has a compact resolvent, the proof of the theorem can be now completed by a standard application of the Leray–Schauder topological degree theory. □

Remark 3.2 Notice that the growth conditions in (2) imply assumption **(G)** after slightly changing the involved constants. Hence, being $\mathcal{S}$ an open set, in the assumption of Theorem 3.1, we can replace assumption **(G)** with (2).

Next, we examine the case when $\mathring{\mathcal{R}} \subseteq \mathcal{S}_N$ for some $N \geqslant 1$, and $\mathcal{R} \cap \Sigma = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$. This situation arises when the resonance condition (18) is satisfied.

Theorem 3.3 (Double Resonance) *Assume* **(G)** *and the existence of a positive integer N such that (18) holds. If for every non-zero T-antiperiodic solution ψ of $\ddot{\psi} + \mu_1\psi^+ - \nu_1\psi^- = 0$ one has*

$$\int_{\{\psi<0\}} \limsup_{x\to-\infty} \big(g(t,x) - \nu_1 x\big)\psi(t)\,dt + \int_{\{\psi>0\}} \liminf_{x\to+\infty} \big(g(t,x) - \mu_1 x\big)\psi(t)\,dt > 0\,, \tag{31}$$

and for every non-zero T-antiperiodic solution χ of $\ddot{\chi} + \mu_2\chi^+ - \nu_2\chi^- = 0$ one has

$$\int_{\{\chi<0\}} \limsup_{x\to-\infty} \big(\nu_2 x - g(t,x)\big)\chi(t)\,dt + \int_{\{\chi>0\}} \liminf_{x\to+\infty} \big(\mu_2 x - g(t,x)\big)\chi(t)\,dt > 0\,, \tag{32}$$

then problem (1) has a solution.

In the above, we have used the standard notation

$$\{\psi < 0\} = \{t \in [0, T] \, : \, \psi(t) < 0\}\,,$$

and similarly for $\{\psi > 0\}$.

Proof Let us set

$$\overline{\mu} = \frac{\mu_1 + \mu_2}{2}\,, \quad \overline{\nu} = \frac{\nu_1 + \nu_2}{2}\,,$$

and consider the family of problems (27), with $\sigma \in [0, 1]$.

Claim There is a $r > 0$ such that, for every solution x of (27) one has $\|x\|_\infty \le r$.

In order to prove the Claim, assume by contradiction that for every positive integer n there exist $\sigma_n \in [0, 1]$ and a solution x_n of (27), with $\sigma = \sigma_n$, such that $\|x_n\|_\infty > n$. Notice that $\sigma_n \neq 0$.

Passing to a subsequence we can assume that $(\sigma_n)_n$ converges to some $\overline{\sigma} \in [0, 1]$. Set $v_n = \frac{x_n}{||x_n||_\infty}$. Then, v_n solves (28), and arguing as in the proof of Theorem 3.1, the sequence $(v_n)_n$ converges, up to a subsequence, to a function v in $C^1([0, T])$ and weakly in $H^2(0, T)$. This function is such that $||v||_\infty = 1$, and it solves (29) for almost every $t \in [0, T]$.

We apply Lemma 2.3 and consider, for definiteness, the first alternative in (19), i.e.,

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

the second one being treated similarly. So, v is a solution of

$$\begin{cases} \ddot{v} + \mu_1 v^+ - \nu_1 v^- = 0\,, \\ v(0) + v(T) = 0\,, \quad \dot{v}(0) + \dot{v}(T) = 0\,. \end{cases} \tag{33}$$

Since we are assuming that (18) holds, from the first equality we deduce that $\mathrm{Rot}(v; [0, T]) = N + \frac{1}{2}$. This is also true for $(v_n(t), \dot{v}_n(t))$, if n is large enough, and so also for $(x_n(t), \dot{x}_n(t))$.

Let us write $(x_n, \dot{x}_n)$ in the following modified polar coordinates:

$$x_n = \begin{cases} \dfrac{1}{\sqrt{\mu_1}}\, r_n \cos\theta_n\,, & \text{if } x_n \geqslant 0\,, \\ \dfrac{1}{\sqrt{\nu_1}}\, r_n \cos\theta_n\,, & \text{if } x_n \leqslant 0\,, \end{cases} \qquad \dot{x}_n = r_n \sin\theta_n\,.$$

We compute the derivatives

$$\dot{\theta}_n = \begin{cases} \sqrt{\mu_1}\, \dfrac{\ddot{x}_n x_n - \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} & \text{if } x_n > 0\,, \\ \sqrt{\nu_1}\, \dfrac{\ddot{x}_n x_n - \dot{x}_n^2}{\nu_1 x_n^2 + \dot{x}_n^2} & \text{if } x_n < 0\,. \end{cases}$$

Since the couple $(x_n(t), \dot{x}_n(t))$ performs precisely $2N+1$ half rotations around the origin in the interval $[0, T]$, two distinct cases arise.

