

Reconstruction of rational ruled surfaces from their silhouettes

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A B S T R A C T

We provide algorithms to reconstruct rational ruled surfaces in three-dimensional projective space from the “apparent contour” of a single projection to the projective plane. We deal with the case of tangent developables and of general projections to \mathbb{P}^3 of rational normal scrolls. In the first case, we use the fact that every such surface is the projection of the tangent developable of a rational normal curve, while in the second we start by reconstructing the rational normal scroll. In both instances we then reconstruct the correct projection to \mathbb{P}^3 of these surfaces by exploiting the information contained in the singularities of the apparent contour.

Keywords:

Rational surface

Projection

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Tangent developable

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1. Introduction

Rational ruled surfaces are rational surfaces that contain a straight line through each point. They have been extensively investigated from the point of view of computer algebra, see for example Chen et al. (2001), Busé et al. (2009), or Shen and Pérez-Díaz (2014). When we project to \mathbb{P}^2 a rational ruled surface $S \subset \mathbb{P}^3$, we get a finite cover of \mathbb{P}^2 branched along a curve, which is the zero set of the discriminant of the equation of S along the direction of projection. In this paper, we study the problem of reconstructing the equation of the surface from the discriminant, up to unavoidable projective automorphisms that preserve the discriminant. We have already investigated this question in Gallet et al. (2019) for a wider class of surfaces, namely the ones admitting at worst ordinary singularities. However, for rational ruled surfaces we are able to create a faster algorithm for reconstruction, based on a completely different technique from the one in Gallet et al. (2019).

We follow the notation from Gallet et al. (2019) and we define the *contour* to be the locus of points on the surface S whose tangent space passes through the center of projection. It consists of the singular locus of the surface and of another curve, which we call the *proper contour*. The projection of the contour is the *silhouette* of S , which splits into the *singular image* and the *proper silhouette*. Our goal is then, starting from the knowledge of the singular image and of the proper silhouette, to reconstruct the surface S , up to projective automorphisms that preserve the center of projection and the lines through it.

In the remainder of the introduction, we discuss the organization of the paper.

A special subcase of rational ruled surfaces is the one of developables, i.e., surfaces that are either a cone over a plane curve or the union of the tangents of a space curve. In Section 2 we consider the latter case. The main idea is that the projection of the space curve appears as a component with multiplicity three in the discriminant. Our task is to lift this projected curve back to space; we do so by exploiting the information contained in particular singularities of the projection of the nodal curve of the surface. In this subcase, the proper silhouette consists of a union of lines, which are projections of special lines of the surface.

In Section 3 we deal with general projections to \mathbb{P}^3 of rational normal scrolls. In this case, we first identify the rational normal scroll and construct a projection from it to \mathbb{P}^2 having the proper silhouette as branching locus. Secondly, we project the rational normal scroll to \mathbb{P}^3 so that we obtain a double curve as prescribed by the singular image. Notice that the two-dimensional picture of a ruled surface tells more than the picture of an arbitrary surface: the lines of the surface map to tangents to the apparent contour, hence we already get (for free!) projections of all lines of the surface. This is an advantage which we try to keep in our algorithm. The rationality of the given ruled surface implies then that the apparent contour is itself rational. There are well-known algorithms to produce a parametrization, and such a parametrization can also be used to parametrize the set of all tangents, namely its dual curve. The μ -basis of the parametrization, which was defined and studied in the context of the implicitization problem by Cox et al. (1998), gives rise to a rational normal scroll surface, embedded in a projective space of higher dimension. The ruled surface that we wish to compute is then simply a projection of this rational scroll, and the most expensive computations go into figuring out this final projection.

We conclude the paper by recalling in Section 4, for the benefit of the reader, a known algorithm for the parametrization of curves which we implement *ad hoc* in our algorithm, since currently available general-purpose algorithms for parametrizations of plane curves do not exploit the special structure of the curves we deal with. This determines a relevant speed-up of our algorithm, since the parametrization of a particular planar curve constitutes its computational bottleneck.

An implementation in Maple of the algorithms developed in this paper is available at

<https://www.risc.jku.at/people/jschicho/pub/ChisiniRuled.mpl>.

2. Tangent developable surfaces

Among ruled surfaces, developable surfaces can be characterized as follows. A line of a ruled surface is called *torsal* if all tangent planes at all smooth points of the line are equal; a *developable surface* is then a ruled surface such that all lines of the surface are torsal.

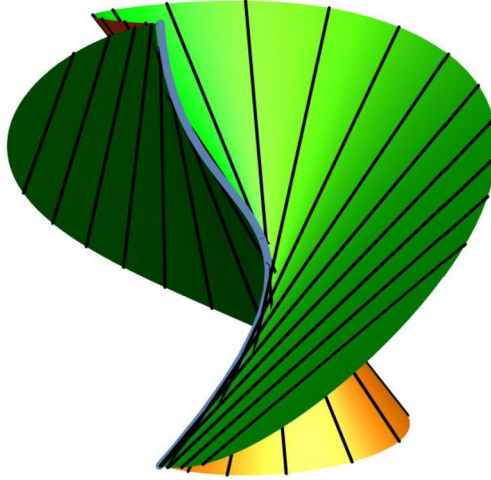


Fig. 1. The tangent developable surface of a twisted cubic, highlighted as a blue thick curve. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

A developable surface is either a *tangent developable*, i.e., the union of all tangents of a space curve (Fig. 1), or a cone over a planar curve — see (Hilbert and Cohn-Vossen, 1952, Chapter IV, Section 30) and (Ushakov, 1999, Theorem 0), or (Arrondo, 1996, Proposition 2.12) for a proof in terms of algebraic geometry. In this section, we recognize general rational tangent developables from their silhouettes with respect to general projections. The case of cones is equivalent to the recognition of a planar algebraic curve from its branching points with respect to a projection to the projective line.

