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Determining an anisotropic conductivity by boundary measurements: Stability at the boundary

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Abstract

We consider the inverse problem of determining, the possibly anisotropic, conductivity of a body $\Omega \subset \mathbb{R}^n$, $n \geq 3$, by means of the so-called local Neumann-to-Dirichlet map on a curved portion Σ of its boundary $\partial\Omega$. Motivated by the uniqueness result for piecewise constant anisotropic conductivities proved in Inverse Problems 33 (2018), 125013, we provide a Hölder stability estimate on Σ when the conductivity is *a-priori* known to be a constant matrix near Σ.

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1. Introduction

Given $\psi : \partial\Omega \longrightarrow \mathbb{R}$, with zero average, consider the Neumann problem

$$\begin{cases} \mathrm{div}(\sigma\nabla u) = 0, & \text{in} \quad \Omega, \\ \sigma\nabla u \cdot \nu = \psi, & \text{on} \quad \partial\Omega, \end{cases}$$

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where $\sigma = \left\{\sigma_{ij}(x)\right\}_{i,j=1}^{n}$, with $x \in \Omega$, satisfies the uniform ellipticity condition

$$\lambda^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \lambda|\xi|^2, \text{ for a.e. } x \in \Omega \text{ , for every } \xi \in \mathbb{R}^n \tag{1}$$

for some positive constant λ. Here, a standard, variational, functional framework is understood. Details will be presented in what follows. Electrical Impedance Tomography (EIT) is the inverse problem of determining the conductivity σ when the Neumann-to-Dirichlet (N-D) map

$$\mathcal{N}_\sigma : \psi \longrightarrow u\big|_{\partial\Omega}$$

is given, [8]. It is well-known that if σ is allowed to be anisotropic, i.e. a full matrix, although symmetric, then it is not uniquely determined by $\mathcal{N}_\sigma$. In fact, if $\Phi : \overline{\Omega} \longrightarrow \overline{\Omega}$ is a diffeomorphism such that $\Phi\big|_{\partial\Omega} = I$, then σ and its push-forward under Φ,

$$\Phi^*\sigma = \frac{(D\Phi)\sigma(D\Phi)^T}{\det(D\Phi)} \circ \Phi^{-1}$$

give rise to the same N-D map. This construction is due to Tartar, as reported by Kohn and Vogelius [11]. A prominent line of research investigates the determination of σ modulo diffeomorphisms which fix the boundary, in this respect we refer to the seminal paper of Lee and Uhlmann [13]. From another point of view, anisotropy cannot be neglected in applications, such as medical imaging or geophysics. It is therefore important to investigate possible kinds of structural assumptions (physically motivated) under which unique determination of σ from $\mathcal{N}_\sigma$ is restored.

In [3] the case of a piecewise constant conductivity σ was treated, and assuming that the interfaces of discontinuity contain portions of curved (non flat) hypersurfaces, uniqueness was proven. Subsequently [4], uniqueness was proven also in cases of a layered structure with unknown interfaces. We refer to these two papers for a bibliography on the relevance of anisotropy in applications.

It is still an open problem, to prove, in such settings, estimates of stability. Indeed, a line of research initiated by Alessandrini and Vessella [5] in the isotropic case (i.e.: $\sigma = \gamma I$, with γ scalar), suggests that also in the setting of Alessandrini-de Hoop-Gaburro [3] a Lipschitz stability estimate might hold. Such a generalization, however, does not appear to be an easy task, because isotropy intervenes in many steps of the proof in [5].

In this note we treat the first step in a program to prove stability for piecewise constant anisotropic conductivity with curved interfaces. More precisely, assuming σ constant in a neighborhood $\mathcal{U}$ of a curved portion Σ of the boundary $\partial\Omega$, we show that $\sigma\big|_{\mathcal{U}}$ depends in a Hölder fashion on the local N-D map $\mathcal{N}_\sigma^\Sigma$, namely

$$\left|\left|\sigma^{(1)}(y) - \sigma^{(2)}(y)\right|\right|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \leq C E^{1-\beta} ||\mathcal{N}_{\sigma^{(1)}}^\Sigma - \mathcal{N}_{\sigma^{(2)}}^\Sigma||^\beta_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\Sigma),({}_0H^{-\frac{1}{2}}(\Sigma))^*\right)}, \tag{2}$$

(see Theorem 2.1 for details). Let us emphasize the presence in explicit form of the term E in (2), the L^∞-norm of the difference $\sigma^{(1)} - \sigma^{(2)}$ in Ω. It is obvious that it could be majorized by the ellipticity constant λ in (1). However, in view of the technique introduced in [5], we leave it in such a form, because it would be instrumental for the goal of global Lipschitz stability under the assumption that σ is piecewise constant on a given finite partition of Ω. See the Example in Section 2 below, for a simple case where this approach succeeds.

Since Kohn and Vogelius [10], [12] and Alessandrini [1], [2], it is customary to treat the uniqueness and stability at the boundary, as a first step towards determination in the interior. And also in the anisotropic case we wish to mention the results of Kang and Yun [9] who proved reconstruction and stability at the boundary up to diffeomorphisms which keep the boundary fixed.

Here, assuming a quantitative formulation of non-flatness of Σ, we are able to stably determine the full conductivity matrix σ (near Σ). More precisely, we shall assume that there exists three points $P_1, P_2, P_3 \in \Sigma$ such that the corresponding unit normal vectors to Σ, $\nu(P_1), \nu(P_2), \nu(P_3)$ are quantitatively pairwise distinct.

As is well-known, in the isotropic case, the boundary determination of the conductivity has a character of stability of Lipschitz type, whereas here only Hölder stability is obtained. The presence of anisotropy has required a new approach, no claim of optimality is made. Let us stress however that, since the original result of boundary Lipschitz stability by Sylvester and Uhlmann [15], this is apparently the first boundary stability estimate in the anisotropic case, which is valid in a fixed coordinates system that is not modulo a suitable diffeomorphism.

The stability result obtained here is given in terms of a local map (the local N-D map). The problem of (stably) determining the conductivity from local measurements has been studied more recently. Results in this direction, limited to piecewise constant coefficients, include [5] and [6] for the isotropic case. The general inverse problem of stably determining anisotropic conductivities in terms of local measurements, however, has remained open.

Our argument here is based on various new features. As noticed in [3], the local N-D map $\mathcal{N}_\sigma^\Sigma$ is an integral operator whose kernel K differs from the well-known Neumann kernel N by a bounded correction term.

The common feature of the two kernels is the character of their singularity which in turn encapsulates information on the tangential part of the metric $\{g_{ij}\}_{i,j=1}^n$ associated to the conductivity σ

$$g_{ij} = (\det \sigma)^{\frac{1}{n-2}} (\sigma_{ij}^{-1}) \tag{3}$$

in dimension $n \geq 3$. By testing $\mathcal{N}_\sigma^\Sigma$ on suitable combination of mollified δ functions, we achieve a quantitative evaluation of the tangential component of $\{g_{ij}\}$. Next, by exploiting the quantitative notion of 'non-flatness' of Σ, we show that the full metric $\{g_{ij}\}$ can be recovered from three tangential samples at three different points with sufficiently distinct tangent spaces, or, equivalently, pairwise distinct normal vectors.

The paper is organized as follows. Section 2 contains the main definitions, including the quantified notion of non-flatness of Σ where we localize the measurements $\mathcal{N}_\sigma^\Sigma$ (Section 2.1). This section also contains the statement of our main result of stability at the boundary of anisotropic conductivities σ that are constant near Σ in terms of $\mathcal{N}_\sigma^\Sigma$ (Section 2.2, Theorem 2.1). We further illustrate in an example how the Hölder stability result at the boundary of Theorem 2.1 can lead to Lispchitz stability in the interior. Section 3 is devoted to the construction of mollified delta functions on Σ. The proof of the main result is a two-step procedure. In the first step, contained in Section 4, we stably recover the tangential component of g in terms of $\mathcal{N}_\sigma^\Sigma$. The argument of this proof is based on estimating the asymptotic behavior of the Neumann kernel $N(\cdot, y)$ of $L = \mathrm{div}(\sigma\nabla\cdot)$ and its derivative, near the pole $y \in \Sigma$, and the sampling $\mathcal{N}_\sigma^\Sigma$ on suitable combinations of the mollifiers given in Section 3. In the second step, discussed in Section 5, exploiting the non-flatness condition on Σ as well as the structure of the metric g, we derive the stability on the boundary for the full metric g which in turn leads to the stable determination of the conductivity σ on Σ.