Case 1. There exist

$$0 < t_0^n < t_1^n < \cdots < t_{2N}^n < T\,,$$

and $\alpha_n \in \,]0, \pi[$, satisfying

$$\begin{aligned} &\theta_n(0) = \alpha_n - \frac{\pi}{2}\,, \\ &x_n(t_l^n) = 0\,, && \text{for } l = 0, 1, 2, \ldots, 2N\,, \\ &\dot{x}_n(t_{2j}^n) < 0\,, && \text{for } j = 0, 1, 2, \ldots, N\,, \\ &\dot{x}_n(t_{2j-1}^n) > 0\,, && \text{for } j = 1, 2, \ldots, N\,. \end{aligned}$$

Case 2. There exist

$$0 < t_1^n < \cdots < t_{2N}^n < t_{2N+1}^n < T\,,$$

and $\alpha_n \in \,]0, \pi[$, satisfying

$$\begin{aligned} &\theta_n(0) = -\alpha_n - \frac{\pi}{2}\,, \\ &x_n(t_l^n) = 0\,, && \text{for } l = 1, 2, \ldots, 2N\,, \\ &\dot{x}_n(t_{2j}^n) < 0\,, && \text{for } j = 1, 2, \ldots, N\,, \\ &\dot{x}_n(t_{2j-1}^n) > 0\,, && \text{for } j = 1, 2, \ldots, N\,. \end{aligned}$$

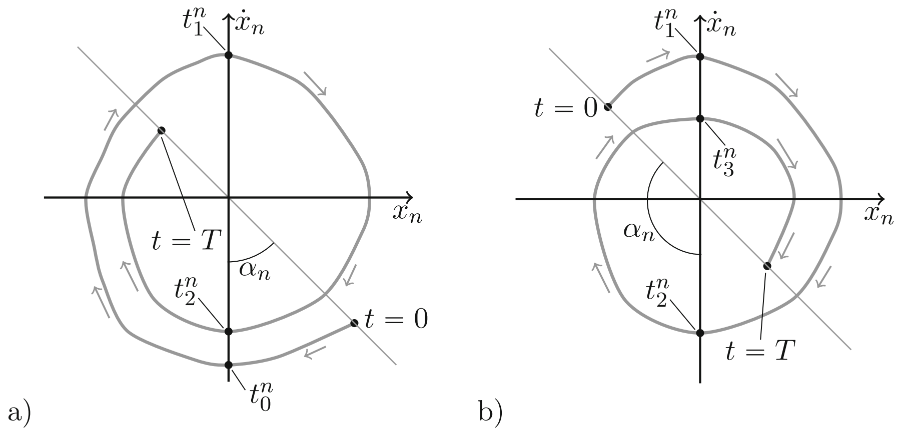

Fig. 5 **a)** Case 1 for $N = 1$. **b)** Case 2 for $N = 1$

We focus our analysis on Case 1, as the second one can be treated analogously. We have the following time estimate (Fig. 5):

$$\begin{aligned}\frac{\alpha_n}{\sqrt{\mu_1}} &= \frac{\theta_n(0) - \theta_n(t_0^n)}{\sqrt{\mu_1}} = \int_0^{t_0^n} \frac{-\dot{\theta}_n}{\sqrt{\mu_1}} \\ &= \int_0^{t_0^n} \frac{[(1-\sigma_n)\overline{\mu} x_n + \sigma_n g(t, x_n)]x_n + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} \\ &\geqslant \int_0^{t_0^n} \frac{[(1-\sigma_n)\mu_1 x_n + \sigma_n g(t, x_n)]x_n + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} \\ &= \int_0^{t_0^n} \frac{\mu_1 x_n^2 + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_0^{t_0^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \\ &= t_0^n + \sigma_n \int_0^{t_0^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \,. \end{aligned} \tag{34}$$

A similar computation gives the following time estimates for $j = 1, \ldots, N$:

$$\begin{aligned}\frac{\pi}{\sqrt{\nu_1}} &\geqslant \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{\nu_1 x_n^2 + \dot{x}_n^2}{\nu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} \\ &= (t_{2j-1}^n - t_{2j-2}^n) + \sigma_n \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} \,, \end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\pi}{\sqrt{\mu_1}} &\geqslant \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{\mu_1 x_n^2 + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \\ &= (t_{2j}^n - t_{2j-1}^n) + \sigma_n \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \,, \end{aligned} \tag{36}$$

and

$$\frac{\pi - \alpha_n}{\sqrt{\nu_1}} \geqslant \int_{t_{2N}^n}^T \frac{\nu_1 x_n^2 + \dot{x}_n^2}{\nu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2N}^n}^T \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2}$$
$$= (T - t_{2N}^n) + \sigma_n \int_{t_{2N}^n}^T \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2}\,. \tag{37}$$