Let $d \geq 3$. Every tangent developable of a rational curve of degree d is a projection of the tangent developable $T_d \subset \mathbb{P}^d$ of the rational normal curve of degree d in \mathbb{P}^d . The surface T_d has degree $2d - 2$ and admits a parametrization $\mathbb{C}^2 \rightarrow T_d$ of the form

$$(s, t) \mapsto (1 : (t + s) : (t^2 + 2st) : (t^3 + 3t^2s) : \dots : (t^d + dt^{d-1}s)).$$

The surface T_d contains the rational normal curve as a cuspidal singularity, namely locally analytically the surface around such a curve looks like a cylinder over a plane cusp. Hence tangent developables have a whole curve composed of cuspidal singularities, which is the projection of the rational normal curve in T_d . We call this curve the *cuspidal curve*, and we notice that it is a rational curve. Tangent developables may have other singularities, coming from the fact that they are projections of T_d , such as self-intersecting curves. Locally analytically around a point of a self-intersection curve, the tangent developable has two branches that intersect transversally. This is why we call these curves of singularities *nodal curves*.

Here we describe the tangent developables in \mathbb{P}^3 we intend to recognize: we prescribe their singularities having in mind the situation of a general projection of T_d to \mathbb{P}^3 . We call a tangent developable in \mathbb{P}^3 *good* if it satisfies the following conditions:

- ▷ the only singularities of the tangent developable are contained in the cuspidal curve, and in the nodal curve;
- ▷ the cuspidal curve is smooth and irreducible;
- ▷ the nodal curve is irreducible and has only ordinary triple points, or singular points as described in the next item;
- ▷ the points of intersection of the nodal curve and the cuspidal curve belong to one of the following two types:
 - points for which the local analytic equation of the surface at the point is equivalent to $(x^2 - y^3)z = 0$; in this local equation, the point is the origin, the cuspidal curve is $x = y = 0$ and the

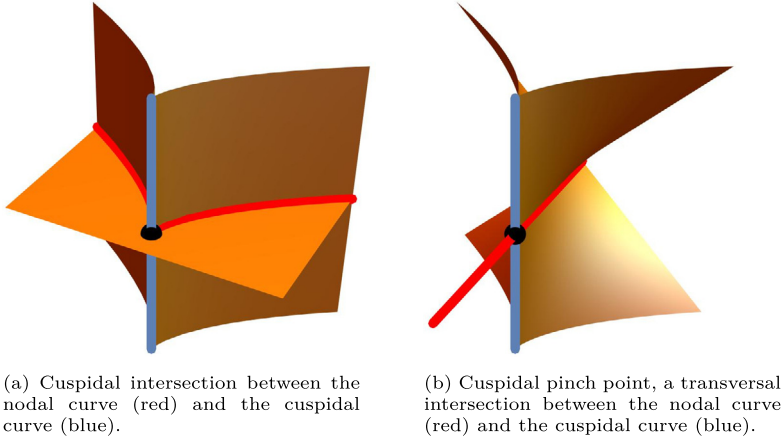


Fig. 2. Two possible types of intersection between the cuspidal and the nodal curve of a tangent developable.

nodal curve is $x^2 - y^3 = z = 0$; as the local equation shows, the nodal curve has a cusp at the intersection point (see Fig. 2a);

- points that are a transversal intersection of the nodal and the cuspidal curves at a “cuspidal pinch point”; the local analytic equation at such a point is equivalent to $z^2 y^3 - x^2 = 0$ (see Fig. 2b);

▷ there are exactly $4(d - 3)$ cuspidal pinch points.

Now we examine the silhouette with respect to a general projection to \mathbb{P}^2 of a good tangent developable surface. In the algorithm we are going to develop (Algorithm `ReconstructTangentDevelopable`), we use the knowledge of the two components of the silhouette to reconstruct the tangent developable.

Definition 2.1. Let $S \subset \mathbb{P}^3$ be a good tangent developable surface. The image of the cuspidal curve of S under a projection to \mathbb{P}^2 is called the *cuspidal image*; the image of the nodal curve of S is called the *nodal image*.

Proposition 2.2. If $S \subset \mathbb{P}^3$ is a good tangent developable surface, then the discriminant of a general projection⁶ of S to \mathbb{P}^2 has the following factors:

- ▷ a factor of multiplicity three, whose zero set is the cuspidal image;
- ▷ a factor of multiplicity two, whose zero set is the nodal image;
- ▷ several linear factors of multiplicity one, which are images of tangent planes passing through the center of projection; their zero set is the proper silhouette of the projection.