2. Main result

2.1. Notation and assumptions

In several places in this paper it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geq 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$ and given $a, b \in \mathbb{R}$, we shall denote with $B_r(x)$, $B_r(x')$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ centered at x, x', respectively, with radius r and by $Q_{a,b}(x)$ the cylinder $\{y = (y', y_n) \in \mathbb{R}^n \ : \ |x' - y'| < a \ ; \ |x_n - y_n| < b\}$. It will also be understood that $B_r = B_r(0)$, $B'_r = B_r(0')$ and $Q_{a,b} = Q_{a,b}(0)$.

We shall assume throughout that $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, is a bounded domain with Lipschitz boundary, as per Definition 2.1 below.

Definition 2.1. We will say that $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$, if for every $P \in \partial\Omega$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap Q_{r_0, r_0 L} = \left\{(x', x_n) \in Q_{r_0, r_0 L} \mid x_n > \varphi(x')\right\},$$

where φ is a Lipschitz continuous function on B'_{r_0} satisfying

$$\varphi(0) = 0$$

and

$$|\varphi(x') - \varphi(y')| \leq L|x' - y'|, \quad \text{for every } \ x', y' \in B'_{r_0}.$$

We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local N-D map are taken). A precise definition of the N-D map and its local version with respect to Σ are given below.

2.1.1. The Neumann-to-Dirichlet map

Denoting by Sym_n the class of $n \times n$ symmetric real valued matrices, we assume that $\sigma \in L^\infty(\Omega\,, Sym_n)$ satisfies the ellipticity condition (1). We consider the function spaces

$$_0H^{\frac{1}{2}}(\partial\Omega) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) | \int_{\partial\Omega} f \ = 0 \right\},$$

$$_0H^{-\frac{1}{2}}(\partial\Omega) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) | \ \langle \psi,\ 1\rangle = 0 \right\}.$$

The global Neumann-to-Dirichlet map is then defined as follows.

Definition 2.2. The Neumann-to-Dirichlet (N-D) map associated with σ,

$$\mathcal{N}_\sigma : {}_0H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow {}_0H^{\frac{1}{2}}(\partial\Omega)$$

is characterized as the selfadjoint operator satisfying

$$\langle \psi, \mathcal{N}_\sigma \psi \rangle = \int_\Omega \sigma(x)\nabla u(x) \cdot \nabla u(x)\, dx,$$

for every $\psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to the Neumann problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0, & \text{in} \quad \Omega, \\ \sigma\nabla u \cdot \nu|_{\partial\Omega} = \psi, & \text{on} \quad \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u = 0 \end{cases}$$

and $\langle\cdot,\cdot\rangle$ denotes the $L^2(\partial\Omega)$-pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega)$.

For the local version of the N-D map, given the open portion Σ introduced above, and, denoting by $\Delta = \partial\Omega \setminus \overline{\Sigma}$ we introduce the closed subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H^{\frac{1}{2}}_{00}(\Delta) = \overline{\left\{ f \in H^{\frac{1}{2}}(\partial\Omega) \mid \operatorname{supp}(f) \subset \Delta \right\}},$$

where the closure is meant in $H^{\frac{1}{2}}(\partial\Omega)$-norm. Let us observe that the space $H^{\frac{1}{2}}_{00}(\Delta)$ might be also characterized as the interpolation space $[H^1_0(\Delta), L^2(\Delta)]_{\frac{1}{2}}$ (see [14], Chapter 1).

We also introduce

$${}_0H^{-\frac{1}{2}}(\Sigma) = \left\{ \psi \in {}_0H^{-\frac{1}{2}}(\partial\Omega) |\ \langle \psi, f\rangle = 0, \quad \text{for any } f \in H^{\frac{1}{2}}_{00}(\Delta) \right\},$$

that is the space of distributions $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ which are supported in $\overline{\Sigma}$ and have zero average on $\partial\Omega$. The local N-D map is then defined as follows.

Definition 2.3. The local Neumann-to-Dirichlet map associated with σ, Σ is the operator $\mathcal{N}^\Sigma_\sigma$: ${}_0H^{-\frac{1}{2}}(\Sigma) \longrightarrow \left({}_0H^{-\frac{1}{2}}(\Sigma)\right)^* \subset {}_0H^{\frac{1}{2}}(\partial\Omega)$ given by

$$\langle \mathcal{N}^\Sigma_\sigma \phi, \psi \rangle = \langle \mathcal{N}_\sigma \phi, \psi \rangle,$$

for every $\phi, \psi \in {}_0H^{-\frac{1}{2}}(\Sigma)$.

Given $\sigma^{(i)} \in L^\infty(\Omega, Sym_n)$, satisfying (1), for $i = 1, 2$, the following equality holds true.

$$\left\langle \psi_1, \left(\mathcal{N}^\Sigma_{\sigma^{(2)}} - \mathcal{N}^\Sigma_{\sigma^{(1)}}\right) \psi_2 \right\rangle = \int_\Omega \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla u_1(x) \cdot \nabla u_2(x),$$

for any $\psi_i \in {}_0H^{-\frac{1}{2}}(\Sigma)$, for $i = 1, 2$ and $u_i \in H^1(\Omega)$ being the unique weak solution to the Neumann problem

$$\begin{cases} \mathrm{div}(\sigma^{(i)}\nabla u_i) = 0, & \text{in} \quad \Omega, \\ \sigma^{(i)}\nabla u_i \cdot \nu|_{\partial\Omega} = \psi_i, & \text{on} \quad \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_i \;= 0. \end{cases}$$

2.1.2. *Non-flatness of* Σ

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be as above. Given α, $\alpha \in (0, 1)$, we say that a portion Σ of $\partial\Omega$ is of class $C^{2,\alpha}$ with constants ρ, $M > 0$ if, up to a rigid transformation of coordinates, Σ is an $(n - 1)$ dimensional $C^{2,\alpha}$ manifold with chart (B'_ρ, φ), where $\varphi : B'_\rho \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is such that

$$\varphi(0) = |\nabla\varphi(0)| = 0 \qquad \|D^2\varphi(x') - D^2\varphi(y')\| \le M|x' - y'|^\alpha \text{ for all } x', y' \in B'_\rho.$$

We will also write

$$\Sigma = \left\{(x', \varphi(x')) \mid |x'| \le \rho\right\}.$$

For any $P \in \partial\Omega$, we will denote by $\nu(P)$ the outer unit normal to $\partial\Omega$ at P.

Definition 2.5. Given Σ as above, we shall say that such a portion of a surface is non-flat (and equivalently the function φ) if, there exist three points $P_1, P_2, P_3 \in \Sigma$ and a constant C_0, $0 < C_0 < 1$, such that

$$\nu(P_1) \cdot \nu(P_2) \le 1 - C_0, \tag{4}$$

$$\nu(P_1) \cdot \nu(P_3) \le 1 - C_0, \tag{5}$$

$$\nu(P_2) \cdot \nu(P_3) \le 1 - C_0\ . \tag{6}$$

2.2. *Local stability at the boundary*

It will be convenient to define throughout this paper the following quantity

$$E := ||\sigma^{(1)} - \sigma^{(2)}||_{L^\infty(\Omega)}.$$

We will assume that there is a point $y \in \partial\Omega$ such that, up to a rigid transformation, $y = 0$, and

$$\Sigma = \partial\Omega \cap B_\rho \tag{7}$$

is a non-flat portion of $\partial\Omega$ of class $C^{2,\alpha}$ with constants $\rho > 0$, $M > 0$ and $C_0 > 0$ as per Definitions 2.4, 2.5.

Definition 2.6. The set of parameters $\{\lambda, r_0,\ L,\ \rho,\ M,\ C_0,\ n\}$ is called the a-priori data.

The following notation will also be adopted throughout the manuscript.

i) A constant C is said to be uniform if it depends on the *a-priori* data only.
ii) We denote by $\mathcal{O}(t)$ a function g such that

$$|g(t)| \leq Ct, \qquad \text{for all}\, t, \quad 0 < t < t_0, \tag{8}$$

where $C, t_0 > 0$ are uniform constants.
iii) We set, for $x' \in \mathbb{R}^{n-1}$,

$$\delta(x') = \begin{cases} Ce^{\frac{1}{(|x'|^2-1)}} & \text{if} \quad |x'| < 1, \\ 0 & \text{if} \quad |x'| \geq 1, \end{cases}$$

where $C > 0$ is the constant such that $\int_{\mathbb{R}^{n-1}} \delta(x')dx' = 1$.