Since the solution completes $2N+1$ half rotations in time T, addition in (34), (35), (36) and (37) for $j = 1, \ldots, N$ leads to

$$\sigma_n \Big[\int_{\{x_n<0\}} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} + \int_{\{x_n>0\}} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \Big] + T$$
$$\leqslant N\mathcal{T}_{\mu_1,\nu_1} + \frac{\alpha_n}{\sqrt{\mu_1}} + \frac{\pi - \alpha_n}{\sqrt{\nu_1}}$$
$$\leqslant N\mathcal{T}_{\mu_1,\nu_1} + \frac{\alpha_n}{\min\{\sqrt{\mu_1}, \sqrt{\nu_1}\}} + \frac{\pi - \alpha_n}{\min\{\sqrt{\mu_1}, \sqrt{\nu_1}\}}$$
$$= N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\min\{\sqrt{\mu_1}, \sqrt{\nu_1}\}}$$
$$= N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T\,,$$

where the last step follows from (18). Being $\sigma_n \neq 0$, we get

$$\int_{\{x_n<0\}} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} + \int_{\{x_n>0\}} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \leqslant 0\,,$$

which can be written as

$$\int_0^T \frac{\big[g(t, x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]x_n}{\mu_1 (x_n^+)^2 + \nu_1 (x_n^-)^2 + \dot{x}_n^2} \leqslant 0\,.$$

Recalling that $v_n = \frac{x_n}{||x_n||_\infty}$, we have

$$\int_0^T \frac{\big[g(t, x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]v_n}{\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2} \leqslant 0\,.$$

Since $\mu_1(v^+(t))^2 + \nu_1(v^-(t))^2 + \dot{v}(t)^2$ is positive and constant in t, and

$$\lim_{n\to\infty} \big(\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2\big) = \mu_1 (v^+)^2 + \nu_1 (v^-)^2 + \dot{v}^2\,,$$

uniformly in $[0, T]$, by Fatou's Lemma,

$$\int_0^T \liminf_n \frac{\big[g(t, x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]v_n}{\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2} \leqslant 0\,.$$

So, it has to be

$$\int_0^T \liminf_n \big[g(t, x_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big] v_n(t)\, dt \leqslant 0\,. \tag{38}$$

Let us now fix $t \in [0, T]$ such that $v(t) < 0$; so $v_n(t) < 0$ for sufficiently large n, and $\lim_n x_n(t) = -\infty$, hence

$$\liminf_n [\nu_1 x_n(t) - g(t, x_n(t))] \geqslant \liminf_{x\to-\infty} [\nu_1 x - g(t, x)]\,,$$

which implies that, for every $t \in [0, T]$ with $v(t) < 0$, we have

$$\begin{aligned}\liminf_n \big[g(t, \|x_n\|_\infty v_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big] v_n(t)\\ \geqslant \liminf_{x\to-\infty} [\nu_1 x - g(t, x)]|v(t)| = \limsup_{x\to-\infty} [g(t, x) - \nu_1 x] v(t)\,.\end{aligned}$$

Similarly, if $v(t) > 0$ for some t, then $v_n(t) > 0$ for sufficiently large n, and $\lim_n x_n(t) = +\infty$, hence

$$\begin{aligned}\liminf_n \big[g(t, \|x_n\|_\infty v_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big] v_n(t)\\ \geqslant \liminf_{x\to+\infty} [g(t, x) - \mu_1 x] v(t)\,.\end{aligned}$$

Thus, by (38),

$$\int_{\{v<0\}} \limsup_{x\to-\infty} \big(g(t, x) - \nu_1 x\big) v(t)\, dt + \int_{\{v>0\}} \liminf_{x\to+\infty} \big(g(t, x) - \mu_1 x\big) v(t)\, dt \leqslant 0\,, \tag{39}$$

a contradiction with (31), thus proving the Claim. The proof of Theorem 3.3 can now be completed arguing as in the proof of Theorem 3.1. □

4 Examples and Final Remarks

In this final section, we discuss some remarks, an example of application, and an open problem in this direction.

Remark 4.1 An existence result holds for the so-called simple resonance, i.e., the case when $\mathring{\mathcal{R}} \subseteq \mathcal{S}$, and $\mathcal{R} \cap \Sigma = \{(\mu_1, \nu_1)\}$ or $\mathcal{R} \cap \Sigma = \{(\mu_2, \nu_2)\}$. Clearly enough, in this case, the Landesman–Lazer condition will be imposed only on one side.

