Optimal identification of cavities in the Generalized Plane Stress problem in linear elasticity

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Abstract

For the Generalized Plane Stress (GPS) problem in linear elasticity, we obtain an optimal stability estimate of logarithmic type for the inverse problem of determining smooth cavities inside a thin isotropic cylinder from a single boundary measurement of traction and displacement. The result is obtained by reformulating the GPS problem as a Kirchhoff-Love plate-like problem in terms of the Airy's function, and by using the strong unique continuation at the boundary for a Kirchhoff-Love plate operator under homogeneous Dirichlet conditions, which has been recently obtained in [A-R-V].

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1 Introduction

In this paper we consider the inverse problem of detecting cavities inside a thin isotropic elastic plate $\Omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where the middle plane Ω is a bounded domain in $\mathbb{R}^2$ and h is the constant thickness, subject to a single experiment consisting in applying in-plane boundary loads and measuring the induced displacement at the boundary. Practical applications concern the use of non-destructive techniques for the identification of possible defects, such as cavities, inside the plate.

The static equilibrium of the plate is described in terms of the classical Generalized Plane Stress (GPS) problem, which allows to reformulate the original three dimensional problem in a two dimensional setting [S]. More precisely, denoting by $D\times \left(-\frac{h}{2}, \frac{h}{2}\right)$ the cavity, with D a possibly disconnected subset of Ω, the in-plane displacement field $a = a_1e_1 + a_2e_2$, solution to the GPS problem, satisfies the following two-dimensional Neumann boundary value problem ($\alpha, \beta = 1, 2$)

$$
\begin{cases}
N_{\alpha\beta,\beta} = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.1)\\
N_{\alpha\beta}n_\beta = \hat{N}_\alpha, & \text{on } \partial\Omega, \quad (1.2)\\
N_{\alpha\beta}n_\beta = 0, & \text{on } \partial D, \quad (1.3)\\
N_{\alpha\beta} = \frac{Eh}{1-\nu^2}((1-\nu)\varrho_{\alpha\beta} + \nu(\varrho_{\gamma\gamma})\delta_{\alpha\beta}), & \text{in } \Omega \setminus \overline{D}, \quad (1.4)\\
\varrho_{\alpha\beta} = \frac{1}{2}(a_{\alpha,\beta} + a_{\beta,\alpha}), & \text{in } \Omega \setminus \overline{D}. \quad (1.5)
\end{cases}
$$

Here, $\hat{N} = \hat{N}_1e_1 + \hat{N}_2e_2$ is the in-plane load field applied to $\partial\Omega$ satisfying the compatibility condition

$$
\int_{\partial\Omega} \hat{N} \cdot r = 0, \quad \text{for every } r \in \mathcal{R}_2, \tag{1.6}
$$

where $\mathcal{R}_2$ is the linear space of infinitesimal two-dimensional rigid displacements. Here, $E = E(x)$ and $\nu = \nu(x)$ are the Young's modulus and the Poisson's coefficient of the material, respectively. Under suitable strong convexity assumptions on the elastic tensor of the material (see Section 3 for details), and assuming $\hat{N} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, problem (1.1)–(1.6) admits a unique solution $a \in H^1(\Omega \setminus \overline{D}, \mathbb{R}^2)$ satisfying the normalization conditions

$$
\int_{\Omega\setminus\overline{D}} a = 0, \qquad \int_{\Omega\setminus\overline{D}} (\nabla a - \nabla^T a) = 0, \tag{1.7}
$$

and such that $\|a\|_{H^1(\Omega\setminus\overline{D})} \leq C\|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}$.

In this work we face the inverse problem of determining the cavity D from a single pair of Cauchy data $\{a, \hat{N}\}$ given on $\partial\Omega$. More precisely, we are interested to obtain quantitative stability estimates, which are useful to control the effect that possible errors on the measurements have on the results of reconstruction procedures. The arbitrariness of the normalization conditions (1.7), which are related to the non-uniqueness of the solution to the direct problem (1.1)–(1.6), leads to the following formulation of the stability issue: given two solutions $a^{(i)} \in H^1(\Omega, \mathbb{R}^2)$, $i = 1, 2$, to the direct problem (1.1)–(1.6) with $D = D_i$, satisfying, for some $\varepsilon > 0$,

$$\min_{r \in \mathcal{R}_2} \|a^{(1)} - a^{(2)} - r\|_{L^2(\Sigma,\mathbb{R}^2)} \leq \varepsilon, \tag{1.8}$$

to control the Hausdorff distance $d_H(\overline{D_1}, \overline{D_2})$ in terms of ε when ε goes to zero, where Σ is an open subset of $\partial\Omega$.

Assuming $D \in C^{6,\alpha}$, $0 < \alpha \leq 1$, we prove

$$d_H(\overline{D_1}, \overline{D_2}) \leq C|\log \varepsilon|^{-\eta}, \tag{1.9}$$

where $C > 0$ and $\eta > 0$ are constants only depending on the a priori data. We refer to Theorem 3.1 for a precise statement. Let us notice that, in view of the counterexamples obtained in the simpler context of electrical conductivity (see, for instance, [Al], [Ma], [DiC-R]), we can infer the optimality of the stability estimate (1.9).

The general scheme of our proof is inspired to the seminal paper [Al-Be-Ro-Ve], which established the first optimal logarithmic estimate for the determination of unknown boundaries in electrostatics. The key tool in [Al-Be-Ro-Ve] was, among others, the polynomial vanishing rate for solutions to the second order elliptic equation of electrostatics, satisfying either homogeneous Dirichlet or homogeneous Neumann boundary conditions, ensured by a doubling inequality at the boundary established in [A-E]. Aiming at obtaining a strong unique continuation property at the boundary (SUCB) for solutions to the GPS elliptic system, in this paper we have exploited the two dimensional character of the problem (1.1)–(1.6) by using the classical Airy's transformation, which (locally) reduces the GPS system with homogeneous Neumann boundary conditions to a scalar fourth order Kirchhoff-Love plate's equation under homogeneous Dirichlet boundary conditions. This reformulation allows us to use the finite vanishing rate at the boundary for homogeneous Dirichlet boundary conditions recently obtained in [A-R-V] in the form of a three spheres inequality at the boundary with optimal exponent, and in [M-R-V3] in the form of a doubling inequality at the boundary.

It is worth noticing that the present approach, here applied to the GPS problem, allows also to cover the analogous inverse problem of detecting cavities in a two-dimensional elastic body made by inhomogeneous Lamé material, thus improving the log − log stability result previously obtained in [M-R]. An optimal log-type estimate in dimension three remains a challenging open problem. Let us mention that the Airy's transformation has been used in [L-U-W] to prove global identifiability of the viscosity in an incompressible fluid governed by the Stokes and the Navier-Stokes equations in the plane by using boundary measurements.

The paper is organized as follows. Notation is presented in Section 2. Section 3 contains the formulation of the inverse problem and the statement of our stability result. The Airy's transformation is illustrated in Section 4. The proof of the main result, given in Section 5, is based on a series of auxiliary propositions concerning Lipschitz propagation of smallness (Proposition 5.1), finite vanishing rate in the interior (Proposition 5.2), finite vanishing rate at the boundary (Proposition 5.3), stability estimate from Cauchy data (Proposition 5.4). Finally, for the sake of completeness, in Section 6 we recall a derivation of the GPS problem from the corresponding three dimensional elasticity problem for a thin plate subject to in-plane boundary loads.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes, of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$.

Definition 2.1. ($C^{k,\alpha}$ regularity) Let Ω be a bounded domain in $\mathbb{R}^2$. Given k, α, with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion S of $\partial\Omega$ is of *class* $C^{k,\alpha}$ *with constants* r_0, $M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{r_0,2M_0r_0} = \{x \in R_{r_0,2M_0r_0} \mid x_2 > g(x_1)\},$$

where g is a $C^{k,\alpha}$ function on $[-r_0, r_0]$ satisfying

$$g(0) = g'(0) = 0,$$

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} \leq M_0 r_0,$$

where

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} = \sum_{i=0}^{k} r_0^i \sup_{[-r_0,r_0]} |g^{(i)}| + r_0^{k+\alpha}|g|_{k,\alpha},$$

$$|g|_{k,\alpha} = \sup_{\substack{t,s\in[-r_0,r_0] \\ t\neq s}} \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t-s|^\alpha}.$$

We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance,

$$\|f\|_{H^1(\Omega)} = r_0^{-1}\left(\int_\Omega f^2 + r_0^2\int_\Omega |\nabla f|^2\right)^{\frac{1}{2}},$$

and so on for boundary and trace norms.

Given a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,\alpha}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is

$$\tau = e_3 \times n, \tag{2.1}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Given any connected component $\mathcal{C}$ of $\partial\Omega$ and fixed a point $P_0 \in \mathcal{C}$, let us define as positive the orientation of $\mathcal{C}$ associated to an arclength parameterization $\psi(s) = (x_1(s), x_2(s))$, $s \in [0, l(\mathcal{C})]$, such that $\psi(0) = P_0$ and $\psi'(s) = \tau(\psi(s))$. Here $l(\mathcal{C})$ denotes the length of $\mathcal{C}$.