In what follows, $\|\cdot\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}$ will denote the operator norm for linear operators between Banach spaces $\mathcal{B}_1, \mathcal{B}_2$. In particular $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ denotes the operator norm induced by the Euclidean vector norm.

Our main result is stated below.

Theorem 2.1. *Let y, ρ and Σ be defined by (7). Let $\sigma^{(i)} = \{\sigma_{l\,m}^{(i)}(x)\}_{l,m=1,\dots,n}$, $x \in \Omega$ satisfy (1), and assume that $\sigma^{(i)}$ is constant on $\overline{\Omega} \cap B_\rho(y)$, for $i = 1, 2$. If $\mathcal{N}_{\sigma^{(i)}}^{\Sigma}$ is the local N-D map corresponding to $\sigma^{(i)}$, for $i = 1, 2$, then*

$$\left\|\sigma^{(1)}(y) - \sigma^{(2)}(y)\right\|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \leq CE^{1-\beta}||\mathcal{N}_{\sigma^{(1)}}^{\Sigma} - \mathcal{N}_{\sigma^{(2)}}^{\Sigma}||^{\beta}_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\Sigma),({}_0H^{-\frac{1}{2}}(\Sigma))^*\right)}, \tag{9}$$

where C is a positive uniform constant and $\beta = \frac{1}{n-1}$.

Remark 2.2. We emphasize that no regularity assumptions are made for σ away from y.

Remark 2.3. As already observed in the Introduction, the presence of the term E in (9) is intended for future global stability studies. In order to clarify this aspect, we present a toy model which exemplifies the approach towards global Lipschitz stability, starting from the above stated Hölder stability at the boundary.

Example. Suppose that Ω is partitioned into finitely many subdomains $D_1, \dots, D_L$. Assume also that each $D_l,\ l = 1, \dots, L$ is such that

$$\partial D_l \cap \partial\Omega \supset \Sigma_l$$

where each

$$\Sigma_l = \partial\Omega \cap B_\rho(y_l) \ \ , \ \ y_l \in \partial\Omega$$

satisfies the non-flatness assumption introduced in Definition 2.5.

Let $\Sigma \subset \partial\Omega$ be such that

$$\cup_{l=1}^{L} \Sigma_l \subset \Sigma \, .$$

Assume

$$\sigma^{(i)} = \sum_{l=1}^{L} \sigma_l^{(i)} \chi_{D_l} \quad , \quad i = 1, 2,$$

where each $\sigma_l^{(i)}$ is constant. By the Theorem 2.1

$$E = \left|\left|\sigma^{(1)} - \sigma^{(2)}\right|\right|_{L^\infty(\Omega)} \leq C E^{1-\beta} ||\mathcal{N}_{\sigma^{(1)}}^{\Sigma} - \mathcal{N}_{\sigma^{(2)}}^{\Sigma}||^{\beta}_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\Sigma), ({}_0H^{-\frac{1}{2}}(\Sigma))^*\right)},$$

which trivially implies that

$$\left|\left|\sigma^{(1)} - \sigma^{(2)}\right|\right|_{L^\infty(\Omega)} \leq C^{\frac{1}{\beta}} ||\mathcal{N}_{\sigma^{(1)}}^{\Sigma} - \mathcal{N}_{\sigma^{(2)}}^{\Sigma}||_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\Sigma), ({}_0H^{-\frac{1}{2}}(\Sigma))^*\right)} .$$

3. Construction of mollifiers on a graph and their $H^{-\frac{1}{2}}$-norm

For any two points $\xi, x \in \Sigma$, with

$$\xi = (\xi', \varphi(\xi')) \quad ; \quad x = (x', \varphi(x'))$$

and $\tau > 0$, we denote

$$\delta_\tau(\xi, x) = C_\tau(\xi', x')\delta\left(\frac{\xi' - x'}{\tau}\right)$$

and we choose C_τ in such a way that

$$\int_\Sigma \delta_\tau(\xi, x) dS(\xi) = 1, \qquad \text{whenever} \quad B'_\tau(x') \subset B'_\rho.$$

To compute C_τ we form

$$1 = \int_\Sigma \delta_\tau(\xi, x) dS(\xi) = \int_{B'_\rho} C_\tau(\xi', x')\delta\left(\frac{\xi' - x'}{\tau}\right)\sqrt{1 + |\nabla\varphi(\xi')|^2} d\xi'.$$

Denoting $q(\xi') = \sqrt{1 + |\nabla\varphi(\xi')|^2}$, we set

$$C_\tau(\xi', x') = \frac{\tau^{1-n}}{q(\xi')},$$

so that

$$\int_{\Sigma} \delta_\tau(\xi, x) dS(\xi) = \int_{\mathbb{R}^{n-1}} \tau^{1-n} \delta\left(\frac{\xi' - x'}{\tau}\right) d\xi' = 1.$$

Remark 3.1. Notice that $q(\xi') = q(x') + \mathcal{O}(\tau^\alpha)$ on $B_\tau(x')$, hence $q(\xi') = \mathcal{O}(1)$ and therefore $C_\tau(\xi', x') = \mathcal{O}(\tau^{1-n})$.

We define the $H^{-\frac{1}{2}}$-norm of an element $f \in H^{-\frac{1}{2}}(\partial\Omega)$ as follows

$$||f||^2_{H^{-\frac{1}{2}}(\partial\Omega)} = \int_{\partial\Omega} h(x)(f(x) - \bar{f})\, dS(x) = \int_{\Omega} |\nabla h(x)|^2\, dx,$$

where

$$\bar{f} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f(x)\, dS(x)$$

and h solves

$$\begin{cases} \Delta h = 0, & \text{in} \quad \Omega, \\ \frac{\partial h}{\partial \nu} = f - \bar{f}, & \text{on} \quad \partial\Omega. \end{cases}$$

Recall that, for $y \in \Omega$, the Neumann kernel $N_0^\Omega(\cdot,\ y)$ for the Laplacian Δ in Ω is defined, to be the distributional solution to

$$\begin{cases} \Delta\, N_0^\Omega(\cdot, y) = -\delta(\cdot - y), & \text{in} \quad \Omega \\ \frac{\partial N_0^\Omega(\cdot, y)}{\partial \nu} = -\frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega, \end{cases}$$

where δ is the $n-$dimensional Dirac delta and we impose the normalization

$$\int_{\partial\Omega} N_0^\Omega(\cdot, y)\, dS(\cdot) = 0,$$

and that for $y \in \partial\Omega$, it solves

$$\begin{cases} \Delta\, N_0^\Omega(\cdot, y) = 0, & \text{in} \quad \Omega \\ \frac{\partial N_0^\Omega(\cdot, y)}{\partial \nu} = \delta(\cdot - y) - \frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega, \end{cases}$$

where here δ is the $(n-1)-$dimensional Dirac delta. We use such convention also in subsequent formulas. Hence we can write

$$h(x) = \int_{\partial\Omega} N_0^\Omega(x, \xi)(f(\xi) - \bar{f})\, dS(\xi) = \int_{\partial\Omega} N_0^\Omega(x, \xi) f(\xi)\, dS(\xi),$$

since $N_0^{\Omega}(x,\cdot)$ and h have zero average on $\partial\Omega$. Hence

$$||f||^2_{H^{-\frac{1}{2}}(\partial\Omega)} = \int\limits_{\partial\Omega\times\partial\Omega} N_0^{\Omega}(x,\xi)f(\xi)f(x)\,dS(x)\,dS(\xi). \tag{10}$$

We are now in the position to estimate the behavior of the $H^{-\frac{1}{2}}(\partial\Omega)$ norm of the mollified delta function $\delta_\tau(\cdot,x)$, with $x\in\Sigma$, in terms of τ.

Lemma 3.2. *Given $x\in\Sigma$ such that $B'_\tau(x')\subset B'_\rho$, we have*

$$||\delta_\tau(\cdot,x)||^2_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\tau^{2-n},$$

where $C>0$ is a uniform constant.