Proof. A general projection of a cusp is a triple zero of the discriminant, hence the cuspidal image is a triple component. Similarly, a general projection of a node is a double zero of the discriminant, hence the nodal image is a double component. Both these two results follow from a local analysis of the situation. In fact, locally around a general point of the cuspidal curve, we can take coordinates so that the equation of the surface is $z^2 - x^3$ and the projection is along z , so the discriminant is $-4x^3$; similarly, locally around a general point of the nodal curve, we can take the equation of the surface to be $(z - x)(z + x)$, and the projection along z , so that the discriminant is $-4x^2$. Since all lines are

⁶ The *discriminant* of a projection $\pi : S \rightarrow \mathbb{P}^2$ is the polynomial defining the locus of points $P \in \mathbb{P}^2$ such that the preimage of P under π is tangent to S . If, in coordinates, the projection is given by the map $\pi : (x : y : z) \mapsto (x : y)$ and $F = 0$ is the equation of S , then the discriminant of π is the discriminant of F with respect to z .

torsal, namely all the tangent planes at points of one of these lines coincide, the proper contour is composed of lines, and their images determine the linear factors of the discriminant. \square

The lines in the proper silhouette are inflection tangent lines of the cuspidal image. The intersections of the cuspidal image and the nodal image are either cusps of the nodal image or transverse intersections coming from transverse intersections of nodal and cuspidal curve, or from pairs of distinct point, each on one of the two curves, collapsed by the projection. Apart from these intersections, and from triple points on the nodal image, we allow the cuspidal image and the nodal image to have only ordinary double points.

The recognition problem of a tangent developable reduces to the construction of the cuspidal curve from the factors of the discriminant. The cuspidal curve is the projection of a rational normal curve of the same degree d . This projection is given by four polynomials of degree d of which we already know three, namely those that define the projection to the cuspidal image in \mathbb{P}^2 . In order to find the fourth polynomial, we have to use the nodal image. Notice, in fact, that the inflection lines do not add any information: any choice of a fourth polynomial would lead to the same linear factors of the discriminant.

The basic idea of Algorithm `ReconstructTangentDevelopable` is to collect linear conditions for the fourth polynomial derived from cuspidal pinch points. Their projections to \mathbb{P}^2 are transversal intersections of the nodal image with the cuspidal image. Hence, we try every possible subset of cardinality $4(d-3)$ of the set of all transversal intersections. Notice that if the cuspidal curve is defined over \mathbb{Q} , then the images of the cuspidal pinch points are all conjugated over \mathbb{Q} , so it is easy to extract them from the set of all intersections. Suppose that the three polynomials giving the parametrization of the cuspidal image are H_0 , H_1 and H_2 . Then the fourth polynomial H_3 can be found as follows: we make a symbolic ansatz for its coefficients and compute the determinant of the matrix

$$\begin{pmatrix} \frac{\partial^3 H_0}{\partial t_0^3} & \frac{\partial^3 H_0}{\partial t_0^2 \partial t_1} & \cdots & \frac{\partial^3 H_0}{\partial t_1^3} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^3 H_3}{\partial t_0^3} & \frac{\partial^3 H_3}{\partial t_0^2 \partial t_1} & \cdots & \frac{\partial^3 H_3}{\partial t_1^3} \end{pmatrix}. \quad (1)$$

This determinant vanishes at the parameters corresponding to images of the cuspidal pinch points (see Remark 2.3). Imposing this vanishing for all points of a subset of cardinality $4(d-3)$ of transversal intersections of the nodal image with the cuspidal image provides linear conditions for the coefficients of H_3 . We show in Lemma 2.4 that the space of solutions of these linear equations is 4-dimensional. This means that the choice of H_3 is unique up to linear combinations of H_0 , H_1 , and H_2 . In turn, this clarifies that the reconstruction of the cuspidal curve (and hence of the tangent developable) performed by Algorithm `ReconstructTangentDevelopable` is unique up to projective transformations of \mathbb{P}^3 that preserve the silhouette.

Algorithm `ReconstructTangentDevelopable`.

Input: The equation of a rational curve $C \subset \mathbb{P}^2$, the image of the cuspidal curve of a good tangent developable, and of another curve D , the image of the nodal curve.

Output: The parametrization of the cuspidal curve of the tangent developable $S \subset \mathbb{P}^3$.

- 1: **Compute** a parametrization $(H_0 : H_1 : H_2)$ of C .
 - 2: **Formulate** a symbolic ansatz for the coefficients of H_3 and compute the determinant of the matrix in Equation (1).
 - 3: **Select** a set T of $4(d-3)$ transverse intersections of C and D (the candidates for the images of cuspidal pinch points).
 - 4: **For** each point x of T **Do**
 - 5: **Evaluate** the determinant at x and collect the linear equations in the coefficients of H_3 .
 - 6: **End For**
 - 7: **Solve** the system of linear equations and obtain H_3 .
 - 8: **Return** the parametrization $(H_0 : H_1 : H_2 : H_3)$.
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We would like to point out that Algorithm `ReconstructTangentDevelopable` relies on non-trivial “basic” operations in computational algebraic geometry, as computing a parametrization of a rational curve, or computing and manipulating the intersection of two plane curves. These are, in general, difficult independent problems for which there is a vast literature, and that are objects of active research; we consider them as “building blocks” of our algorithm.

Notice that, in order to perform Step 3, one may plug the parametrization of C into the equation of D , hence obtaining an equation in a single variable. Then this equation may be solved (symbolically, which may need the introduction of algebraic numbers, or numerically), and the solutions provide the points of intersection of C and D .

Remark 2.3. Consider a nondegenerate rational curve C of degree d in \mathbb{P}^n , and a point $P \in C$, which we can suppose to be $(1 : 0 : \dots : 0)$. A local parametrization of C around P is of the form $(f_0(t) : f_1(t) : \dots : f_n(t))$ with $\text{ord}(f_i) = \alpha_i$. We may assume that $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n$ up to linear changes of coordinates. We say that P is *special* if $\alpha_n > n$. The *multiplicity* of a special point is defined as $\sum_{i=0}^n \alpha_i - \binom{n+1}{2}$, and one has that the sum of multiplicities of special points of C equals $(n+1)(d-n)$ (see (Gallet and Schicho, 2017, Definition 3.4 and Lemma 3.6)). A general rational curve C of degree d in \mathbb{P}^3 has exactly $4(d-3)$ special points of multiplicity 1 (see (Gallet and Schicho, 2017, Lemma 3.5)), and a local parametrization of C at these points is of the form $(1 + \dots : t + \dots : t^2 + \dots : t^4 + \dots)$. These points determine the cuspidal pinch points of tangent developable of C .