Throughout the paper, we denote by $w_{,\alpha}$, $\alpha = 1, 2$, $w_{,s}$, and $w_{,n}$ the derivatives of a function w with respect to the x_α variable, to the arclength s and to the normal direction n, respectively, and similarly for higher order derivatives.

We denote by $\mathbb{M}^n$ the space of $n \times n$ real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

Given A, $B \in \mathbb{M}^n$ and $\mathbb{K} \in \mathcal{L}(\mathbb{M}^n, \mathbb{M}^n)$, we use the following notation:

$$(\mathbb{K}A)_{ij} = \sum_{k,l=1}^{2} K_{ijkl}A_{kl}, \tag{2.2}$$

$$A \cdot B = \sum_{i,j=1}^{2} A_{ij}B_{ij}, \tag{2.3}$$

$$|A| = (A \cdot A)^{\frac{1}{2}}, \tag{2.4}$$

$$\hat{A} = \frac{1}{2}(A + A^T). \tag{2.5}$$

We denote by I_n the $n \times n$ identity matrix, and by $\mathrm{tr}(A)$ the trace of A.

When $n = 2$, we replace the Latin indexes with Greek ones.

The linear space of the infinitesimal rigid displacements, for $n = 2, 3$, is defined as

$$\mathcal{R}_n = \left\{ r(x) = c + Wx,\ c \in \mathbb{R}^n,\ W \in \mathbb{M}^n,\ W + W^T = 0 \right\}. \tag{2.6}$$

3 Inverse problem and main result

i) A priori information on the geometry.

Let Ω be a bounded domain in $\mathbb{R}^2$ and let us assume that the cavity D is an open subset compactly contained in Ω, such that

$$\Omega \setminus D \text{ is connected.} \tag{3.1}$$

Moreover, let us assume that, given positive numbers r_0, M_0, M_1, with $M_0 \geq \frac{1}{2}$, we have

$$\mathrm{diam}(\Omega) \leq M_1 r_0, \tag{3.2}$$

$$\mathrm{dist}(D, \partial\Omega) \geq 2M_0 r_0, \tag{3.3}$$

$$\partial\Omega \text{ is of } \textit{class } C^{1,\alpha} \textit{ with constants } r_0, M_0, \tag{3.4}$$

$$\partial D \text{ is of } \textit{class } C^{6,\alpha} \textit{ with constants } r_0, M_0, \tag{3.5}$$

with α such that $0 < \alpha \leq 1$.

Let us denote by Σ the open portion of $\partial\Omega$ where measurements are taken. We assume that there exists $P_0 \in \Sigma$ such that

$$\partial\Omega \cap R_{r_0, 2M_0 r_0}(P_0) \subset \Sigma, \tag{3.6}$$

and

$$\Sigma \text{ is of } \textit{class } C^{2,\alpha} \textit{ with constants } r_0, M_0. \tag{3.7}$$

Let us notice that, without loss of generality, we have chosen $M_0 \geq \frac{1}{2}$ to ensure that $B_{r_0}(P) \subset R_{r_0, 2M_0 r_0}(P)$ for every $P \in \partial\Omega$.

ii) A priori information on the Neumann boundary data.

We assume that

$$\hat{N} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2), \quad \hat{N} \not\equiv 0, \tag{3.8}$$

$$\int_{\partial\Omega} \hat{N} \cdot r = 0, \quad \text{for every } r \in \mathcal{R}_2, \tag{3.9}$$

$$\mathrm{supp}(\hat{N}) \subset\subset \Sigma, \tag{3.10}$$

and that, for a given constant $F > 0$,

$$\frac{\|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}{\|\hat{N}\|_{H^{-1}(\partial\Omega,\mathbb{R}^2)}} \leq F, \tag{3.11}$$

iii) A priori information on the elasticity tensor.

The constitutive equation (1.4) can be written as

$$N_{\alpha\beta}(x) = C_{\alpha\beta\gamma\delta}(x)\varrho_{\gamma\delta}, \tag{3.12}$$

where the elasticity tensor $\mathbb{C} = (C_{\alpha\beta\gamma\delta})$ is defined as

$$\mathbb{C}(x)A = \frac{Eh}{1 - \nu^2(x)}((1 - \nu(x))\hat{A} + \nu(\mathrm{tr}(A))I_2), \tag{3.13}$$

for every 2×2 matrix A, where the Young's modulus E and the Poisson's coefficient ν are given in terms of the Lamé moduli as follows

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \qquad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))}. \tag{3.14}$$

On the Lamé coefficients $\mu = \mu(x)$, $\lambda = \lambda(x)$, $\mu : \overline{\Omega} \to \mathbb{R}$, $\lambda : \overline{\Omega} \to \mathbb{R}$, we assume

$$\mu(x) \geq \alpha_0, \qquad 2\mu(x) + 3\lambda(x) \geq \gamma_0, \qquad \text{in } \overline{\Omega}, \tag{3.15}$$

for positive constants α_0 and γ_0.

The above assumptions ensure that $\mathbb{C}$ satisfies the minor and major symmetries

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma}, \quad C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}, \quad \text{for every } \alpha, \beta, \gamma, \delta = 1, 2, \text{ in } \overline{\Omega}, \tag{3.16}$$

and that it is strongly convex in $\overline{\Omega}$, precisely

$$\mathbb{C}A \cdot A \geq h\xi_0|A|^2, \qquad \text{in } \Omega, \tag{3.17}$$

for every 2×2 symmetric matrix A, where $\xi_0 = \min\{2\alpha_0, \gamma_0\}$ (see [M-R-V1, Lemma 3.5] for details). Moreover, $E(x) > 0$ and $-1 < \nu(x) < \frac{1}{2}$ in $\overline{\Omega}$.

We further assume that

$$\|\lambda\|_{C^4(\overline{\Omega})}, \quad \|\mu\|_{C^4(\overline{\Omega})} \leq \Lambda_0, \tag{3.18}$$

for some positive constant Λ_0.

We note that the equilibrium problem (1.1)–(1.5) can be written in compact form as

$$\begin{cases} \operatorname{div}(\mathbb{C}\nabla a) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (3.19)\\ (\mathbb{C}\nabla a)n = \hat{N}, & \text{on } \partial\Omega, \quad (3.20)\\ (\mathbb{C}\nabla a)n = 0, & \text{on } \partial D. \quad (3.21) \end{cases}$$

The weak formulation of (3.19)–(3.21) consists in finding $a = a(x) \in H^1(\Omega \setminus \overline{D})$ satisfying

$$\int_{\Omega\setminus\overline{D}} \mathbb{C}\nabla a \cdot \nabla v = \int_{\partial\Omega} \hat{N} \cdot v, \quad \text{for every } v \in H^1(\Omega \setminus \overline{D}). \tag{3.22}$$

Under our assumptions, there exists a unique solution to (3.22) up to addition of a rigid displacement. In order to select a single solution, we shall assume the normalization conditions

$$\int_{\Omega\setminus\overline{D}} a = 0, \quad \int_{\Omega\setminus\overline{D}} (\nabla a - \nabla^T a) = 0, \tag{3.23}$$

which imply the following stability estimate for the direct problem (3.19)–(3.21)

$$\|a\|_{H^1(\Omega\setminus\overline{D})} \leq C r \|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}, \tag{3.24}$$

where $C > 0$ is a constant only depending on h, α_0, γ_0, M_0 and M_1.

In what follows, we shall refer to the set of constants h, α_0, γ_0, Λ_0, α, M_0, M_1 and F as the a priori data.

Theorem 3.1 (Stability result). *Let Ω be a domain satisfying* (3.2), (3.4) *and let Σ be an open portion of $\partial\Omega$ satisfying* (3.6)–(3.7). *Let the elasticity tensor $\mathbb{C} = \mathbb{C}(x) \in \mathbb{L}(\mathbb{M}^2, \mathbb{M}^2)$ given by* (3.13), *with Lamé moduli $\lambda = \lambda(x)$, $\mu = \mu(x)$ satisfying* (3.15) *and* (3.18). *Let $\hat{N} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, $\hat{N} \not\equiv 0$, satisfying* (3.9)–(3.11). *Let D_i, $i = 1, 2$, be two open subsets of Ω satisfying* (3.1), (3.3), (3.5), *and let $a^{(i)} \in H^1(\Omega \setminus D_i, \mathbb{R}^2)$ be the solution to* (3.19)–(3.21), *satisfying* (3.23), *when $D = D_i$, $i = 1, 2$. If, given $\varepsilon > 0$, we have*

$$\min_{r\in\mathcal{R}_2} \|a^{(1)} - a^{(2)} - r\|_{L^2(\Sigma,\mathbb{R}^2)} \leq r_0\varepsilon, \tag{3.25}$$

then we have

$$(3.26)\qquad d_H(\overline{D_1},\overline{D_2}) \le C r_0 \left|\log\left(\frac{\varepsilon}{\|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\right)\right|^{-\eta},$$

where C, η, $C>0$, $\eta>0$, only depend on the a priori data.