Proof. By (10) and the fact that δ_τ is compactly supported on Σ, we have

$$||\delta_\tau(\cdot,x)||^2_{H^{-\frac{1}{2}}(\partial\Omega)} = \int\limits_{\Sigma\times\Sigma} N_0^{\Omega}(\xi,\eta)\delta_\tau(\xi,x)\delta_\tau(\eta,x)\,dS(\xi)\,dS(\eta)$$
$$\le C\int\limits_{B'_\rho\times B'_\rho} |\xi'-\eta'|^{2-n}\tau^{2(1-n)}\delta\left(\frac{\xi'-x'}{\tau}\right)\delta\left(\frac{\eta'-x'}{\tau}\right)d\xi'\,d\eta'.$$

The change of variables

$$\zeta'=\frac{\xi'-x'}{\tau} \qquad ; \qquad \theta'=\frac{\eta'-x'}{\tau},$$

together with the fact that $|\xi'-\eta'|=\tau|\zeta'-\theta'|$, leads to

$$\begin{aligned}||\delta_\tau(\cdot,x)||^2_{H^{-\frac{1}{2}}(\partial\Omega)} &\le C\int\limits_{B'_1\times B'_1}\tau^{2-n}|\zeta'-\theta'|^{2-n}\tau^{2(1-n)}\delta\left(\zeta'\right)\delta\left(\theta'\right)\tau^{2(n-1)}\,d\zeta'\,d\theta'\\ &= C\int\limits_{B'_1\times B'_1}\tau^{2-n}|\zeta'-\theta'|^{2-n}\delta\left(\zeta'\right)\delta\left(\theta'\right)d\zeta'\,d\theta' \le C\tau^{2-n}. \quad \square\end{aligned}$$

We recall that for $\sigma(x)=\left\{\sigma_{ij}(x)\right\}_{i,j=1,\dots,n}$, $x\in\Omega$, symmetric, positive definite matrix-valued function satisfying (1), we denote by L the operator

$$L=\operatorname{div}(\sigma\nabla\cdot) \tag{11}$$

and that if in dimension $n>2$ we define the matrix

$$g=(\det\sigma)^{\frac{1}{n-2}}\sigma^{-1}, \tag{12}$$

then

$$\frac{1}{\sqrt{detg}} L = \Delta_g,$$

on the open set Ω endowed with the Riemannian metric g, see for instance [7], [16]. We emphasize that, being $n > 2$, the knowledge of σ is equivalent to the knowledge of g.

4. Stability of the tangential part of g

We start by observing that (12), together with the uniform ellipticity assumption (1) on σ, implies the following uniform ellipticity of g

$$\lambda^{-\frac{2n-2}{n-2}} |\xi|^2 \le g(x)\xi \cdot \xi \le \lambda^{\frac{2n-2}{n-2}} |\xi|^2, \ \ for\ almost\ every\ x \in \Omega, \quad for\ every\ \xi \in \mathbb{R}^n, \tag{13}$$

where $\lambda > 0$ has been introduced in (1).

We also recall below few facts from [3] about the Neumann kernel to make this paper self-contained. The Neumann kernel N_σ^Ω for the boundary value problem associated with the operator (11) and Ω, for any $y \in \Omega$, $N_\sigma^\Omega(\cdot, y)$, is defined to be the distributional solution to

$$\begin{cases} L\, N_\sigma^\Omega(\cdot, y) = -\delta(\cdot - y), & \text{in} \quad \Omega \\ \sigma \nabla N_\sigma^\Omega(\cdot, y) \cdot \nu = -\frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega. \end{cases}$$

Note that N_σ^Ω is uniquely determined up to an additive constant. For simplicity we impose the normalization

$$\int_{\partial\Omega} N_\sigma^\Omega(\cdot, y)\, dS(\cdot) = 0.$$

With this convention we obtain by Green's identities that

$$N_\sigma^\Omega(x, y) = N_\sigma^\Omega(y, x), \qquad \text{for all} \quad x, y \in \Omega, \quad x \neq y.$$

$N_\sigma^\Omega(x, y)$ extends continuously up to the boundary $\partial\Omega$ (provided that $x \neq y$) and in particular, when $y \in \partial\Omega$, it solves

$$\begin{cases} L\, N_\sigma^\Omega(\cdot, y) = 0, & \text{in} \quad \Omega \\ \sigma \nabla N_\sigma^\Omega(\cdot, y) \cdot \nu = \delta(\cdot - y) - \frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega. \end{cases}$$

Theorem 4.1. *Let y, ρ and Σ be defined by (7). If L is the operator (11), with coefficients matrix $\sigma \in C^\alpha(B_\rho(y) \cap \overline{\Omega})$, with $0 < \alpha < 1$, then the Neumann kernel $N_\sigma^\Omega(\cdot, y)$ satisfies*

$$N_\sigma^\Omega(x, y) = 2\Gamma_\sigma(x, y) + \mathcal{O}(|x-y|^{2-n+\alpha}), \tag{14}$$

as $x \to y$, $x \in \overline{\Omega} \setminus \{y\}$. Here

$$\Gamma_\sigma(x,y) := C_n \left(g(y)(x-y)\cdot(x-y)\right)^{\frac{2-n}{2}}$$

and $C_n = \frac{1}{n(n-2)\omega_n}$ *with* ω_n *denoting the volume of the unit ball in* $\mathbb{R}^n$.

Proof. See [3] for a proof. □

Therefore, we have

Lemma 4.2. *Let* y, ρ *and* Σ *be defined by* (7). *If* L *is the operator* (11), *with coefficients matrix* $\sigma \in C^\alpha(B_\rho(y)\cap\overline{\Omega})$, *with* $0<\alpha<1$, *then the knowledge of* $N_\sigma^\Omega(x,y)$, *for every* $x \in \partial\Omega \cap B_\rho(y)$ *uniquely determines*

$$g_{(n-1)}(y) = \left\{g(y)v_i \cdot v_j\right\}_{i,j=1,\dots,(n-1)},$$

where $v_1,\dots,v_{n-1}$ *is a basis for* $T_y(\partial\Omega)$, *the tangent space to* $\partial\Omega$ *at* y.

Such a uniqueness result obtained in [3], will guide us towards a stability estimate.

In what follows, for $y \in \Sigma$, we set for $i = 1, 2$:

$$\Gamma_i(x,y) := \Gamma_{\sigma^{(i)}(y)}(x,y) = C_n\left(g^{(i)}(y)(x-y)\cdot(x-y)\right)^{\frac{2-n}{2}},$$
$$N_i(x,y) := N^{\Omega}_{\sigma_i}(x,y)$$
$$L_i = \operatorname{div}\left(\sigma^{(i)}\nabla\cdot\right)$$

and

$$L_{i;y} = \operatorname{div}\left(\sigma^{(i)}(y)\nabla\cdot\right). \tag{15}$$

Lemma 4.3. *Under the same hypotheses of Theorem 2.1, for any* $y \in \Sigma$ *and any* $x \in \overline{\Omega}\setminus\{y\}$ *we have*

$$|(N_1 - N_2)(x,y)| \le CE|x-y|^{2-n}, \tag{16}$$
$$|\nabla_x(N_1 - N_2)(x,y)| \le CE\,|x-y|^{1-n}, \tag{17}$$

where $C>0$ *is a uniform constant.*

Proof. From Theorem 4.1 for any $y \in \Sigma$, for any $x \in \Omega$, for $i = 1, 2$, we have

$$N_i(x,y) = 2\Gamma_i(x,y) + R_i(x,y), \tag{18}$$

with

$$|R_i(x,y)| \le C|x-y|^{2-n+\alpha},$$
$$|\nabla_x R_i(x,y)| \le C|x-y|^{1-n+\alpha}.$$

Recalling that for $y \in \Sigma$, $N_i(\cdot, y)$ is the distributional solution to

$$\begin{cases} L_i\ N_i(\cdot, y) = 0, & \text{in} \quad \Omega \\ \sigma \nabla N_i(\cdot, y) \cdot \nu = \delta(\cdot - y) - \frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega, \end{cases}$$

for any $\eta \in H^1(\Omega) \cap C^1(\overline{\Omega} \cap B_\rho(y))$, we have

$$\begin{aligned} \int_\Omega \sigma^{(i)}(x) \nabla_x N_i(x, y) \cdot \nabla_x \eta(x) dx &= \int_{\partial\Omega} \eta(x) \left(\delta(x - y) - \frac{1}{|\partial\Omega|} \right) dS(x) \\ &= \eta(y) - \bar{\eta}, \end{aligned}$$

where $\bar{\eta} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \eta(x) dS(x)$. Recalling the decomposition (18) we obtain

$$\int_\Omega \sigma^{(i)}(x) \nabla_x 2\Gamma_i(x, y) \cdot \nabla_x \eta(x) dx + \int_\Omega \sigma^{(i)}(x) \nabla_x R_i(x, y) \cdot \nabla_x \eta(x) dx = \eta(y) - \bar{\eta},$$

which leads to

$$\begin{aligned} &\int_\Omega \sigma^{(i)}(y) \nabla_x 2\Gamma_i(x, y) \cdot \nabla_x \eta(x) dx \\ &+ \int_\Omega \left(\sigma^{(i)}(x) - \sigma^{(i)}(y) \right) \nabla_x 2\Gamma_i(x, y) \cdot \nabla_x \eta(x) dx \\ &+ \int_\Omega \sigma^{(i)}(x) \nabla_x R_i(x, y) \cdot \nabla_x \eta(x) dx = \eta(y) - \bar{\eta}. \end{aligned} \tag{19}$$