Lemma 2.4. *The dimension of the solution space to the linear system in Algorithm `ReconstructTangentDevelopable` is 4.*

Proof. The dimension is at least 4 since we suppose that we start from a projection of a tangent developable. In particular, there exists a rational curve C of degree d in \mathbb{P}^3 parametrized by $(H_0 : H_1 : H_2 : H_3)$ where the H_i are linearly independent elements in the solution space. By assumption, the curve C admits exactly $4(d-3)$ special points of multiplicity 1: in fact, if C has a special point P of multiplicity 1, we see from its local parametrization that the tangent developable of C has a cuspidal pinch point at P . Suppose, by contradiction, that the dimension of the solution space is at least 5. This implies that there exists a rational curve C' of degree d in \mathbb{P}^4 projecting to C . The curve C' has at least $4(d-3)$ special points that remain special after projection to \mathbb{P}^3 . Therefore, these points must be of multiplicity at least 2. This follows from the fact that special points of multiplicity 1 project to non-special points, since any multiplicity 1 point on the curve C' has local parametrization of type $(1 : t + \dots : t^2 + \dots : t^3 + \dots : t^5 + \dots)$; these points project to points in \mathbb{P}^3 around which the parametrization is $(1 : t + \dots : t^2 + \dots : t^3 + \dots)$, and so they are non-special. The sum of multiplicities of special points in C' must give $5(d-4)$, but this is a contradiction, since $2 \cdot 4(d-3) > 5(d-4)$ for $d \geq 3$. \square

Remark 2.5. Notice that there is no nodal curve in the tangent developable of a twisted cubic, and thus no special points in its cuspidal curve. However, Lemma 2.4 is trivially true for $d = 3$ since the space of univariate polynomials of degree at most 3 is 4-dimensional.

3. General projections of rational normal scrolls

In this section, we provide a reconstruction algorithm for rational ruled surfaces that admit particularly simple singularities. Recall from the introduction that, given a projection $S \rightarrow \mathbb{P}^2$, we call *contour* the locus of points on S whose tangent plane passes through the center of projection. The contour is the union of singular locus of the surface and of the *proper contour*. The image of the contour under the projection is the *silhouette*; it is constituted of the *singular image* (the image of the singular locus) and the *proper silhouette* (the image of the proper contour) (Fig. 3). In this section, we consider *good* rational ruled surfaces $S \subset \mathbb{P}^3$ and *good* projections $S \rightarrow \mathbb{P}^2$, namely we ask that:

- ▷ S has at most ordinary singularities: an irreducible self-intersection curve, self-intersection triple points, and pinch points;

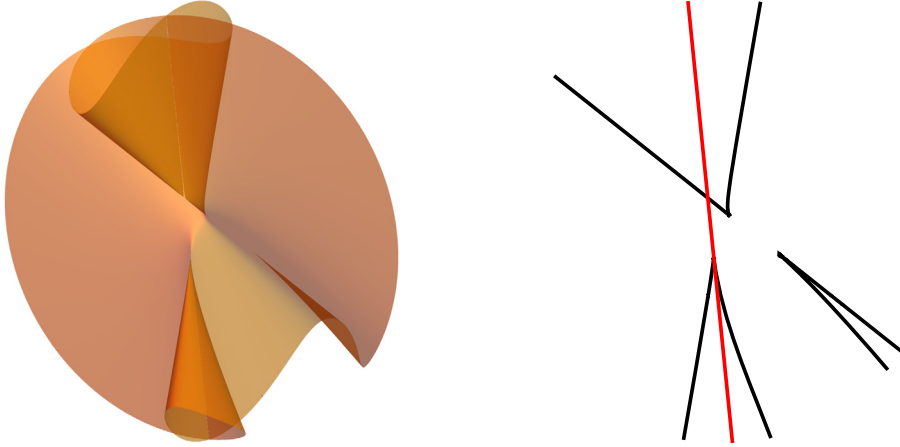


Fig. 3. A singular ruled cubic and its silhouette, constituted of the singular image (straight line in red) and the proper silhouette (in black).

- ▷ the proper silhouette has only nodes and cusps;
- ▷ the singular image has only nodes and ordinary triple points;
- ▷ the proper contour is irreducible and projects birationally to the proper silhouette;
- ▷ the singular curve projects birationally to the singular image.

Recall that every rational ruled surface is a projection of a rational normal scroll. Our assumptions imply that the restriction of this projection to its ramification locus is generically injective. Notice that all our assumptions are fulfilled when we consider projections from general centers (both when we project from the rational normal scroll to \mathbb{P}^3 , and when we project from the surface S to \mathbb{P}^2). In particular, irreducibility of the singular curve is Franchetta's Theorem (see (Mezzetti and Portelli, 1997, Theorem 6)). The fact that good rational ruled surfaces have ordinary singularities implies that they have finitely many torsal lines, each passing through exactly one pinch point.