Remark 3.2. Let us notice that, as it will be clear from the proof, the above stability result holds true also when the domain Ω contains a finite number of connected cavities $D^{(j)}$, $j=1,\dots,J$, such that $\partial D^{(j)} \in C^{6,\alpha}$ with constants r_0, M_0, and $\mathrm{dist}(\partial D^{(j)}, \partial D^{(k)}) \ge r_0$, for $j \neq k$.

4 Airy's transformation

It is known that the boundary value problem in plane linear elasticity can be formulated in terms of an equivalent Kirchhoff-Love *plate-like* problem involving a scalar-valued function called *Airy's function*. Although this argument is well established, see, for instance, [G] and [Fic], for reader convenience in what follows we recall the essential points of the analysis.

For the sake of completeness, we consider a mixed boundary value problem, in order to describe the transformation of both Dirichlet and Neumann boundary conditions. Let $a = a_1e_1 + a_2e_2$, $a \in H^1(\mathrm{U}, \mathbb{R}^2)$, be the solution to the GPS problem

$$\begin{cases} N_{\alpha\beta,\beta} = 0, & \text{in } \mathrm{U}, & (4.1)\\ N_{\alpha\beta}n_\beta = \hat{N}_\alpha, & \text{on } \partial_t\mathrm{U}, & (4.2)\\ a_\alpha = \overline{a}_\alpha, & \text{on } \partial_u\mathrm{U}, & (4.3)\\ N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\left((1-\nu)\varrho_{\alpha\beta} + \nu(\varrho_{\gamma\gamma})\delta_{\alpha\beta}\right), & \text{in } \mathrm{U}, & (4.4)\\ \varrho_{\alpha\beta} = \frac{1}{2}(a_{\alpha,\beta}+a_{\beta,\alpha}), & \text{in } \mathrm{U}, & (4.5)\end{cases}$$

where $\hat{N} \in H^{-1/2}(\partial_t\mathrm{U}, \mathbb{R}^2)$ and $\overline{a} \in H^{1/2}(\partial_u\mathrm{U}, \mathbb{R}^2)$ are given Neumann and Dirichlet data, respectively. Here, $\partial_u\mathrm{U}$, $\partial_t\mathrm{U}$ are two disjoint connected open subsets of $\partial\mathrm{U}$, with $\partial\mathrm{U} = \partial_u\mathrm{U} \cup \partial_t\mathrm{U}$.

The equilibrium equations (4.1), and the simply connectness of U, ensure the existence of a single-valued function $\phi = \phi(x_1, x_2)$, $\phi \in H^2(\mathrm{U})$, such that

$$(4.6)\qquad N_{\alpha\beta} = e_{\alpha\gamma}e_{\beta\delta}\phi_{,\gamma\delta},$$

where the matrix $e_{\alpha\gamma}$ is defined as follows: $e_{11} = e_{22} = 0$, $e_{12} = 1$, $e_{21} = -1$; see [Ai]. We recall that, by construction, the function ϕ and its first partial derivatives $\phi_{,1}$, $\phi_{,2}$ are uniquely determined up to an additive arbitrary constant.

It is convenient to introduce the *strain functions* $K_{\alpha\beta}$, $\alpha, \beta = 1, 2$, associated to the infinitesimal strain $\varrho_{\alpha\beta}$:

(4.7) $$K_{\alpha\beta} = e_{\delta\alpha} e_{\gamma\beta} \varrho_{\delta\gamma}, \qquad \alpha, \beta = 1, 2.$$

By inverting the constitutive equation (4.4), we get

(4.8) $$\varrho_{\alpha\beta} = \frac{1+\nu}{Eh} N_{\alpha\beta} - \frac{\nu}{Eh}(N_{\gamma\gamma})\delta_{\alpha\beta},$$

and using (4.6) we obtain

(4.9) $$\varrho_{\alpha\beta} = \frac{1+\nu}{Eh} e_{\alpha\gamma} e_{\beta\delta} \phi_{,\gamma\delta} - \frac{\nu}{Eh}(\phi_{,\gamma\gamma})\delta_{\alpha\beta}.$$

Inserting this expression of $\varrho_{\alpha\beta}$ into (4.7), we have

(4.10) $$K_{\alpha\beta} = L_{\alpha\beta\gamma\delta} \phi_{,\gamma\delta},$$

where the Cartesian components $L_{\alpha\beta\gamma\delta}$ of the fourth order tensor L are

(4.11) $$L_{\alpha\beta\gamma\delta} = \frac{1+\nu}{Eh} \delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{\nu}{Eh}\delta_{\alpha\beta}\delta_{\gamma\delta}.$$

The strain $\varrho_{\alpha\beta}$ obviously satisfies the well-known two-dimensional *Saint-Venant compatibility equation*

(4.12) $$\varrho_{11,22} + \varrho_{22,11} - 2\varrho_{12,12} = 0, \qquad \text{in } \mathrm{U}.$$

Inverting (4.7), we have

(4.13) $$\varrho_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} K_{\gamma\delta},$$

and the equation (4.12), written in terms of $K_{\gamma\delta}$, becomes

(4.14) $$\operatorname{div}(\operatorname{div}(\mathrm{L}\nabla^2\phi)) = 0, \qquad \text{in } \mathrm{U},$$

or, more explicitly,

(4.15) $$\Delta^2\phi + 2Eh\nabla \frac{1}{Eh} \cdot \nabla(\Delta\phi) - Eh\Delta \frac{\nu}{Eh} \Delta\phi + \\ + Eh\nabla^2 \frac{1+\nu}{Eh} \cdot \nabla^2\phi = 0, \qquad \text{in } \mathrm{U}.$$

The above partial differential equation expresses the form assumed by the field equation (4.1) in terms of the Airy's function ϕ.

We now consider the transformation of the Neumann boundary condition (4.2) on $\partial_t U$. By (4.6), the condition on $\partial_t U$ can be written as

$$e_{\alpha\gamma} e_{\beta\delta} \phi_{,\gamma\delta} n_\beta = \hat{N}_\alpha, \tag{4.16}$$

that is, recalling that $\tau_\delta = e_{\beta\delta} n_\beta$ on ∂U,

$$(\phi_{,1})_{,s} = -\hat{N}_2, \quad (\phi_{,2})_{,s} = \hat{N}_1, \quad \text{on } \partial_t U, \tag{4.17}$$

where s is an arc length parametrization on ∂U. By integrating the above equations with respect to s, from $P_0 \in \partial_t U$ to $P \in \partial_t U$, with $s(P_0) = 0$ and $s(P) = s$, the gradient of ϕ on $\partial_t U$ can be determined up to an additive constant vector $c = c_1 e_1 + c_2 e_2$, namely

$$\nabla\phi(s) = c + \hat{g}(s), \quad \text{on } \partial_t U, \tag{4.18}$$

where $\hat{g}(s) = \hat{g}_1(s)e_1 + \hat{g}_2(s)e_2$, $\hat{g}_1(s) = -\int_0^s \hat{N}_2(\xi)d\xi$, $\hat{g}_2(s) = \int_0^s \hat{N}_1(\xi)d\xi$. It follows that the normal derivative of ϕ on $\partial_t U$ is prescribed in terms of the Neumann data $\hat{N}$, that is,

$$\phi_{,n} = (c + \hat{g}(s)) \cdot n, \quad \text{on } \partial_t U, \tag{4.19}$$

whereas, integrating once more (4.18) from P_0 to P, we have

$$\phi(s) = C + \hat{G}(s), \quad \text{on } \partial_t U, \tag{4.20}$$

where $C = \phi(0)$ =constant, and $\hat{G}(s) = \int_0^s (c + g(\xi)) \cdot \tau(\xi)d\xi$. We notice that it is always possible to select the two arbitrary constants occurring in the construction of $\nabla\phi$ such that $c_1 = c_2 = 0$ (see, for example, [S] for details). In particular, if the Neumann data $\hat{N}$ vanishes on $\partial_t U$, then we can also choose the third constant $C = 0$, so that $\phi(s) = 0$ on $\partial_t U$. In this case, the homogeneous Neumann boundary conditions for the GPS problem are transformed into the homogeneous Dirichlet boundary conditions for the Airy's function:

$$\phi = 0, \quad \phi_{,n} = 0, \quad \text{on } \partial_t U. \tag{4.21}$$

The determination of the boundary conditions satisfied by ϕ on $\partial_u U$ is less obvious, since the corresponding boundary conditions in the original two-dimensional elasticity problem are not explicitly expressed in terms of the Airy's function or its derivatives. In dealing with this boundary condition, we need to assume $C^{1,1}$-regularity for ∂U. We adopt a variational-like approach. Without loss of generality, we can assume $\partial_u U = \partial U$.