Noticing that $2\Gamma_i(\cdot, y)$ solves the boundary value problem

$$\begin{cases} L_{i;y}\ 2\Gamma_i(\cdot, y) = 0, & \text{in} \quad \Omega \\ \sigma^{(i)}(y) \nabla \left(2\Gamma_i(\cdot, y) \right) \cdot \nu = \delta(\cdot - y) + f_i(\cdot, y), & \text{on} \quad \partial\Omega, \end{cases}$$

with

$$|f_i(x, y)| \le C|x - y|^{1-n+\alpha},$$

where $C > 0$ is a uniform constant and $L_{i;y}$ has been defined in (15), therefore

$$\int_\Omega \sigma^{(i)}(y) \nabla_x 2\Gamma_i(x, y) \cdot \nabla_x \eta(x) dx = \eta(y) + \int_{\partial\Omega} f_i(x, y) \eta(x) dS(x), \tag{20}$$

and defining

$$F_i(x, y) := \left(\sigma^{(i)}(x) - \sigma^{(i)}(y) \right) \nabla_x 2\Gamma_i(x, y),$$

we can rewrite (19) as

$$\int_{\Omega} \sigma^{(i)}(x)\nabla_x R_i(x,y)\cdot\nabla_x\eta(x)dx = -\int_{\Omega} F_i(x,y)\cdot\nabla_x\eta(x)dx - \int_{\partial\Omega} f_i(x,y)\eta(x)dS(x) - \bar{\eta}, \tag{21}$$

where $F_i(x,y)$ is zero for $x \in B_\rho(y)$, therefore bounded for $x \in \Omega$. Therefore by combining (20) and (21), we have that for any $\eta \in H^1(\Omega) \cap C^1(\overline{B_\rho(y)})$

$$\begin{aligned}
&\int_{\Omega} \sigma^{(1)}(x)\nabla_x\left(R_1 - R_2\right)(x,y)\cdot\nabla_x\eta(x)dx \\
&= \int_{\Omega}\left(\sigma^{(2)} - \sigma^{(1)}\right)(x)\nabla_x R_2(x,y)\cdot\nabla_x\eta(x)dx \\
&- \int_{\Omega}\left(F_1 - F_2\right)(x,y)\cdot\nabla_x\eta(x)dx - \int_{\partial\Omega}\left(f_1 - f_2\right)(x,y)\eta(x)dS(x).
\end{aligned} \tag{22}$$

By denoting for any $x \in \Omega$, for any $y \in \Sigma$,

$$F(x,y) := \left(F_1 - F_2\right)(x,y),$$

we have

$$|F(x,y)| \le CE.$$

Moreover, by denoting for any $x \in \partial\Omega$, for any $y \in \Sigma$, $x \neq y$

$$f(x,y) := \left(f_1 - f_2\right)(x,y) = 2[\sigma^{(1)}(y)\nabla\Gamma_1(x,y) - \sigma^{(2)}(y)\nabla\Gamma_2(x,y)]\cdot\nu(x)$$

and by the explicit expression of Γ_1 and Γ_2, the ellipticity condition (1) and the regularity of the portion Σ we have

$$f(x,y) = \mathcal{O}(E|x-y|^{-n})\mathcal{O}(|x-y|^{1+\alpha}),$$

which leads to

$$|f(x,y)| \le CE|x-y|^{1-n+\alpha}.$$

Next, by denoting

$$G(x,y) := \left(\sigma^{(2)} - \sigma^{(1)}\right)(x)\nabla_x R_2(x,y) - F(x,y),$$

we have

$$|G(x,y)| \le CE|x-y|^{1-n+\alpha},$$

where $C>0$ denotes a uniform constant.
For any $z \in \overline{\Omega} \setminus \{y\}$ and any $k \in \mathbb{N}$, we define $\eta_k(x) = \min\{N_1(x,z), k\}$ with $x \in \Omega$. By choosing $\eta(x) = \eta_k(x)$ in (22) and by the dominated convergence theorem we obtain

$$\begin{aligned}&\int\limits_{\Omega} \sigma^{(1)}(x)\nabla_x (R_1 - R_2)(x,y) \cdot \nabla_x N_1(x,z)dx \\ &= \int\limits_{\Omega} G(x,y) \cdot \nabla_x N_1(x,z)dx - \int\limits_{\partial\Omega} f(x,y)N_1(x,z)dS(x).\end{aligned}$$

By performing integration by parts on the integral appearing on the left hand side of (22) we have

$$\begin{aligned}&\frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega} (R_1 - R_2)(x,y)dS(x) + (R_1 - R_2)(z,y) \\ &= \int\limits_{\Omega} G(x,y) \cdot \nabla_x N_1(x,z)dx - \int\limits_{\partial\Omega} f(x,y)N_1(x,z)dS(x),\end{aligned} \tag{23}$$

where the right hand side of equality (23) can be estimated as follows

$$\begin{aligned}&\left|\int\limits_{\Omega} G(x,y) \cdot \nabla_x N_1(x,z)dx - \int\limits_{\partial\Omega} f(x,y)N_1(x,z)dS(x)\right| \\ &C||\sigma^{(1)} - \sigma^{(2)}||_{L^\infty(\Omega)}\left\{\int\limits_{\Omega} |x-y|^{1-n+\alpha}\,|x-z|^{1-n}dx \right. \\ &\left. + \int\limits_{\partial\Omega} |x-y|^{1-n+\alpha}\,|x-z|^{2-n}dS(x)\right\} \\ &\le CE|z-y|^{2-n+\alpha},\end{aligned}$$

where $C>0$ is a uniform constant. To estimate $\int_{\partial\Omega}(R_1 - R_2)(x,y)dS(x)$ in (23) recall that $N_i(\cdot,y)$ is uniquely determined by imposing the condition

$$\int\limits_{\partial\Omega} N_i(x,y)dS(x) = 0,$$

which leads to

$$\int\limits_{\partial\Omega} R_i(x,y)dS(x) = -\int\limits_{\partial\Omega} 2\Gamma_i(x,y)dS(x).$$

Therefore

$$\left|\int_{\partial\Omega} (R_1 - R_2)(x, y) dS(x)\right| = \left|\int_{\partial\Omega} 2(\Gamma_1 - \Gamma_2)(x, y) dS(x)\right| \le CE,$$

where $C > 0$ is a uniform constant, which leads to

$$|(R_1 - R_2)(z, y)| \le CE\left\{1 + |z - y|^{2-n+\alpha}\right\} \le CE|z - y|^{2-n+\alpha}, \tag{24}$$

where $C > 0$ is a uniform constant. (16) follows from (24). To prove (17) we observe that $\sigma^{(i)}$ is constant on $\overline{\Omega} \cap B_\rho(y)$, for $i = 1, 2$, therefore

$$\mathrm{div}\left(\sigma^{(i)}\nabla N_i(\cdot, y)\right) = \mathrm{tr}\left(\sigma^{(i)} D^2 N_i(\cdot, y)\right) = 0, \qquad \text{for} \quad i = 1, 2,$$

therefore

$$\mathrm{tr}\left(\sigma^{(1)} D^2 (N_1 - N_2)(\cdot, y)\right) = \mathrm{tr}\left((\sigma^{(2)} - \sigma^{(1)}) D^2 N_2(\cdot, y)\right).$$

By fixing $r > 0$, with $r < \frac{\rho}{4}$ and defining

$$C_r^+ := \left(B_{2r}(y) \setminus \overline{B_r(y)}\right) \cap \Omega,$$
$$C_{2r}^+ := \left(B_{4r}(y) \setminus \overline{B_{\frac{r}{2}}(y)}\right) \cap \Omega,$$

we have for $i = 1, 2$

$$||D^2 N_i||_{C^\alpha(C_r^+)} \le \frac{C}{r^2}||N_i||_{C^\alpha(C_{2r}^+)} \le Cr^{-n-\alpha}$$

and

$$||D^2(N_1 - N_2)||_{C^\alpha(C_r^+)} \le CEr^{-n-\alpha}. \tag{25}$$

(17) follows by (16), (25) and an interpolation argument. $\square$

Proposition 4.4 *(Stability of the tangential part of g). For any $y \in \Sigma$,*

$$\left|\left|g^{(1;n-1)}(y) - g^{(2;n-1)}(y)\right|\right|_{\mathcal{L}(\mathbb{R}^{n-1},\ \mathbb{R}^{n-1})} \le CE^{1-\beta}||\mathcal{N}_{\sigma_1}^{\Sigma} - \mathcal{N}_{\sigma_2}^{\Sigma}||^{\beta}_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\partial\Omega), {}_0H^{\frac{1}{2}}(\partial\Omega)\right)}, \tag{26}$$

where C is a positive uniform constant, $\beta = \frac{1}{n-1}$, $g^{(i;n-1)}(y)$ is the $(n-1) \times (n-1)$ upper left sub-matrix of $g^{(i)}(y)$, for $i = 1, 2$.