We divide the reconstruction process in two steps: first, we determine the rational normal scroll of which the surface is a projection, together with the projection from this rational normal scroll to \mathbb{P}^2 , by computing a parametrization of the dual of the proper silhouette, which is a rational curve; second, we construct the projection map from the rational normal scroll to the surface in \mathbb{P}^3 . To explain how the first part of the reconstruction works, we recall some facts about rational normal scrolls.

For our purposes, we use the description of *rational normal scrolls* provided by (Cox et al., 2011, Example 2.3.16): given two natural numbers $d_1, d_2 \in \mathbb{N}$, the rational scroll Σ_{d_1, d_2} is the Zariski closure of the image of the map

$$p_{d_1, d_2}: (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^{d_1+d_2+1}, \quad (s, t) \mapsto (1 : t : \dots : t^{d_2} : s : st : \dots : st^{d_1}).$$

In this way, a linear projection $\rho: \Sigma_{d_1, d_2} \longrightarrow \mathbb{P}^2$ can be identified with two vectors of polynomials $Q_1, Q_2 \in \mathbb{C}[t]^3$ of degrees d_1 and d_2 so that the map $r: (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^2$

$$(s, t) \mapsto (Q_{20}(t) + s Q_{10}(t) : Q_{21}(t) + s Q_{11}(t) : Q_{22}(t) + s Q_{12}(t)) \quad (2)$$

fits into the commutative diagram

$$\begin{array}{ccc} \Sigma_{d_1, d_2} & \xrightarrow{\rho} & \mathbb{P}^2 \\ p_{d_1, d_2} \uparrow & \nearrow r & \\ (\mathbb{C}^*)^2 & & \end{array}$$

Since the projection $\rho: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ is a map between smooth varieties, the notions analogous to the ones of contour and silhouette for projections $S \rightarrow \mathbb{P}^2$ are usually called, in this case, the *ramification locus* and the *branching locus*, respectively. Hereafter, we collect some pieces of information about the branching locus of such a projection. The following result is well-known, but we report the proof, since we could not find a reference asserting exactly the fact we need.

Lemma 3.1. *Let $\rho: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ be a projection from a rational normal scroll whose restriction to the ramification locus $R \subset \Sigma_{d_1, d_2}$ is generically injective. Then every line in Σ_{d_1, d_2} is projected by ρ to a tangent line of the branching locus.*

Proof. Let $L \subset \Sigma_{d_1, d_2}$ be a line. By hypothesis, we know that $\rho|_L$ and $\rho|_R$ are both generically injective because any projection whose center is not in Σ_{d_1, d_2} is generically injective on the lines of the surface. Let $p \in L \cap R$ be a smooth point of R , then $\rho(L)$ is tangent to $\rho(R)$ at $\rho(p)$. In fact, by definition, the tangent plane of Σ_{d_1, d_2} at p intersects the center of projection. The tangent line of R at p and L are both contained in this tangent plane. The projection either collapses the tangent line of R . In the first case, since $\rho|_R$ is generically injective, and there are no birational maps between smooth curves collapsing tangent vectors, it follows that $\rho(p)$ is singular in $\rho(R)$. Therefore the statement is trivially true. In the second case, both L and the tangent line of R are projected to a single line in \mathbb{P}^2 , which equals $\rho(L)$. Hence $\rho(L)$ is tangent to $\rho(R)$ at $\rho(p)$. \square

The next proposition provides the first part of the reconstruction algorithm. We are going to use the concept of μ -bases, introduced by Cox, Sederberg and Chen in Cox et al. (1998): these are particular sets of generators of the module of syzygies of a parametrization of a curve or of a surface; we refer to the original paper for a more precise definition and for their properties.

Proposition 3.2. *Let $B \subset \mathbb{P}^2$ be the branching locus of a projection $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ whose restriction to the ramification locus is generically injective. Let (Q_1, Q_2) be a μ -basis of a parametrization of \check{B} , the dual curve of B . Then the projection $\rho: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ induced by (Q_1, Q_2) has B as branching locus. Moreover, every general projection $\hat{\rho}: \Sigma_{\delta_1, \delta_2} \rightarrow \mathbb{P}^2$ having B as branching locus is projectively equivalent to ρ over B , namely there exists a projective automorphism $\alpha: \Sigma_{\delta_1, \delta_2} \rightarrow \Sigma_{d_1, d_2}$ such that α fixes all linear spaces of dimension $d_1 + d_2 - 1$ through the center of projection, which has dimension $d_1 + d_2 - 2$.*

Proof. We first show that B is the branching locus of the map $\rho: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ obtained from (Q_1, Q_2) . By the properties of μ -bases (see (Song and Goldman, 2009, Theorem 2)), the vector $Q_1 \times Q_2$ gives a parametrization of \check{B} . This implies that the projection ρ sends the lines of Σ_{d_1, d_2} to the family of tangent lines of B . In fact, the Plücker coordinates of the line between $Q_1(t)$ and $Q_2(t)$ are given by $(Q_1 \times Q_2)(t)$. By Lemma 3.1, the curve B is the branching locus of ρ . Suppose that $\phi: \Sigma_{\delta_1, \delta_2} \rightarrow \mathbb{P}^2$ is another general projection having B as branching locus. Let $\tilde{Q}_1, \tilde{Q}_2 \in \mathbb{C}[t]^3$ be the two vectors of polynomials defining ϕ as explained in Equation (2). By Lemma 3.1, the images of the lines of $\Sigma_{\delta_1, \delta_2}$ are lines tangent to B . Hence, by the same argument as before, $\tilde{Q}_1 \times \tilde{Q}_2$ is a parametrization of \check{B} . Thus $(\tilde{Q}_1, \tilde{Q}_2)$ is a μ -basis of this parametrization. By the uniqueness of μ -bases, it follows that $\delta_1 = d_1$ and $\delta_2 = d_2$, and that ϕ and ρ differ by an automorphism over B . \square

Proposition 3.2 tells us that, in order to reconstruct the rational normal scroll of which the desired surface in \mathbb{P}^3 is a projection it is enough to compute a μ -basis of a parametrization of the dual of the proper silhouette.