Let $\tilde{\phi}$, $\tilde{\phi} : \overline{U} \to \mathbb{R}$, be a C^∞-test function, and define the *associated Airy stress field*

$$\tilde{N}_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} \tilde{\phi}_{,\gamma\delta}, \quad \text{in } \overline{U}, \tag{4.22}$$

which obviously satisfies the equilibrium equations

$$\tilde{N}_{\alpha\beta,\beta} = 0, \quad \text{in } U. \tag{4.23}$$

Multiplying (4.23) by the displacement field $a = a_1 e_1 + a_2 e_2$ solution to (4.1)–(4.5), and integrating by parts, we obtain

$$\int_U \Phi_{,\gamma\delta} K_{\gamma\delta} = \int_{\partial U} \tilde{N}_{\alpha\beta} n_\beta \hat{a}_\alpha. \tag{4.24}$$

We first work on the integral of the left hand side of (4.24). After two integrations by parts, we obtain

$$\int_U \tilde{\phi}_{,\gamma\delta} K_{\gamma\delta} = \int_U K_{\gamma\delta,\gamma\delta} \Phi + \int_{\partial U} \Phi_{,\gamma} K_{\gamma\delta} n_\delta - \int_{\partial U} \tilde{\phi} K_{\gamma\delta,\delta} n_\gamma. \tag{4.25}$$

We elaborate the second integral I on the right hand side of the above equation in terms of the local coordinates. Recalling that $\tau_\alpha = e_{\beta\alpha} n_\beta$ on ∂U and $\tilde{\phi}_{,\alpha} = n_\alpha \tilde{\phi}_{,n} + \tau_\alpha \tilde{\phi}_{,s}$ on ∂U, $\alpha, \beta = 1, 2$, we have

$$I = \int_{\partial U} (\tilde{\phi}_{,n} K_{nn} + \tilde{\phi}_{,s} K_{\tau n}), \tag{4.26}$$

where, to simplify the notation, we have introduced on ∂U the two functions

$$K_{nn} = K_{\gamma\delta} n_\delta n_\gamma, \qquad K_{n\tau} = K_{\gamma\delta} n_\delta \tau_\gamma (= K_{\tau n}). \tag{4.27}$$

Since ∂U is of $C^{1,1}$-class, integrating by parts the second term in (4.26) gives

$$I = \int_{\partial U} (\tilde{\phi}_{,n} K_{nn} - \tilde{\phi} K_{\tau n,s}). \tag{4.28}$$

Therefore, the left hand side of (4.24) takes the form

$$\int_U \tilde{\phi}_{,\gamma\delta} K_{\gamma\delta} = \int_U K_{\gamma\delta,\gamma\delta} \Phi + \int_{\partial U} (K_{nn} \tilde{\phi}_{,n} - (K_{\gamma\delta,\delta} n_\gamma + K_{\tau n,s}) \tilde{\phi}). \tag{4.29}$$

We next elaborate the integral appearing on the right hand side of (4.24). Let us introduce the *boundary displacement functions* associated to the Dirichlet data $\hat{a}$:

$$\hat{U}_\gamma = e_{\alpha\gamma} \hat{a}_\alpha, \quad \text{on } \partial U. \tag{4.30}$$

Passing to local coordinates, after an integration by parts, we have

$$(4.31)\qquad \int_{\partial U} \tilde{N}_{\alpha\beta} n_\beta \hat{a}_\alpha = \int_{\partial U} \tilde{\phi}_{,\gamma\delta}\tau_\delta \hat{U}_\gamma = \int_{\partial U} (\tilde{\phi}_{,\gamma})_{,s}\hat{U}_\gamma = -\int_{\partial U} \tilde{\phi}_{,\gamma}\hat{U}_{\gamma,s}.$$

Expressing again $\nabla\tilde{\phi}$ in terms of local coordinates, and integrating by parts, by the regularity of ∂U we obtain

$$(4.32)\qquad \int_{\partial U} \tilde{N}_{\alpha\beta} n_\beta \hat{a}_\alpha = \int_{\partial U} (-\tilde{\phi}_n \hat{U}_{\gamma,s} n_\gamma + \tilde{\phi}(\tau_\gamma \hat{U}_{\gamma,s})_{,s}).$$

Finally, by rewriting (4.24) using (4.29) and (4.32), the strain functions $K_{\gamma\delta}$ satisfy the condition

$$(4.33)\qquad \int_U K_{\gamma\delta,\gamma\delta}\tilde{\phi} + \int_{\partial U} (K_{nn} + \hat{U}_{\gamma,s} n_\gamma)\tilde{\phi}_{,n} - \int_{\partial U} (K_{\gamma\delta,\delta} n_\gamma + K_{\tau n,s} + (\tau_\gamma \hat{U}_{\gamma,s})_{,s})\tilde{\phi} = 0,$$

for every $\tilde{\phi} \in C^\infty(\overline{U})$. By the arbitrariness of the test function $\tilde{\phi}$, and of the traces of $\tilde{\phi}$ and $\tilde{\phi}_{,n}$ on ∂U, we determine the conditions satisfied by $K_{\gamma\delta}$, namely, the field equation

$$(4.34)\qquad K_{\gamma\delta,\gamma\delta} = 0, \quad \text{in } U,$$

which coincides with (4.14), and the two boundary conditions

$$(4.35)\qquad K_{nn} = -\hat{U}_{\gamma,s} n_\gamma, \quad \text{on } \partial U,$$

$$(4.36)\qquad K_{\gamma\delta,\delta} n_\gamma + K_{\tau n,s} = -(\tau_\gamma \hat{U}_{\gamma,s})_{,s}, \quad \text{on } \partial U.$$

The above equations (4.34) and (4.35), (4.36) are known as *compatibility field equation* and *compatibility boundary conditions* for the strain functions $K_{\gamma\delta}$, respectively. In conclusion, under the assumption $\hat{N} = 0$ on $\partial_t U$, the two-dimensional elasticity problem (4.1)–(4.5) can be formulated in terms of the Airy's function as follows:

$$\begin{cases} K_{\gamma\delta,\gamma\delta} = 0, & \text{in } U, & (4.37)\\ \phi = 0, & \text{on } \partial_t U, & (4.38)\\ \frac{\partial\phi}{\partial n} = 0, & \text{on } \partial_t U, & (4.39)\\ K_{\alpha\beta} n_\alpha n_\beta = -\hat{U}_{\gamma,s} n_\gamma, & \text{on } \partial_u U, & (4.40)\\ K_{\alpha\beta,\beta} n_\alpha + (K_{\alpha\beta} n_\beta \tau_\alpha)_{,s} = -(\tau_\gamma \hat{U}_{\gamma,s})_{,s}, & \text{on } \partial_u U, & (4.41)\\ K_{\alpha\beta} = \frac{1}{Eh}\left((1+\nu)\phi_{,\alpha\beta} - \nu(\Delta\phi)\delta_{\alpha\beta}\right), & \text{in } \overline{U}. & (4.42) \end{cases}$$

There is an important analogy connected with the above boundary value problem. Equations (4.37)–(4.42) describe the conditions satisfied by the transversal displacement $\phi = \phi(x_1, x_2)$ of the middle surface U of a Kirchhoff-Love thin elastic plate made by isotropic material. The plate is clamped on ∂_tU, and subject to a couple field $\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n \tau$ assigned on ∂_uU, with $\widehat{M}_n = -\widehat{U}_{\gamma,s} n_\gamma$ and $\widehat{M}_\tau = \tau_\gamma \widehat{U}_{\gamma,s}$, see, for example, [M-R-V1]. Within this analogy, the strain functions $K_{\alpha\beta} = K_{\alpha\beta}(x_1, x_2)$ play the role of the *bending moments* (for $\alpha = \beta$) and the *twisting moments* (for $\alpha \neq \beta$) of the plate at $(x_1, x_2) \in \overline{\Omega}$ (per unit length), and the *bending stiffness* of the plate is equal to $(Eh)^{-1}$.

Let us observe that the geometry of the inverse problem here considered, that is U $= \Omega \setminus \overline{D}$ does not ensure the existence of a globally defined Airy's function, since the hypotheses of simple connectedness is missing. For this reason, in the following Section 5 we shall make use of local Airy's functions, defined either in interior discs (see the proof of Proposition 5.2) or in neighbourhoods of the boundary of the cavity (see the proof of Proposition 5.3).