Proof. Let $d > 0$ be such that $d < (1+M)\rho$. Given distinct points $x, y, w, z \in \Sigma$, we recall from [3] the following definition

$$K_\sigma(x, y, w, z) = N_\sigma(x, y) - N_\sigma(x, w) - N_\sigma(z, y) + N_\sigma(z, w).$$

We also recall that knowing $\mathcal{N}_\sigma^\Sigma$ is equivalent to knowing K_σ, for any $x, y, w, z \in \Sigma$ [3, Lemma 3.8].

We also note that, fixing $w, z \in \Sigma$, K_σ, as a function of x, y, has the same asymptotic behavior of $N_\sigma(x, y)$ as $x \to y$.

Given $y \in \Sigma$, we choose $x, w, z \in \Sigma$ such that

$$\begin{aligned} |x - y| &\leq \frac{d}{4}, \\ |x - w| &\geq \frac{d}{4}, \\ |x - z| &\geq \frac{d}{4}, \\ |w - z| &\geq \frac{d}{4}. \end{aligned}$$

Let

$$\tau := h|x - y|, \qquad \text{with} \quad 0 < h < \frac{1}{16} \tag{27}$$

and let $\delta_\tau(\cdot, \cdot)$ be the approximate Dirac's delta functions on Σ introduced in Section 3, centered on the second argument. Then we have

$$\begin{aligned} &\langle \delta_\tau(\cdot\, , x) - \delta_\tau(\cdot\, , z), \left(\mathcal{N}_{\sigma_1}^\Sigma - \mathcal{N}_{\sigma_2}^\Sigma\right)(\delta_\tau(\cdot\, , y) - \delta_\tau(\cdot\, , w))\rangle \\ &= \int_{\Sigma\times\Sigma} (N_1 - N_2)(\xi, \eta)\, (\delta_\tau(\xi\, , x) - \delta_\tau(\xi\, , z))\, (\delta_\tau(\eta\, , y) - \delta_\tau(\eta\, , w))\, dS(\xi) dS(\eta). \end{aligned} \tag{28}$$

The integral appearing on the right hand side of (28) depends on x, y, w, z and to estimate how close this quantity is to $(K_1 - K_2)(x, y, w, z)$ we form

$$\begin{aligned} &(K_1 - K_2)(x, y, w, z) \\ &- \int_{\Sigma\times\Sigma} (N_1 - N_2)(\xi, \eta)\, (\delta_\tau(\xi, x) - \delta_\tau(\xi, z))\, (\delta_\tau(\eta, y) - \delta_\tau(\eta, w))\, dS(\xi) dS(\eta) \\ &= (N_1 - N_2)(x, y) - \int_{\Sigma\times\Sigma} (N_1 - N_2)(\xi, \eta)\delta_\tau(\xi, x)\delta_\tau(\eta, y) dS(\xi) dS(\eta) \\ &- (N_1 - N_2)(x, w) + \int_{\Sigma\times\Sigma} (N_1 - N_2)(\xi, \eta)\delta_\tau(\xi, x)\delta_\tau(\eta, w) dS(\xi) dS(\eta) \end{aligned} \tag{29}$$

$$+(N_1-N_2)(z,w)-\int_{\Sigma\times\Sigma}(N_1-N_2)(\xi,\eta)\delta_\tau(\xi,z)\delta_\tau(\eta,w)dS(\xi)dS(\eta)$$
$$-(N_1-N_2)(z,y)+\int_{\Sigma\times\Sigma}(N_1-N_2)(\xi,\eta)\delta_\tau(\xi,z)\delta_\tau(\eta,y)dS(\xi)dS(\eta).$$

We estimate each term on the right hand side of (29) as follows

$$\begin{aligned}
&\left|(N_1-N_2)(x,y)-\int_{\Sigma\times\Sigma}(N_1-N_2)(\xi,\eta)\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\right|\\
&=\left|\int_{\Sigma\times\Sigma}\Big((N_1-N_2)(x,y)-(N_1-N_2)(\xi,\eta)\Big)\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\right|\\
&\le\int_{\Sigma\times\Sigma}\Big|(N_1-N_2)(x,y)-(N_1-N_2)(\xi,y)\Big|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\\
&+\int_{\Sigma\times\Sigma}\Big|(N_1-N_2)(\xi,y)-(N_1-N_2)(\xi,\eta)\Big|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\\
&\le\int_{\Sigma\times\Sigma}\Big|\nabla_\xi(N_1-N_2)(\hat{\xi},y)\Big||x-\xi|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\\
&+\int_{\Sigma\times\Sigma}\Big|\nabla_\eta(N_1-N_2)(\xi,\hat{\eta})\Big||\eta-y|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\\
&\le\int_{\Sigma\times\Sigma}\frac{CE}{|\hat{\xi}-y|^{n-1}}|x-\xi|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\\
&+\int_{\Sigma\times\Sigma}\frac{CE}{|\xi-\hat{\eta}|^{n-1}}|\eta-y|\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta),
\end{aligned}$$

where $\hat{\xi}=(1-t)x+t\xi$ and $\hat{\eta}=(1-s)y+s\eta$ for some $t,s\in(0,1)$.

Notice that, given the choice of τ in (27), we have that

$$\tau<\frac{d}{64};\qquad |x-y|=\mathcal{O}(|\hat{\xi}-y|);\qquad |x-y|=\mathcal{O}(|\xi-\hat{\eta}|),$$

therefore

$$\left|(N_1-N_2)(x,y)-\int_{\Sigma\times\Sigma}(N_1-N_2)(\xi,\eta)\delta_\tau(\xi,x)\delta_\tau(\eta,y)dS(\xi)dS(\eta)\right|$$

$$
\begin{aligned}
&\leq \int\limits_{\Sigma\times\Sigma} \frac{2\tau C E}{|x-y|^{n-1}} \delta_\tau(\xi, x)\delta_\tau(\eta, y) dS(\xi) dS(\eta) \\
&\leq \int\limits_{\Sigma\times\Sigma} \frac{C E h}{|x-y|^{n-2}} \delta_\tau(\xi, x)\delta_\tau(\eta, y) dS(\xi) dS(\eta) \\
&= \frac{CEh}{|x-y|^{n-2}}.
\end{aligned}
$$

Each of the remaining three terms appearing on the right hand side of (29) involve at most one of the poles $x, y \in \Sigma$ and are therefore bounded by CEh. Therefore we have

$$
\begin{aligned}
&\Bigg| (K_1 - K_2)(x, y, w, z) \\
&- \int\limits_{\Sigma\times\Sigma} (N_1 - N_2)(\xi, \eta) \left(\delta_\tau(\xi, x) - \delta_\tau(\xi, z)\right)\left(\delta_\tau(\eta, y) - \delta_\tau(\eta, w)\right) dS(\xi) dS(\eta) \Bigg| \\
&\leq \frac{CEh}{|x-y|^{n-2}} + CEh \\
&\leq \frac{CEh + CEhd^{n-2}}{|x-y|^{n-2}} \leq \frac{CEh}{|x-y|^{n-2}}.
\end{aligned}
$$