This concludes the first part of the reconstruction process, and proves the correctness of Algorithm `ReconstructRationalScroll`: there, instead of computing the map ρ , we compute the map r as in Equation (2), which provides the same information. In fact, the proper silhouette of a projection $S \rightarrow \mathbb{P}^2$ is the branching locus of the corresponding projection $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$.

Algorithm ReconstructRationalScroll.

Input: A curve $B \subset \mathbb{P}^2$, the proper silhouette of a projection S of a rational normal scroll Σ_{d_1, d_2} whose restriction to the ramification locus is generically injective.

Output: Two numbers $d_1, d_2 \in \mathbb{N}$ and a map $r: (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is B .

- 1: **Compute** the dual curve \check{B} of B .
- 2: **Parametrize** the curve \check{B} .
- 3: **Compute** a μ -basis (Q_1, Q_2) of the parametrization of \check{B} .
- 4: **Set** $(d_1, d_2) = (\deg(Q_1), \deg(Q_2))$.
- 5: **Set** r to be the map

$$(s, t) \mapsto (Q_{20}(t) + s Q_{10}(t) : Q_{21}(t) + s Q_{11}(t) : Q_{22}(t) + s Q_{12}(t)).$$

- 6: **Return** d_1, d_2 and r .
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As a consequence of Algorithm ReconstructRationalScroll, we obtain a characterization of proper silhouettes of good projections $S \rightarrow \mathbb{P}^2$, where S is a good rational ruled surface.

Proposition 3.3. *Proper silhouettes of good projections $S \rightarrow \mathbb{P}^2$ of good rational ruled surfaces S are rational plane curves with maximal number of cusps for a given degree and only ordinary nodes as the other singularities.*

Proof. Let B be a proper silhouette of a good projection $S \rightarrow \mathbb{P}^2$, where S is a good rational ruled surface. We know that the degree of the proper silhouette B is $n = 2d - 2$, where $d = \deg(S)$. We can obtain this number by considering the degree of the silhouette, which is $d(d - 1)$, and subtracting from it twice the degree of the singular image, which is $\frac{1}{2}(d - 1)(d - 2)$ (see (Piene, 2005, Section 6)). If we denote by δ the number of nodes of B and by κ the number of cusps of B , then from the rationality of B and the Plücker formulas we get

$$\frac{(n - 1)(n - 2)}{2} - \delta - \kappa = 0 \quad \text{and} \quad n^* = n(n - 1) - 2\delta - 3\kappa,$$

where n^* is the degree of the dual of B . Since the dual of B is in fact a plane section of the dual of S , and the dual of a ruled surface is a ruled surface of the same degree, we conclude that $n^* = d$. These two equations imply that $\kappa = \frac{3}{2}(n - 2)$. On the other hand, we know that for a rational curve of degree n the number of cusps must be less than or equal to $\frac{3}{2}(n - 2)$ (see (Lefschetz, 1913, Section 5) and (Hollcroft, 1937, Section 4)). Hence proper silhouettes of rational ruled surfaces are rational curves with maximal number of cusps for a given degree.

Conversely, if we are given a rational curve of degree n , where n is of the form $2d - 2$ for some d , having $\frac{3}{2}(n - 2)$ ordinary cusps and only ordinary nodes as the other singularities, then we can apply Algorithm ReconstructRationalScroll to it, thus showing that such a curve is the silhouette of the projection of a rational ruled surface whose restriction to the ramification locus is generically injective. \square

Now that, recalling Equation (2), we reconstructed the map $\rho: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ via Algorithm ReconstructRationalScroll, we proceed by recovering the projection $\Sigma_{d_1, d_2} \rightarrow S$. To do so, we need to use the information provided by the singular image, namely the projection on \mathbb{P}^2 of the singular locus of S . We formulate two different algorithms to construct this projection: one (Algorithm CollapseMates) imposes the projection to collapse pairs of points in Σ_{d_1, d_2} in order to create a prescribed double curve; the other (Algorithm UsePinchPoints), instead, forces the differential of the projection to be rank-deficient at the preimages of pinch points.

Recall that the double curve of S is irreducible by assumption. This implies that, in presence of pinch points, the curve in Σ_{d_1, d_2} that is mapped 2 : 1 to the singular curve of S is also irreducible, because the two sheets of the double cover meet precisely at the preimages of the pinch points. Let $W \subset \mathbb{P}^2$ be the singular image of a good projection $S \rightarrow \mathbb{P}^2$, where S is a good rational ruled sur-

face. Let $W' \subset \rho^{-1}(W)$ be the curve that is mapped 2 : 1 to the singular curve Z of S . The curve W' has bidegree

$$(d_1 + d_2 - 2, (d_1 + d_2 - 2)(d_2 - 1)).$$

In fact, by the so-called *double point formula* (see (Fulton, 1998, Section 9.3) for the general formula, and (Dolgachev, 2012, Chapter 10, Equation 10.52) for a specialization of the formula in our case) its divisor class is given by $(d_1 + d_2 - 2)(H - L)$, where H is the class of a hyperplane section of Σ_{d_1, d_2} and L is the class of a line; the bidegree of W' then follows from the fact that H has bidegree $(1, d_2)$ and L has bidegree $(0, 1)$.