Proposition 4.1. *Under the above notation and assumptions, we have*

$$\frac{(1-|\nu|)^2}{E^2h^2}|\nabla^2\phi|^2 \le |\hat{\nabla} a|^2 \le \frac{(1+|\nu|)^2}{E^2h^2}|\nabla^2\phi|^2 \tag{4.43}$$

Proof. By (4.6), we have $N_{11} = \phi_{,22}$, $N_{22} = \phi_{,11}$, $N_{12} = N_{21} = -\phi_{,12}$, so that

$$|\nabla^2\phi|^2 = \sum_{\alpha,\beta=1}^{2} N_{\alpha\beta}^2. \tag{4.44}$$

By (4.8), we have $\varrho_{11} = \frac{1}{Eh}N_{11} - \frac{\nu}{Eh}N_{22}$, $\varrho_{22} = \frac{1}{Eh}N_{22} - \frac{\nu}{Eh}N_{11}$, $\varrho_{12} = \varrho_{21} = \frac{1+\nu}{Eh}N_{12}$, so that

$$|\hat{\nabla} a|^2 = \sum_{\alpha,\beta=1} \varrho_{\alpha\beta}^2 = \frac{1}{(Eh)^2}\{(1+\nu^2)(N_{11}^2 + N_{22}^2) + 2(1+\nu)^2 N_{12}^2 - 4\nu N_{11}N_{22}\}. \tag{4.45}$$

Let us estimate the term $-4\nu N_{11}N_{22}$ by using the elementary inequalities

$$\pm 2N_{11}N_{22} \le N_{11}^2 + N_{22}^2. \tag{4.46}$$

I) Estimate from below.

i) $0 < \nu < \frac{1}{2}$

If $N_{11}N_{22} < 0$, then $-4\nu N_{11}N_{22} > 0$, whereas if $N_{11}N_{22} \geq 0$, then, by (4.46), $-4\nu N_{11}N_{22} \geq -2\nu(N_{11}^2 + N_{22}^2)$. Since $-2\nu(N_{11}^2 + N_{22}^2) \leq 0$, we have, independently of the sign of $N_{11}N_{22}$,

$$-4\nu N_{11}N_{22} \geq -2\nu(N_{11}^2 + N_{22}^2). \tag{4.47}$$

ii) $\nu = 0$

In this case,

$$-4\nu N_{11}N_{22} = 0. \tag{4.48}$$

iii) $-1 < \nu < 0$ ($\Leftrightarrow 0 < -\nu < 1$)

If $N_{11}N_{22} \geq 0$, then $-4\nu N_{11}N_{22} \geq 0$, whereas if $N_{11}N_{22} < 0$, then, by (4.46), $-4\nu N_{11}N_{22} \geq 2\nu(N_{11}^2 + N_{22}^2)$. Since $2\nu(N_{11}^2 + N_{22}^2) \leq 0$, we have, independently of the sign of $N_{11}N_{22}$,

$$-4\nu N_{11}N_{22} \geq 2\nu(N_{11}^2 + N_{22}^2). \tag{4.49}$$

Therefore, collecting together the three cases, we have

$$-4\nu N_{11}N_{22} \geq -2|\nu|(N_{11}^2 + N_{22}^2). \tag{4.50}$$

From (4.45) and (4.50), we have

$$|\hat{\nabla} a|^2 \geq \frac{1}{(Eh)^2}\{(1+|\nu|^2 - 2|\nu|)(N_{11}^2 + N_{22}^2) + 2(1+\nu)^2 N_{12}^2\} \geq \\ \geq \frac{(1-|\nu|)^2}{E^2h^2}\sum_{\alpha,\beta=1}^{2} N_{\alpha\beta}^2 = \frac{(1-|\nu|)^2}{E^2h^2}|\nabla^2\phi|^2. \tag{4.51}$$

II) Estimate from above.

By distinguishing the three cases as above, we get similarly

$$-4\nu N_{11}N_{22} \leq 2|\nu|(N_{11}^2 + N_{22}^2). \tag{4.52}$$

From (4.45) and (4.52), we get the right hand side of (4.43).

□

5 Proof of the main result

Proposition 5.1 (Lipschitz Propagation of Smallness). *Let* Ω *be a domain satisfying* (3.2), (3.4). *Let* D *be an open subset of* Ω *satisfying* (3.1), (3.3), (3.5). *Let* $a \in H^1(\Omega \setminus \overline{D}, \mathbb{R}^2)$ *be the solution to* (3.19)–(3.21), *satisfying* (3.23). *Let the elasticity tensor* $\mathbb{C} = \mathbb{C}(x) \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ *given by* (3.13), *with Lamé moduli* $\lambda = \lambda(x)$, $\mu = \mu(x)$ *satisfying* (3.15) *and* (3.18). *Let* $\hat{N} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, $\hat{N} \not\equiv 0$, *satisfying* (3.9)–(3.11). *Then, there exists* $s > 1$, *only depending on* α_0, γ_0, Λ_0 *and* M_0, *such that for every* $\rho > 0$ *and every* $\bar{x} \in (\Omega \setminus \overline{D})_{s\rho}$, *we have*

$$\int_{B_\rho(\bar{x})} |\widehat{\nabla} a|^2 \geq \frac{C r_0^2}{\exp\left[A\left(\frac{r_0}{\rho}\right)^B\right]} \|\hat{N}\|^2_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}, \tag{5.1}$$

where $A, B, C > 0$ *are positive constants only depending on* α_0, γ_0, Λ_0, M_0, M_1 *and* F.

Proof. The proof follows by merging the Lipschitz Propagation of Smallness estimate (3.5) contained in [M-R, Proposition 3.1], Korn inequalities (see, for instance, [Fr], [A-M-R]), trace inequalities ([L-M]) and equivalence relations for the $H^{-\frac{1}{2}}$ and H^{-1}-norms of the Neumann data $\hat{N}$ (see (3.9)–(3.10) in [M-R, Remark 3.4]). □

Proposition 5.2 (Finite Vanishing Rate in the Interior). *Under the hypotheses of Proposition 5.1, there exist* $\tilde{c}_0 < \frac{1}{2}$ *and* $C > 0$, *only depending on* α_0, γ_0 *and* Λ_0, *such that, for every* $\bar{r} \in (0, r_0)$ *and for every* $\bar{x} \in \Omega \setminus \overline{D}$ *such that* $B_{\bar{r}}(\bar{x}) \subset \Omega \setminus \overline{D}$, *and for every* $r_1 < \tilde{c}_0 \bar{r}$, *we have*

$$\int_{B_{r_1}(\bar{x})} |\widehat{\nabla} a|^2 \geq C \left(\frac{r_1}{\bar{r}}\right)^{\tau_0} \int_{B_{\bar{r}}(\bar{x})} |\widehat{\nabla} a|^2, \tag{5.2}$$

where $\tau_0 \geq 1$ *only depends on* α_0, γ_0, Λ_0, M_0, M_1, $\frac{r_0}{\bar{r}}$ *and* F.

Proof. We can introduce in $B_{\bar{r}}(\bar{x})$ a locally defined Airy's function ϕ associated to the solution a. The proof follows by adapting the arguments in the proof of the analogous Proposition 3.5 in [M-R-V2] which applies to Kirchhoff-Love plate equation. The main difference consists in estimating the L^2 norms of ϕ and $|\nabla\phi|$ appearing in (3.21) of [M-R-V2] in terms of the L^2 norm of $|\nabla^2\phi|$ and using (4.43), the stability estimate (3.24) and Proposition 5.1.

□

Proposition 5.3 (Finite Vanishing Rate at the Boundary). *Under the hypotheses of Proposition 5.1, there exist $\bar{c}_0 < \frac{1}{2}$ and $C > 0$, only depending on α_0, γ_0, Λ_0, M_0, α, such that, for every $\bar{x} \in \partial D$ and for every $r_1 < \bar{c}_0 r_0$, we have*

$$(5.3) \qquad \int_{B_{r_1}(\bar{x})\cap(\Omega\setminus\overline{D})} |\hat{\nabla} a|^2 \geq C \left(\frac{r_1}{r_0}\right)^{\tau} \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} |\hat{\nabla} a|^2,$$

where $\tau \geq 1$ only depends on α_0, γ_0, Λ_0, M_0, α, M_1 and F.

Proof. Let us consider the Airy's function ϕ associated to the solution a and defined in $R_{r_0,2M_0r_0}(\bar{x}) \cap \Omega \setminus \overline{D}$, which satisfies the partial differential equation

$$(5.4) \qquad \mathrm{div}\,(\mathrm{div}\,(\mathbb{L}\nabla^2\phi)) = 0, \quad \text{in } R_{r_0,2M_0r_0}(\bar{x}) \cap \Omega \setminus \overline{D},$$

or, equivalently,

$$(5.5) \quad \Delta^2\phi + 2Eh\nabla\left(\frac{1}{Eh}\right)\cdot\nabla(\Delta\phi) - Eh\Delta\left(\frac{\nu}{Eh}\right)\Delta\phi+$$
$$+ Eh\nabla^2\left(\frac{1+\nu}{Eh}\right)\cdot\nabla^2\phi = 0, \quad \text{in } R_{r_0,2M_0r_0}(\bar{x}) \cap \Omega \setminus \overline{D},$$

and the homogeneous Dirichlet conditions

$$(5.6) \qquad \phi = \phi_{,n} = 0, \quad \text{on } \partial D \cap R_{r_0,2M_0r_0}(\bar{x}).$$