Remark 4.2 Notice that Theorem 3.3 generalizes Theorem 3.1. Indeed, if we focus our attention on $\mathcal{R} \subseteq \mathcal{S}$, it is possible to find $\epsilon_1, \epsilon_2 > 0$ such that

$$N\mathcal{T}_{\mu_1-\epsilon_1,\nu_1-\epsilon_1} + M_{\mu_1-\epsilon_1,\nu_1-\epsilon_1} = T = (N+1)\mathcal{T}_{\mu_2+\epsilon_2,\nu_2+\epsilon_2} + m_{\mu_2+\epsilon_2,\nu_2+\epsilon_2}\,.$$

Setting $\tilde{\mu}_1 = \mu_1 - \epsilon_1$, $\tilde{\mu}_2 = \mu_2 + \epsilon_2$, $\tilde{\nu}_1 = \nu_1 - \epsilon_1$ and $\tilde{\nu}_2 = \nu_2 + \epsilon_2$, we have

$$\lim_{x\to-\infty} \big(g(t,x) - \tilde{\nu}_1 x\big) = -\infty\,, \qquad \lim_{x\to+\infty} \big(g(t,x) - \tilde{\mu}_1 x\big) = +\infty\,,$$

and

$$\lim_{x\to-\infty} \big(\tilde{\nu}_2 x - g(t,x)\big) = -\infty\,, \qquad \lim_{x\to+\infty} \big(\tilde{\mu}_2 x - g(t,x)\big) = +\infty\,,$$

uniformly in t, from which we easily verify that the Landesman–Lazer conditions (31) and (32), respectively, hold.

Remark 4.3 The results contained in this paper could be rephrased in a L^2-Carathéodory setting. The ideas in the proofs remain unchanged. We preferred not to enter in this formalism, in order to avoid excessive technicalities.

As an example of application, we consider the problem

$$\begin{cases} \ddot{x} + \mu x^+ - \nu x^- + h(t,x) = 0\,, \\ x(0) + x(T) = 0\,, \quad \dot{x}(0) + \dot{x}(T) = 0\,, \end{cases} \tag{40}$$

where h is a bounded continuous function. If the couple (μ, ν) does not belong to the Fučík spectrum, as a consequence of Theorem 3.1, we get easily the existence of a solution for problem (40). So, let us focus our attention on a situation at resonance. To this aim, we assume for definiteness

$$\mu \neq \nu\,, \qquad (k+1)\frac{\pi}{\sqrt{\mu}} + k\frac{\pi}{\sqrt{\nu}} = T\,,$$

for some positive integer k. Recalling the definition of the function $\varphi_{\mu,\nu}$ in (7), we can find the following existence result as a consequence of Theorem 3.3.

Corollary 4.4 *In the above setting, if either*

$$\int_{\{\varphi_{\mu,\nu}<0\}} \limsup_{x\to-\infty} h(t,x)\varphi_{\mu,\nu}(t)\,dt + \int_{\{\varphi_{\mu,\nu}>0\}} \liminf_{x\to+\infty} h(t,x)\varphi_{\mu,\nu}(t)\,dt > 0\,,$$

or

$$\int_{\{\varphi_{\mu,\nu}<0\}} \liminf_{x\to-\infty} h(t,x)\varphi_{\mu,\nu}(t)\,dt + \int_{\{\varphi_{\mu,\nu}>0\}} \limsup_{x\to+\infty} h(t,x)\varphi_{\mu,\nu}(t)\,dt < 0\,,$$

then problem (40) admits at least one solution.

Proof As it has been shown in Section 2.1, the nontrivial solutions of problem (5) are of the type $x(t) = \rho\varphi_{\mu,\nu}(t)$ for every $\rho > 0$. If the first alternative holds in the statement, we simply need to choose $(\mu_1, \nu_1) = (\mu, \nu)$ and $(\mu_2, \nu_2) = (\mu+\varepsilon, \nu+\varepsilon)$, with $\varepsilon > 0$, and apply Theorem 3.3. In the other case, we need to fix $(\mu_1, \nu_1) = (\mu - \varepsilon, \nu - \varepsilon)$ and $(\mu_2, \nu_2) = (\mu, \nu)$, with $\varepsilon > 0$ sufficiently small, and apply again Theorem 3.3. □

The reader can easily compare the assumptions in the above corollary with the general Landesman–Lazer condition introduced in (3).

Let us briefly mention how to recover a $2T$-periodic solution from a given T-antiperiodic solution $x : [0, T] \to \mathbb{R}$ of $\ddot{x} + g(t, x) = 0$. Setting

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } t \in [0, T], \\ -x(t - T) & \text{if } t \in [T, 2T], \end{cases}$$

the function $\tilde{x} : [0, 2T] \to \mathbb{R}$ solves the same differential equation provided that $g(t - T, -x) = -g(t, x)$ holds for every $t \in [T, 2T]$.

It would be interesting to investigate how our results can be adapted to differential equations ruled by other differential operators, like the p-Laplacian, following, e.g., the ideas in [6].

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Declarations

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