For any general point $p \in W'$ there is a unique $q \in W'$ such that $\{p, q\}$ is a fiber of the double cover of Z . Note that the restriction ρ' of ρ to W' is also a 2 : 1 map whose fibers are equal to the fibers of $W' \rightarrow Z$. This allows us to compute q in terms of p using ρ . We say that q is the *mate* of p .

In order to express the conditions imposed by our algorithm more explicitly, we can assume without loss of generality that the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is the map forgetting the last coordinate. Notice that the map $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^3$ is defined by four linear forms, three of which we already know from the map $\rho : \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^2$ obtained via Algorithm `ReconstructRationalScroll`. Let F_0, F_1 , and F_2 be the known forms, and let F be the form to be determined. For every general point $p \in W'$ and its mate q , we get a linear condition in the coefficients of F by requiring that the map $(F_0 : F_1 : F_2 : F)$ collapses p and q , namely by asking that

$$F_i(q)F(p) - F_i(p)F(q) = 0 \quad \text{for every } i \in \{0, 1, 2\}.$$

The forms F_0, F_1 , and F_2 also satisfy this linear condition, so we look for a solution of the linear system that is linearly independent from F_0, F_1 , and F_2 .

We claim that the dimension of the solution space for the linear system above is not bigger than 4. Assume, indirectly, that there is a fifth linearly independent form G satisfying the linear equations defined above. Then, the image of the map $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^4$ defined by F_0, F_1, F_2, F , and G is a nondegenerate rational ruled surface with a double curve of the same degree as the double curve of S , namely $\frac{1}{2}(d-1)(d-2)$. Moreover, the projection from \mathbb{P}^4 to \mathbb{P}^3 is birational once restricted to the two surfaces. This contradicts the following lemma.

Lemma 3.4. *Let $f : \Sigma_{d_1, d_2} \rightarrow X \subset \mathbb{P}^3$ be a projection of a rational normal scroll satisfying our assumptions, namely X is a good rational ruled surface. Suppose that f factors through a projection \tilde{f} to \mathbb{P}^4 , namely that $f = \pi \circ \tilde{f}$ where $\tilde{f} : \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^4$ and $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Then $\tilde{X} := \tilde{f}(\Sigma_{d_1, d_2})$ has at most isolated singularities.*

Proof. Assume by contradiction that \tilde{X} has a singular curve. Consider a general plane section C of X and its preimage $\tilde{C} \subset \tilde{X}$. Both \tilde{C} and C have the same geometric genus because they are birational. By Bertini's theorem, since X is the projection of a smooth projective variety, the singularities of C are exactly at the intersection with the singular locus Z of X , and their number is $\deg(Z)$. Since $\deg(Z) = \frac{1}{2}(d-1)(d-2)$ (see (Piene, 2005, Section 6)), and we know that C is rational, it follows from the geometric genus formula (see (Hartshorne, 1977, Chapter IV, Exercise 1.8(a))) that the delta invariant of each of the singularities of C is 1. The sum of delta invariants of singularities of the space curve \tilde{C} must be strictly smaller than the sum of the delta invariants of the singularities of C . This follows from the geometric genus formula, since the arithmetic genus $\frac{1}{2}(d-1)(d-2)$ of C is bigger than the arithmetic genus of \tilde{C} , which cannot be greater than $\frac{1}{4}d^2 - d + 1$ (see (Hartshorne, 1977, Chapter IV, Theorem 6.4 and Figure 18)). It follows that the singular locus $\tilde{Z} \subset \tilde{X}$ must have strictly lower degree than $Z \subset X$, because every intersection of \tilde{C} and \tilde{Z} determines a singularity of \tilde{C} . This implies the image of \tilde{Z} under the projection to \mathbb{P}^3 is a component of Z , but by assumption Z is irreducible. This contradiction concludes the proof. \square

From an algorithmic point of view, to obtain the desired projection $\Sigma_{d_1, d_2} \longrightarrow \mathbb{P}^3$, we need to find a solution F of the infinitely many linear equations above (one for each pair of mates) that is linearly independent from F_0 , F_1 , and F_2 . We could just collect sufficiently many points on W and solve the linear equations arising from them and their mates. However, finding points on W is not trivial. What we do instead is to compute the mate of a point with coordinates in a transcendental field extension of the base field that is isomorphic to the function field of W . More concretely, the equations

$$F_i(q)F_j(p) - F_i(p)F_j(q) = 0 \quad \text{for every } i, j \in \{0, 1, 2\}$$

allow one to write the coordinates (u, v) of the mate q of a point $p = (s, t)$ as rational functions of s and t . This is a consequence of the fact that whenever we have a $2 : 1$ map $C \longrightarrow D$ between two curves, then there exists a birational automorphism of C swapping the two points in any fiber. This leads to a single linear equation for F with coefficients in this function field. Using Gröbner bases, we can eliminate from this single equation the generators of the function field and obtain an equivalent system of linear equations with scalar coefficients.

The discussion so far proves the correctness of Algorithm `CollapseMates`.

Algorithm `CollapseMates`.

Input: A map $r: (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is B , and the singular image W of a good projection $S \longrightarrow \mathbb{P}^2$ with proper silhouette B , where S is a good rational ruled surface.

Output: A parametrization of the surface S .