Let us notice that, under our assumptions, the fourth order tensor $\mathbb{L}$ satisfies the strong convexity condition

$$(5.7) \qquad \mathbb{L}A \cdot A \geq \frac{1}{5h\Lambda_0}|A|^2, \qquad \text{in } \Omega,$$

for every 2×2 symmetric matrix A. We also notice that the coefficients of the terms involving second and third-order derivatives of ϕ in (5.5) are of class C^2 and C^3 in $R_{r_0,2M_0r_0}(\bar{x}) \cap \Omega \setminus \overline{D}$, respectively, with corresponding C^2 and C^3-norm bounded by a constant only depending on h, α_0, γ_0 and Λ_0. Therefore, we can apply the results obtained in [A-R-V]. Precisely, by Corollary 2.3 in [A-R-V], there exist $c < 1$, only depending on M_0 and α, and $C > 1$, only depending on α_0, γ_0, Λ_0, M_0 and α, such that, for every $r_1 < r_2 < cr_0$, we have

$$(5.8) \qquad \int_{B_{r_1}(\bar{x})\cap(\Omega\setminus\overline{D})} \phi^2 \geq C \left(\frac{r_1}{r_0}\right)^{\frac{\log B}{\log\frac{cr_0}{r_2}}} \int_{B_{r_0}(x)\cap(\Omega\setminus\overline{D})} \phi^2,$$

where $B > 1$ is given by

$$B = C\left(\frac{r_0}{r_2}\right)^C \frac{\int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} \phi^2}{\int_{B_{r_2}(\bar{x})\cap(\Omega\setminus\overline{D})} \phi^2}. \tag{5.9}$$

Let us choose $r_2 = \bar{c}_0 r_0$, with $\bar{c}_0 = \frac{c}{2}$. We need to estimate the quantity B. By applying Poincaré inequality (see, for instance, [A-M-R, Example 4.4]) and (4.43), we have

$$\int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} \phi^2 \leq C r_0^4 \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} |\nabla^2\phi|^2 = C r_0^4 \int_{B_{r_0}(\bar{x})\cap(\Omega\setminus\overline{D})} |\hat{\nabla} a|^2 \tag{5.10}$$

where $C > 0$ only depends on α_0, γ_0, Λ_0, M_0 and α. Moreover, by applying Lemma 4.7 in [A-R-V] and (4.43), and recalling the choice of r_2, we have

$$\int_{B_{r_2}(\bar{x})\cap(\Omega\setminus\overline{D})} \phi^2 \geq C r_2^4 \int_{B_{\frac{r_2}{2}}(\bar{x})\cap(\Omega\setminus\overline{D})} |\nabla^2\phi|^2 = C r_0^4 \int_{B_{\frac{cr_0}{4}}(\bar{x})\cap(\Omega\setminus\overline{D})} |\hat{\nabla} a|^2. \tag{5.11}$$

By (5.10)–(5.11), using the stability estimate of the direct problem (3.24) and Proposition 5.1, we can estimate $B \leq C$, with C only depending on α_0, γ_0, Λ_0, M_0, α, M_1 and F. By using again Poincaré inequality, Lemma 4.7 in [A-R-V] and (4.43), we obtain the thesis. □

From now on, we shall denote by G the connected component of $\Omega \setminus (D_1 \cup D_2)$ such that $\Sigma \subset \partial G$.

Proposition 5.4 (Stability Estimate of Continuation from Cauchy Data). *Under the hypotheses of Theorem 3.1, we have*

$$\int_{(\Omega\setminus\overline{G})\setminus D_1} |\hat{\nabla} a^{(1)}|^2 \leq r_0^2 \|\hat{N}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}\, \omega\left(\frac{\varepsilon}{\|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.12}$$

$$\int_{(\Omega\setminus\overline{G})\setminus D_2} |\hat{\nabla} a^{(2)}|^2 \leq r_0^2 \|\hat{N}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}\, \omega\left(\frac{\varepsilon}{\|\hat{N}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.13}$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(t) \leq C(\log|\log t|)^{-\frac{1}{2}}, \qquad \text{for every } t < e^{-1}, \tag{5.14}$$

with $C > 0$ only depending on α_0, γ_0, Λ_0, M_0, α and M_1. Moreover, there exists $d_0 > 0$, with $\frac{d_0}{r_0}$ only depending on M_0 and α, such that if $d_H(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}) \leq d_0$ then (5.12)–(5.13) hold with ω given by

$$\omega(t) \leq C|\log t|^{-\sigma}, \qquad \text{for every } t < 1, \tag{5.15}$$

where $\sigma > 0$ and $C > 0$ only depend on α_0, γ_0, Λ_0, M_0, α, M_1.

Proof. The proof can be easily obtained by adapting the proof of the analogous estimates contained in Proposition 3.5 and Proposition 3.6 in [M-R]. The only difference consists in replacing the auxiliary function $w = a^{(1)} - a^{(2)}$ with $w = a^{(1)} - a^{(2)} - r$, where $r \in \mathcal{R}_2$ is the minimizer of problem (3.25), and noticing that $\hat{\nabla} r = 0$. □

Proof of Theorem 3.1. It is convenient to introduce the following auxiliary distances:

$$d = d_H(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}), \tag{5.16}$$

$$d_m = \max \left\{ \max_{x \in \partial D_1} \mathrm{dist}(x, \overline{\Omega \setminus D_2}), \max_{x \in \partial D_2} \mathrm{dist}(x, \overline{\Omega \setminus D_1}) \right\}. \tag{5.17}$$

Let $\eta > 0$ such that

$$\max_{i=1,2} \int_{(\Omega \setminus \overline{G}) \setminus \overline{D_i}} |\hat{\nabla} a^{(i)}|^2 \leq \eta. \tag{5.18}$$

Step 1. Let us assume $\eta \leq r_0^2 \|\hat{N}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}$. We have

$$d_m \leq C r_0 \left(\frac{\eta}{r_0^2 \|\hat{N}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right)^{\frac{1}{\tau}}, \tag{5.19}$$

where τ has been introduced in Proposition 5.3 and C is a positive constant only depending on the a priori data.

Proof. Without loss of generality, let $x_0 \in \partial D_1$ such that

$$\mathrm{dist}(x_0, \overline{\Omega \setminus D_2}) = d_m > 0. \tag{5.20}$$

Since $B_{d_m}(x_0) \subset D_2 \subset \Omega \setminus \overline{G}$, we have

$$B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1}) \subset (\Omega \setminus \overline{G}) \setminus \overline{D_1} \tag{5.21}$$

and then, by (5.18),

$$\int_{B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1})} |\hat{\nabla} a^{(1)}|^2 \leq \eta. \tag{5.22}$$

Let us distinguish two cases. First, let

$$d_m < \overline{c}_0 r_0, \tag{5.23}$$

where $\bar{c}_0$ is the positive constant appearing in Proposition 5.3. By applying this proposition, we have

$$\eta \geq C \left(\frac{d_m}{r_0}\right)^{\tau} \int_{B_{r_0}(x_0)\cap(\Omega\setminus \overline{D_1})} |\widehat{\nabla} a^{(1)}|^2, \tag{5.24}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0, α, M_0, M_1 and F.

By Proposition 5.1, we have

$$\eta \geq C \left(\frac{d_m}{r_0}\right)^{\tau} r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}, \tag{5.25}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0, α, M_0, M_1, F, from which we can estimate d_m, obtaining (5.19).

As second case, let

$$d_m \geq \bar{c}_0 r_0. \tag{5.26}$$

By starting again from (5.22), applying Proposition 5.1 and recalling $d_m \leq M_1 r_0$, we have

$$d_m \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{5.27}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0, M_0, M_1, F. Since we have assumed $\eta \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$, also in this case we obtain (5.19). $\square$

Step 2. Let us assume $\eta \leq r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}$. *We have*

$$d \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}}\right)^{\frac{1}{\tau_1}}, \tag{5.28}$$

with $\tau_1 = \max\{\tau, \tau_0\}$ *and* $C > 0$ *only depends on* α_0, γ_0, Λ_0, α, M_0, M_1 *and* F.

Proof. We may assume that $d > 0$ and there exists $y_0 \in \overline{\Omega \setminus D_1}$ such that

$$\mathrm{dist}(y_0, \overline{\Omega \setminus D_2}) = d. \tag{5.29}$$

Since $d > 0$, we have $y_0 \in D_2 \setminus D_1$. Let

$$h = \mathrm{dist}(y_0, \partial D_1), \tag{5.30}$$

possibly $h = 0$.

There are three cases to consider:

I) $h \le \frac{d}{2}$;
II) $h > \frac{d}{2}$, $h \le \frac{d_0}{2}$;
III) $h > \frac{d}{2}$, $h > \frac{d_0}{2}$.

Here the number d_0, $0 < d_0 < r_0$, is such that $\frac{d_0}{r_0}$ only depends on M_0, and it is the same constant appearing in Proposition 5.4. In particular, Proposition 3.6 in [Al-Be-Ro-Ve] shows that there exists an absolute constant $C > 0$ such that if $d \le d_0$, then $d \le C d_m$.

Case i).