Recalling that

$$
\left| \langle \delta_\tau(\cdot\,, x) - \delta_\tau(\cdot\,, z), \left(\mathcal{N}^\Sigma_{\sigma_1} - \mathcal{N}^\Sigma_{\sigma_2}\right)\left(\delta_\tau(\cdot\,, y) - \delta_\tau(\cdot\,, w)\right)\rangle \right| \leq C\tau^{2-n}\varepsilon,
$$

where $\varepsilon = ||\mathcal{N}^\Sigma_{\sigma^{(1)}} - \mathcal{N}^\Sigma_{\sigma^{(2)}}||_{\mathcal{L}\left({}_0H^{-\frac{1}{2}}(\partial\Omega), {}_0H^{\frac{1}{2}}(\partial\Omega)\right)}$. Hence, recalling the definition of τ in (27), we obtain the following pointwise estimate for $K_1 - K_2$

$$
\begin{aligned}
|(K_1 - K_2)(x, y, w, z)| &\leq C\varepsilon\tau^{2-n} + \frac{CEh}{|x-y|^{n-2}} \\
&= C\varepsilon \frac{h^{2-n}}{|x-y|^{n-2}} + \frac{CEh}{|x-y|^{n-2}} \\
&= \frac{C\varepsilon h^{2-n} + CEh}{|x-y|^{n-2}}, \qquad 0 < h < \frac{1}{16}.
\end{aligned}
\tag{30}
$$

Minimization of (30) with respect to h leads to

$$
|(K_1 - K_2)(x, y, w, z)| \leq \frac{C\varepsilon^\beta E^{1-\beta}}{|x-y|^{n-2}},
$$

that is

$$
|x-y|^{n-2}|(K_1 - K_2)(x, y, w, z)| \leq C\varepsilon^\beta E^{1-\beta}, \tag{31}
$$

with $\beta = \frac{1}{n-1}$. Inequality (31) is a uniform bound with respect to $x, y \in \Sigma$. Setting $y = 0$, $\nu(0) = -e_n$, where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, and writing $x \in \Sigma$ as $x = (x', \varphi(x'))$, with $x' = r\xi'$, with $\xi' \in \mathbb{R}^{n-1}$ and $||\xi'|| = 1$, (14) leads to

$$\left|\left(g^{(1)}(0)\xi' \cdot \xi'\right)^{\frac{2-n}{2}} - \left(g^{(2)}(0)\xi' \cdot \xi'\right)^{\frac{2-n}{2}}\right| \leq C_n \left|\lim_{r\to 0} r^{n-2} \left(K_1 - K_2\right)(x, 0, z, w)\right|, \tag{32}$$

where we identified ξ' with $(\xi', 0)$. By Lagrange's mean value theorem and (13), we have

$$\left|\left(g^{(1)}(0) - g^{(2)}(0)\right)\xi' \cdot \xi'\right| \leq C \left|\left(g^{(1)}(0)\xi' \cdot \xi'\right)^{\frac{2-n}{2}} - \left(g^{(2)}(0)\xi' \cdot \xi'\right)^{\frac{2-n}{2}}\right| \tag{33}$$

and by combining (33) with (31) together with (32), we obtain

$$\left|\left(g^{(1)}(0) - g^{(2)}(0)\right)\xi' \cdot \xi'\right| \leq C E^{1-\beta}\varepsilon^{\beta}, \quad \text{for any} \quad \xi' \in \mathbb{R}^{n-1}, \quad ||\xi'|| = 1,$$

which concludes the proof, since, as it is well known

$$\begin{aligned}&\left|\left|g^{(1;n-1)}(0) - g^{(2;n-1)}(0)\right|\right|_{\mathcal{L}(\mathbb{R}^{n-1},\ \mathbb{R}^{n-1})}\\ &= \sup\left\{\left|\left(g^{(1)}(0) - g^{(2)}(0)\right)\xi' \cdot \xi'\right| : \xi' \in \mathbb{R}^{n-1},\ \|\xi'\| = 1\right\}. \quad \square\end{aligned}$$

5. Stability of the full metric g

In the following we shall prove that up to a suitable condition on the geometry of Ω and on the structure of the metric g, the knowledge of the Neumann kernel in a neighborhood of Σ allows us to recover the full metric g on Σ.

We assume that Σ is $C^{2,\alpha}$ and non-flat as per Definition 2.5. This means that we can find $P_1, P_2, P_3 \in \Sigma$ and a constant C_0, $0 < C_0 < 1$ satisfying (4) - (6). If we denote by $\{e_1, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$, we can assume, without loss of generality, that $P_1 = 0 \in \Sigma$, that the tangent space to $\partial\Omega$ at $0 \in \Sigma$ is $T_0(\partial\Sigma) = \langle e_1, \ldots, e_{n-1}\rangle$ and the outer unit normal to $\partial\Omega$ at 0 is $\nu(P_1) = -e_n$. Hence we have

$$\nu(0) \cdot \nu(P_2) \leq 1 - C_0, \tag{34}$$

$$\nu(0) \cdot \nu(P_3) \leq 1 - C_0, \tag{35}$$

$$\nu(P_2) \cdot \nu(P_3) \leq 1 - C_0 \tag{36}$$

and without loss of generality, we can assume that there exists some $0 < \gamma_1 < \frac{1}{2}$ such that

$$\nu(P_2) = \frac{1}{\sqrt{1 + {\gamma_1}^2}}\left(-e_n + \gamma_1 e_{n-1}\right),$$

and some $0 \leq \gamma_2, \gamma_3 \leq 1$ such that

$$\nu(P_3) = \frac{1}{\sqrt{1+{\gamma_2}^2+{\gamma_3}^2}}\left(-e_n+\gamma_2 e_{n-1}+\gamma_3 e_{n-2}\right).$$

Let Π_1, Π_2, Π_3 be the tangent spaces to $\partial\Omega$ at 0, P_2, P_3, respectively, with orthonormal basis $\{v_1^1, \dots, v_{n-1}^1\}$, $\{v_1^2, \dots, v_{n-1}^2\}$, $\{v_1^3, \dots, v_{n-1}^3\}$ and a linear application T,

$$T : Sym_n \to \mathbb{R}^{3(n-1)^2},$$

defined by

$$Tg = \left\{ g v_i^k \cdot v_j^k \mid k = 1, 2, 3, \quad i, j = 1, \dots, n-1 \right\}, \quad \text{for any } g \in Sym_n\ .$$

Proposition 5.1. *In the above setting for Σ, if $\sigma \in L^\infty(\Omega\ , Sym_n)$ satisfies (1) and it is constant on $\overline{\Omega} \cap B_\rho$, then the knowledge of $N_\sigma^\Omega(x, y)$, for every $x, y \in \Sigma$ uniquely determines $g(0)$. Moreover T is a linear and injective application such that*

$$\|g\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} \leq C\|Tg\|, \tag{37}$$

where $C > 0$ is a constant only depending on C_0 and $\|Tg\|$ is the Euclidean norm of Tg in $\mathbb{R}^{3(n-1)^2}$.

Proof. We reformulate conditions (34) - (36) in terms of $\gamma_1, \gamma_2, \gamma_3$. From condition (34) we get

$$\nu(0) \cdot \nu(P_2) = \frac{1}{\sqrt{1+\gamma_1^2}} \leq 1 - C_0,$$

which leads to

$$\gamma_1 \geq k_0,$$

where $k_0 = \sqrt{\frac{1-(1-C_0)^2}{(1-C_0)^2}}$. Furthermore, from (35) we obtain

$$\nu(0) \cdot \nu(P_3) = \frac{1}{\sqrt{1+\gamma_2^2+\gamma_3^2}} \leq 1 - C_0,$$

which leads to

$$\gamma_2^2 + \gamma_3^2 \geq {k_0}^2\ .$$

Hence we get that

$$\gamma_2 \geq \frac{k_0}{\sqrt{2}} \quad \text{or} \quad \gamma_3 \geq \frac{k_0}{\sqrt{2}}\ .$$

Finally, by condition (36), we have

$$\nu(P_2)\cdot\nu(P_3) = \frac{1+\gamma_1\gamma_2}{\sqrt{1+\gamma_1^2}\sqrt{1+\gamma_2^2}}\frac{\sqrt{1+\gamma_2}^2}{\sqrt{1+\gamma_2{}^2+\gamma_3^2}} \leq 1-C_0\,. \tag{38}$$

By (38) we have that

$$\begin{aligned}
&(1+\gamma_1\gamma_2)^2\\
&\leq (1-C_0)^2(1+\gamma_1^2)(1+\gamma_2^2+\gamma_3^2)\\
&\leq (1-C_0)^2(\gamma_1-\gamma_2)^2+(1-C_0)^2(1+2\gamma_1\gamma_2+\gamma_1^2\gamma_2^2+\gamma_3^2+\gamma_1^2\gamma_3^2).
\end{aligned}$$