By definition, there exists $z_0 \in \partial D_1$ such that $|z_0 - y_0| = h$. By applying the triangle inequality, we get $\mathrm{dist}\left(z_0, \overline{\Omega \setminus D_2}\right) \ge \frac{d}{2}$. Since, by definition, $\mathrm{dist}\left(z_0, \overline{\Omega \setminus D_2}\right) \le d_m$, we obtain $d \le 2d_m$.

Case ii).

It turns out that $d < d_0$ and then, by the above recalled property, again we have that $d \le C d_m$, for an absolute constant C.

Case iii).

Let $\tilde{h} = \min\{h, r_0\}$. We obviously have that $B_{\tilde{h}}(y_0) \subset \Omega \setminus \overline{D_1}$ and $B_d(y_0) \subset D_2$. Let us set

$$d_1 = \min\left\{\frac{d}{2}, \frac{\tilde{c}_0 d_0}{4}\right\},$$

where $\tilde{c}_0$ is the positive constant appearing in Proposition 5.2. Since $d_1 < d$ and $d_1 < \frac{\tilde{h}}{2}$, we have that $\overline{B_{d_1}(y_0)} \subset D_2 \setminus D_1$ and therefore $\eta \ge \int_{B_{d_1}(y_0)} |\nabla \hat{a}^{(1)}|^2$.

Since $\frac{d_0}{2} < \tilde{h}$, $B_{\frac{d_0}{2}}(y_0) \subset \Omega \setminus \overline{D_1}$ so that we can apply Proposition 5.2 with $d = d_1$, $r = \frac{d_0}{2}$, obtaining $\eta \ge C \left(\frac{2d_1}{d_0}\right)^{\tau_0} \int_{B_{\frac{d_0}{2}}(y_0)} |\nabla \hat{a}^{(1)}|^2$, with $C > 0$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1$ and F. Next, by Proposition 5.1, recalling that $\frac{d_0}{r_0}$ only depends on M_0, we derive that

$$d_1 \le C r_0 \left(\frac{\eta}{r_0^2 \|\hat{M}\|^2_{H^{-1/2}(\partial\Omega, \mathbb{R}^2)}} \right)^{\frac{1}{\tau_0}},$$

where $C > 0$ only depends on α_0, γ_0, Λ_0, M_0, M_1 and F. For η small enough, $d_1 < \frac{\tilde{c}_0 d_0}{4}$, so that $d_1 = \frac{d}{2}$ and

$$d \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}} \right)^{\frac{1}{\tau_0}},$$

where $C > 0$ only depends on α_0, γ_0, Λ_0, M_0, M_1 and F. Collecting the three cases, the thesis follows. □

Step 3. We have

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq \sqrt{1 + M_0^2}\, d. \tag{5.31}$$

Proof. The proof is based on purely geometrical arguments, we refer to[M-R-V2, Proof of Theorem 3.1, Step 3]. □

Conclusion. By Proposition 5.4,

$$d \leq C r_0 \left(\log \left| \log \left(\frac{\varepsilon}{\|\widehat{N}\|^2_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)}} \right) \right| \right)^{-\frac{1}{2\tau_1}}, \tag{5.32}$$

with $\tau_1 \geq 1$ and $C > 0$ only depends on α_0, γ_0, Λ_0, α, M_0, M_1 and F. By this first rough estimate, there exists $\varepsilon_0 > 0$, only depending on on α_0, γ_0, Λ_0, α, M_0, M_1 and F, such that, if $\varepsilon \leq \varepsilon_0$, then $d \leq d_0$. Therefore, we can apply the second statement of Proposition 5.4, obtaining the thesis. □

6 Generalized Plane Stress problem

In this section we derive the Generalized Plane Stress (GPS) problem for the statical equilibrium of a thin elastic plate under in-plane boundary loads. Our analysis follows the classical approach of the theory of structures, according to the original idea introduced by Filon [Fil]. Alternative, more formal derivations have been proposed to justify the GPS problem. The interested reader can refer, among others, to the contributions [C-D], [A-B-P] and [P].

Let $\mathcal{U}$ be a bounded domain in $\mathbb{R}^2$, and consider the cylinder $\mathcal{C} = \mathcal{U} \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ with middle plane $\mathcal{U} \times \{x_3 = 0\}$ (which we will simply denote by $\mathcal{U}$ in what follows) and thickness h. Here, $\{O, x_1, x_2, x_3\}$ is a Cartesian coordinate system, with origin O belonging to the plane $x_3 = 0$ and axis x_3 orthogonal to U. Such cylinder is called *plate* if h is small with respect to the linear dimensions of U, e.g., $h << diam(\mathrm{U})$.

Let us suppose that the faces $\mathrm{U} \times \{x_3 = \pm \frac{h}{2}\}$ of the plate are free of applied loads, and all external surface forces acting on the lateral surface $\partial \mathrm{U} \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ lie in planes parallel to the middle plane U, and are independent of x_3. We shall further assume that body forces vanish in C. The plate is assumed to be made by linearly elastic isotropic material, with Lamé moduli independent of the x_3-coordinate, e.g., $\lambda = \lambda(x_1, x_2)$, $\mu = \mu(x_1, x_2)$ for every $(x_1, x_2, 0) \in \overline{\mathrm{U}}$. Moreover, let $\lambda, \mu \in C^{0,1}(\overline{\mathrm{U}})$ and such that $\mu \geq \alpha_0$, $2\mu + 3\lambda \geq \gamma_0$ in $\overline{\mathrm{U}}$, with α_0, γ_0 positive constants.

Under the above assumptions, the problem of elastostatics consists in finding a displacement u solution to

$$
\begin{cases}
T_{ij,j} = 0, & \text{in } \mathrm{C}, \quad (6.1)\\
T_{i3} = 0, & \text{on } \mathrm{U} \times \{x_3 = \pm\frac{h}{2}\}, \quad (6.2)\\
T_{\alpha\beta} n_\beta = \hat{t}_\alpha, & \text{on } \partial\mathrm{U} \times \left(-\frac{h}{2}, \frac{h}{2}\right), \quad (6.3)\\
T_{3\beta} n_\beta = 0, & \text{on } \partial\mathrm{U} \times \left(-\frac{h}{2}, \frac{h}{2}\right), \quad (6.4)\\
T_{ij} = 2\mu E_{ij} + \lambda (E_{kk})\delta_{ij}, & \text{in } \mathrm{C}, \quad (6.5)\\
E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), & \text{in } \mathrm{C}, \quad (6.6)
\end{cases}
$$

where the force field $\hat{t} = (\hat{t}_1, \hat{t}_2, 0)$, with $\hat{t}_\alpha = \hat{t}_\alpha(x_1, x_2)$, $\alpha = 1, 2$, assigned on $\partial\mathrm{U} \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ satisfies the compatibility conditions

$$
(6.7) \qquad \int_{\partial\mathrm{U}\times(-\frac{h}{2},\frac{h}{2})} \hat{t} = 0, \qquad \int_{\partial\mathrm{U}\times(-\frac{h}{2},\frac{h}{2})} x \times \hat{t} = 0,
$$

‾

$2 \quad 2 \qquad\qquad 2 \; 2$

see, for example, [G, §45]. The above boundary value problem is called *plane*

problem of elastostatics. It is known that, under our assumptions and for

— 1 2 1 3

$\hat{t}_\alpha \in H^{\frac{1}{2}}(\partial U,$

is unique up to an infinitesimal rigid displacement $r(x) = a + b \times x$, with *a, b* $\in \mathrm{R}^3$ constant vectors.

We now formulate the Generalized Plane Stress (GPS) problem associated to (6.1)–(6.6). The GPS problem is a two-dimensional boundary value problem formulated in terms of the thickness averages of u, E and T, under the a priori assumption

$$T_{33} = 0, \quad \text{in } C. \tag{6.8}$$

For a physically plausible justification of the above assumption under the hypothesis of small h, we refer to [S, §67] and to the paper [Fil] by Filon, who first derived the GPS problem.

Given a function $f : \mathcal{C} \to \mathbb{R}^3$, $f \in H^1(\mathcal{C})$, let us define the function $\tilde{f} : \mathcal{C} \to \mathbb{R}^3$ as follows:

$$
\begin{cases}
\tilde{f}_1(x_1, x_2, x_3) = f_1(x_1, x_2, -x_3), & (6.9)\\
\tilde{f}_2(x_1, x_2, x_3) = f_2(x_1, x_2, -x_3), & (6.10)\\
\tilde{f}_3(x_1, x_2, x_3) = -f_3(x_1, x_2, -x_3). & (6.11)
\end{cases}
$$

By definition of the plane problem, if u is a solution to (6.1)–(6.6), then also $\tilde{u}$ is a solution of the same problem. Moreover, $(u - \tilde{u})$ is a solution to (6.1)–(6.6) with $\hat{t} = 0$ and, therefore, $(u - \tilde{u}) \in R_3$. Noticing that $(u_1 - \tilde{u}_1)|_{x_3=0} = (u_2 - \tilde{u}_2)|_{x_3=0} = 0$, we have $u - \tilde{u} = a_3 e_3 + (b_1 e_1 + b_2 e_2) \times \sum_{i=1}^{3} x_i e_i$, with $a_3, b_1, b_2 \in \mathbb{R}$. Now, it is easy to see that, choosing $r' \in R_3$ as $r' = \sum_{i=1}^{3} a'_i e_i + \sum_{i=1}^{3} b'_i e_i \times \sum_{i=1}^{3} x_i e_i$, with $a'_3 = -\frac{a_3}{2}$, $b'_1 = -\frac{b_1}{2}$, $b'_2 = -\frac{b_2}{2}$, the solution $u + r'$ to (6.1)–(6.6) satisfies the condition $u + r' = (\widetilde{u + r'})$, for every $a'_1, a'_2, b'_3 \in \mathbb{R}$.

We next introduce the *thickness average* $\bar{f}$ of a function $f : \mathcal{C} \to \mathbb{R}^3$, $\bar{f} : \mathcal{U} \to \mathbb{R}$, defined as

$$
\bar{f}(x_1, x_2) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x_1, x_2, x_3)\, dx_3. \tag{6.12}
$$

Taking into account that the thickness average of an x_3-odd function is zero, and the x_3-derivative of an x_3-even function is x_3-odd, for every point $(x_1, x_2) \in \mathcal{U}$ we have

$$
\begin{cases}
\bar{u}_3 = \bar{E}_{\alpha 3} = \bar{T}_{\alpha 3} = 0, \quad \alpha = 1, 2, & (6.13)\\
\bar{E}_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha}), \quad \alpha, \beta = 1, 2, & (6.14)\\
\bar{T}_{\alpha\beta} = 2\mu \bar{E}_{\alpha\beta} + \lambda(\bar{E}_{\gamma\gamma} + \bar{E}_{33})\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, & (6.15)\\
\bar{T}_{33} = 2\mu \bar{E}_{33} + \lambda(\bar{E}_{\gamma\gamma} + \bar{E}_{33}), & (6.16)
\end{cases}
$$

where the solution $u + r'$ is denoted by u. Using the a priori assumption (6.8) in (6.16), the function $\bar{E}_{33}$ can be expressed in terms of $\bar{E}_{\gamma\gamma}$, and the *two-dimensional* constitutive equation can be written as

$$
\bar{T}_{\alpha\beta} = 2\mu \bar{E}_{\alpha\beta} + \lambda^* \bar{E}_{\gamma\gamma}\delta_{\alpha\beta}, \tag{6.17}
$$

with

$$
\lambda^* = \frac{2\mu\lambda}{\lambda + 2\mu}. \tag{6.18}
$$

Integrating on the thickness in (6.1)–(6.6), and neglecting those equations which yield to identities, we obtain the averaged equations of equilibrium and the corresponding boundary conditions, and $\bar{u} \in H^1(\mathcal{U}, \mathbb{R}^2)$ is a solution to

$$
\begin{cases}
\overline{T}_{\alpha\beta,\beta} = 0, & \text{in } U, \quad (6.19)\\
\overline{T}_{\alpha\beta} n_\beta = \hat{t}_\alpha, & \text{on } \partial U, \quad (6.20)\\
\overline{T}_{\alpha\beta} = 2\mu \overline{E}_{\alpha\beta} + \lambda^*(\overline{E}_{\gamma\gamma})\delta_{\alpha\beta}, & \text{in } U, \quad (6.21)\\
\overline{E}_{\alpha\beta} = \frac{1}{2}(\overline{u}_{\alpha,\beta} + \overline{u}_{\beta,\alpha}), & \text{in } U, \quad (6.22)
\end{cases}
$$

where the force field $\hat{t} = \hat{t}_1 e_1 + \hat{t}_2 e_2$ applied on ∂U satisfies the compatibility conditions

$$\int_{\partial U} \hat{t} = 0, \qquad \int_{\partial U} x \times \hat{t} = 0. \tag{6.23}$$

Let us notice that the constitutive equation (6.21) can be written as

$$\overline{T}_{\alpha\beta} = \frac{E}{1-\nu^2}\left((1-\nu)\overline{E}_{\alpha\beta} + \nu(\overline{E}_{\gamma\gamma})\delta_{\alpha\beta}\right), \tag{6.24}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{6.25}$$

where E, ν are the Young's modulus and the Poisson's coefficient of the material, respectively. Finally, by defining

$$a_\alpha = \overline{u}_\alpha, \quad \varrho_{\alpha\beta} = \overline{E}_{\alpha\beta} = \hat{\nabla} a, \quad N_{\alpha\beta} = h\overline{T}_{\alpha\beta}, \quad \hat{N}_\alpha = h\hat{t}_\alpha, \quad \alpha, \beta = 1, 2, \tag{6.26}$$

we obtain the GPS problem

$$
\begin{cases}
N_{\alpha\beta,\beta} = 0, & \text{in } U, \quad (6.27)\\
N_{\alpha\beta} n_\beta = \hat{N}_\alpha, & \text{on } \partial U, \quad (6.28)\\
N_{\alpha\beta} = \frac{Eh}{1-\nu^2}\left((1-\nu)\varrho_{\alpha\beta} + \nu(\varrho_{\gamma\gamma})\delta_{\alpha\beta}\right), & \text{in } \overline{U}, \quad (6.29)\\
\varrho_{\alpha\beta} = \frac{1}{2}(a_{\alpha,\beta} + a_{\beta,\alpha}), & \text{in } U, \quad (6.30)
\end{cases}
$$

with

$$\int_{\partial U} \hat{N} \cdot r = 0, \quad \text{for every } r \in \mathbf{R}_2. \tag{6.31}$$

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References

[A-E] V. Adolfsson, L. Escauriaza, $C^{1,\alpha}$ domains and unique continuation at the boundary, Comm. Pure Appl. Math. L (1997) 935–969.

[Ai] G.B. Airy, On the strains in the interior of beams, Phil. Trans. Roy. Soc. Lond. 153 (1863) 49–80.

[Al] G. Alessandrini, Examples of instability in inverse boundary-value problems, Inverse Problems 13 (1997) 887–897.

[Al-Be-Ro-Ve] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 29(4) (2000) 755–806.

[A-M-R] G. Alessandrini, A. Morassi, E. Rosset, The linear constraints in Poincaré and Korn type inequalities, Forum Math. 20 (2008) 557–569.

[A-R-V] G. Alessandrini, E. Rosset, S. Vessella, Optimal three spheres inequality at the boundary for the Kirchhoff-Love plate's equations with Dirichlet conditions, Arch. Ration. Mech. Anal. 231 (2019) 1455–1486.

[A-B-P] G. Anzellotti, S. Baldo, D. Percivale, Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity, Asymptotic Anal. 9 (1994) 61–100.

[C-D] P.G. Ciarlet, P. Destuynder, A justification of the two-dimensional linear plate model, J. Mécanique 18 (1979) 227–258.

[DiC-R] M. Di Cristo, L. Rondi, Examples of exponential instability for elliptic inverse problems, Inverse Problems 19 (2003) 685–701.

[Fic] G. Fichera, *Problemi analitici nuovi nella fisica matematica classica*, Quaderni del Consiglio Nazionale delle Ricerche, Gruppo Nazionale di Fisica Matematica, Napoli, 1985.

[Fil] L.N.G. Filon, On an approximative solution for the bending of a beam of rectangular cross-section under any system of load, with special reference to points of concentrated or discontinuous loading, Phil. Trans. Roy. Soc. Lond., Ser. A 201 (1903) 63–155.

[Fr] K.O. Friedrichs, On the boundary value problems of the theory of elasticity and Korn's inequality, Annals of Math. 48 (1947) 441–471.

[G] M. E. Gurtin, *The Linear Theory of Elasticity*, Handbüch der Physik, vol. VI, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[L-U-W] R.-Y. Lai, G. Uhlmann, J.-N. Wang, Inverse boundary value problem for the Stokes and the Navier-Stokes equations in the plane, Arch. Ration. Mech. Anal. 215 (2015) 811–829.

[L-M] J. L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications I*, Springer-Verlag, Berlin, 1972.

[Ma] N. Mandache, Exponential instability in an inverse problem for the Schrödinger equation, Inverse Problems 17 (2001) 1435–1444.

[M-R] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, Inverse Problems 20 (2004) 453–480.

[M-R-V1] A. Morassi, E. Rosset, S. Vessella, Size estimates for inclusions in an elastic plate by boundary measurements, Indiana Univ. Math. J. 56(5) (2007) 2535–2384.

[M-R-V2] A. Morassi, E. Rosset, S. Vessella, Optimal stability in the identification of a rigid inclusion in an isotropic Kirchhoff-Love plate, SIAM J. Math. Anal. 51(2) (2019) 731–747.

[M-R-V3] A. Morassi, E. Rosset, S. Vessella, Doubling inequality at the boundary for the Kirchhoff-Love plate's equation with Dirichlet conditions, arXiv:1906.08642*v*1 (2019).

[P] R. Paroni, The equations of motion of a plate with residual stress, Meccanica 41 (2006) 1–21.

[S] I.S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1956.