Hence

$$[1-(1-C_0)^2](1+\gamma_1\gamma_2)^2-(1-C_0)^2\gamma_3^2(1+\gamma_1^2) \leq (1-C_0)^2(\gamma_1-\gamma_2)^2. \tag{39}$$

If $\gamma_3 \leq \frac{k_0}{\sqrt{2}}$, from (39) and the fact that $\gamma_1 \leq \frac{1}{2}$ we get that

$$[1-(1-C_0)^2](1+\gamma_1\gamma_2)^2-\frac{5}{8}(1-C_0)^2{k_0}^2 \leq (1-C_0)^2(\gamma_1-\gamma_2)^2,$$

which leads to

$$(\gamma_1-\gamma_2)^2 \geq \frac{3}{8}\frac{[1-(1-C_0)^2]}{2(1-C_0)^2}\,.$$

In conclusion, we have the following two cases:

$$\gamma_1 \geq k_0\,, \quad \gamma_3 \geq \frac{k_0}{\sqrt{2}} \tag{40}$$

or

$$\gamma_1 \geq k_0\,, \quad \gamma_2 \geq \frac{k_0}{\sqrt{2}} \quad \text{and} \quad |\gamma_1-\gamma_2| \geq \sqrt{\frac{3[1-(1-C_0)^2]}{16(1-C_0)^2}}. \tag{41}$$

We observe that the spaces Π_1, Π_2, Π_3 are generated by

$$\begin{aligned}
&\{e_1,\dots,e_{n-1}\}\,,\\
&\left\{e_1,\dots,e_{n-2},\frac{e_{n-1}+\gamma_1 e_n}{\sqrt{1+\gamma_1^2}}\right\},\\
&\{e_1,\dots,e_{n-3},e_{n-2}+\gamma_3 e_n,e_{n-1}+\gamma_2 e_n\}\,,
\end{aligned}$$

respectively, although the latter is not an orthonormal basis. Let $\{v_1,\dots,v_{n-1}\}$ be an orthonormal basis for Π_3 and let

$$w_1 = e_{n-2}+\gamma_3 e_n, \qquad w_2 = e_{n-1}+\gamma_2 e_n\,.$$

By using the repeated index notation we have

$$w_i = (w_i \cdot v_j)v_j, \qquad \text{for} \quad i = 1, 2$$

and

$$g w_i \cdot w_j = (g v_l \cdot v_k)(w_i \cdot v_l)(w_j \cdot v_k), \tag{42}$$

for $i, j = 1, 2$ and $k,\ l = 1, \dots, n-1$. Noticing that

$$\sum_{j=1}^{n-1} (w_i \cdot v_j)^2 = \|w_i\|^2 \le \max\{1+\gamma_2^2, 1+\gamma_3^2\} \le 2,$$

by (42) we have that

$$|g w_i \cdot w_j| \le C\|Tg\|,$$

where C is an absolute constant. Now, from the tangential component of g over Π_1, we recover the upper left submatrix of g, in particular

$$|g_{i,j}| = |g e_i \cdot e_j| \le \|Tg\|, \tag{43}$$

for any $i, j = 1, \dots, n-1$. From the tangential component of g over Π_2 we know the following quantities

$$g e_i \cdot \frac{e_{n-1} + \gamma_1 e_n}{\sqrt{1+\gamma_1^2}} = \frac{1}{\sqrt{1+\gamma_1^2}}(g_{n-1,i} + \gamma_1 g_{n,i}),$$

with $i = 1, \dots, n-2$. Hence since by (43) we can control $g_{n-1,i}$, we get that

$$|g_{i,n}| \le C\|Tg\|, \tag{44}$$

for any $i = 1, \dots, n-2$. To estimate the remaining entries $g_{n-1,n}$, g_{nn} of $g \in Sym_n$, we consider the following known quantity

$$g\left(\frac{e_{n-1} + \gamma_1 e_n}{\sqrt{1+\gamma_1^2}}\right) \cdot \left(\frac{e_{n-1} + \gamma_1 e_n}{\sqrt{1+\gamma_1^2}}\right) = \frac{1}{1+\gamma_1^2}(g_{n-1,n-1} + 2\gamma_1 g_{n-1,n} + \gamma_1^2 g_{n,n}).$$

Hence, since by (43) we can control $g_{n-1,n-1}$, we get

$$|2g_{n-1,n} + \gamma_1 g_{n,n}| = |F_1| \le C\|Tg\|, \qquad \text{for} \quad i = 1, \dots, n-2,$$

where

$$F_1 = \frac{1}{\gamma_1}\Big((g(e_{n-1}+\gamma_1 e_n)\cdot e_{n-1}+\gamma_1 e_n) - g_{n-1,n-1}\Big).$$

From the tangential component of g over Π_3 we know the quantity

$$g\left(\frac{e_{n-2}+\gamma_3 e_n}{\sqrt{1+\gamma_3^2}}\right)\cdot\left(\frac{e_{n-2}+\gamma_3 e_n}{\sqrt{1+\gamma_3^2}}\right) = \frac{1}{1+\gamma_3^2}(g_{n-2,n-2}+2\gamma_3 g_{n-2,n}+\gamma_3^2 g_{n,n}).$$

Hence by (43) and by (44) we have that

$$|\gamma_3^2 g_{n,n}| = |F_2| \le \|Tg\|,$$

where

$$F_2 = g(e_{n-2}+\gamma_3 e_n)\cdot(e_{n-2}+\gamma_1 e_n) - g_{n-2,n-2} - 2\gamma_3 g_{n-2,n}.$$

Finally, we also know

$$g\left(\frac{e_{n-2}+\gamma_2 e_n}{\sqrt{1+\gamma_2^2}}\right)\cdot\left(\frac{e_{n-2}+\gamma_2 e_n}{\sqrt{1+\gamma_2^2}}\right) = \frac{1}{(1+\gamma_2^2)}\big(g_{n-1,n-1}+2\gamma_2 g_{n-1,n}+\gamma_2^2 g_{n,n}\big).$$

Hence by (43) we have that

$$|2\gamma_2 g_{n-1,n}+\gamma_2^2 g_{n,n}| = |F_3| \le \|Tg\|,$$

where

$$F_3 = g\,(e_{n-2}+\gamma_2 e_n)\cdot(e_{n-2}+\gamma_2 e_n) - g_{n-1,n-1}.$$

Collecting together the above calculations lead to the linear system

$$\mathcal{A}G = F,$$

where

$$\mathcal{A} = \begin{pmatrix} 2 & \gamma_1 \\ 0 & \gamma_3^2 \\ 2\gamma_2 & \gamma_2^2, \end{pmatrix},$$

$G = (g_{n-1,n} \quad g_{n,n})^T$ and $F = (F_1\ F_2\ F_3)^T$. If case (40) holds, then we recover G by inverting the square matrix

$$\mathcal{A}_1 = \begin{pmatrix} 2 & \gamma_1 \\ 0 & \gamma_3^2 \end{pmatrix}.$$

Otherwise, if case (41) holds, then we recover G by inverting the square matrix

$$\mathcal{A}_2 = \begin{pmatrix} 2 & \gamma_1 \\ 2\gamma_2 & \gamma_2^2 \end{pmatrix},$$

which concludes the proof. □

Proof of Theorem 2.1. Let y, ρ and Σ be defined by (7). From (12) we have

$$\sigma_i(y) = (\det g_i(y))^{\frac{1}{2}}\, g_i^{-1}(y), \qquad \text{for} \quad i = 1, 2,$$

therefore

$$\begin{aligned}\sigma_1(y) - \sigma_2(y) &= \left[(\det g_1(y))^{\frac{1}{2}} - (\det g_2(y))^{\frac{1}{2}}\right] g_1^{-1}(y) \\ &\quad + (\det g_2(y))^{\frac{1}{2}} \left(g_1^{-1}(y) - g_2^{-1}(y)\right) \\ &= \frac{\det g_1(y) - \det g_2(y)}{(\det g_1(y))^{\frac{1}{2}} + (\det g_2(y))^{\frac{1}{2}}}\, g_1^{-1}(y) \\ &\quad + (\det g_2(y))^{\frac{1}{2}}\, g_1^{-1}(y)\,(g_2(y) - g_1(y))\, g_2^{-1}(y),\end{aligned}$$

which leads to

$$||\sigma_1(y) - \sigma_2(y)||_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)} \le C||g_1(y) - g_2(y)||_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)}.$$

In view of Proposition 4.4 and Proposition 5.1 the proof is complete. □

Data availability

No data was used for the research described in the article.

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