- 1: **Compute** the preimage of the singular image W under r . Let h be the polynomial defining such preimage.
- 2: **Select** a factor H of h of bidegree $(d_1 + d_2 - 2, (d_1 + d_2 - 2)(d_2 - 1))$.
- 3: **Construct** the system of equations for the mate $q = (u, v)$ of a point $p = (s, t)$. Let F_0 , F_1 and F_2 be the components of the map r . The equations for mates are

$$F_0(u, v) F_2(s, t) - F_2(u, v) F_0(s, t) = 0$$

$$F_1(u, v) F_2(s, t) - F_2(u, v) F_1(s, t) = 0$$

$$H(u, v) = 0$$

- 4: **Write** u and v as rational functions $U(s, t)$ and $V(s, t)$ using the previous equations by computing a Gröbner basis with an elimination term order.
- 5: **Set up** a system of equations for the coefficients of a polynomial F_3 of the form $F_{23}(t) + sF_{13}(t)$ with F_{i3} of degree d_i with indeterminate coefficients as follows:

$$F_0(U(s, t), V(s, t)) F_3(s, t) - F_3(U(s, t), V(s, t)) F_0(s, t) = 0$$

- 6: **Solve** the linear system for the coefficients of F_3 .

- 7: **Return** the parametrization $(F_0 : F_1 : F_2 : F_3)$.
-

The second algorithm we propose is based on the observation that the fourth unknown polynomial F satisfies particularly simple equations coming from pinch points: the Jacobian of the projection $\Sigma_{d_1, d_2} \longrightarrow S$ is rank-deficient at the preimages of pinch points, more precisely the tangent line of W' at those points is collapsed to a point by the map defined by $(F_0 : F_1 : F_2 : F)$.

For this algorithm to work, we need to suppose that the images of pinch points under the projection $S \longrightarrow \mathbb{P}^2$ are transversal intersections of the proper silhouette and the singular image. This is true for projections from general centers (see (Gallet et al., 2019, Proposition 2.1) and the discussion before for a more thorough analysis).

The algorithm works as follows: for each intersection P of the proper silhouette and the singular image coming from a pinch point, one considers the corresponding point $P' \in \Sigma_{d_1, d_2}$ that is sent to P by $\rho: \Sigma_{d_1, d_2} \longrightarrow \mathbb{P}^2$. Notice that P' is unique by assumption, since P comes from a pinch point. The point P' lies on the curve W' which is mapped to the singular image by ρ . We compute the tangent line of W' at P' , and we impose that the differential of the map $(F_0 : F_1 : F_2 : F)$ sends it to zero. In this way, we obtain linear conditions for the coefficients of F . Solving these linear systems provides the desired parametrization of S . Notice that we do not know *a priori* which transversal intersections

of the proper silhouette and the singular image come from pinch points, therefore in principle one may have to repeat the previous procedure for all possible subsets of $4(d-3)$ transversal intersections of proper silhouette and singular image (recall that we suppose from the beginning that the number of pinch points is $4(d-3)$). We hence get the following algorithm.

Algorithm UsePinchPoints.

Input: A map $r: (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is B , the singular image W of a good projection $S \rightarrow \mathbb{P}^2$ where S is a good rational ruled surface of degree d , and the images in \mathbb{P}^2 of the pinch points of S .

Output: A parametrization of the surface S .

- 1: **Pick** $4(d-3)$ transverse intersections of B and W .
 - 2: **Compute** the preimages in $(\mathbb{C}^*)^2$ of these $4(d-3)$ points.
 - 3: **Compute** the preimage \tilde{W}' in $(\mathbb{C}^*)^2$ of the singular image W .
 - 4: **Define** a polynomial F_3 of the form $F_{32}(t) + sF_{31}(t)$ with F_{3i} of degree d_i with indeterminate coefficients.
 - 5: **For** each preimage P' of the images of the pinch points **Do**
 - 6: **Compute** a tangent vector of \tilde{W}' at P' .
 - 7: **Add** linear equations for the coefficients of F_3 obtained by imposing that the map $(F_0 : F_1 : F_2 : F_3)$ sends the tangent vector to zero.
 - 8: **End For**
 - 9: **Solve** the linear system for F_3 .
 - 10: **Return** the parametrization $(F_0 : F_1 : F_2 : F_3)$.
-

We show that the solution space for the linear equations in Step 8 is exactly four-dimensional, thus proving the correctness of the algorithm. The dimension of this solution space is at least 4 because we know by hypothesis that there is a good projection $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^3$ whose components satisfy the linear conditions. Proposition 3.2 implies that the dimension cannot be bigger than 4.

We propose an alternative proof of this fact, based on the torsal lines on the ruled surface, which reveals some of the underlying geometry of these surfaces. The proof that the dimension cannot be bigger goes in two steps: first, we show that lines in S passing through critical values⁷ of the projection are torsal, and secondly we prove that a surface in \mathbb{P}^4 with too many torsal lines must be degenerate.

Recall from Equation (2) that a projection $\alpha: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^4$ can be encoded in two vectors of polynomials $Q_1, Q_2 \in \mathbb{C}[t]^5$ via the map

$$(\mathbb{C}^*)^2 \rightarrow \mathbb{P}^4, \quad (s, t) \mapsto Q_2(t) + sQ_1(t)$$

in such a way that $Q_1(t)$ and $Q_2(t)$ are linearly independent for every t . The map α is singular at the point (s, t) if and only if the matrix

$$\left(Q_1(t) \mid Q_2(t) \mid \frac{\partial Q_2}{\partial t}(t) + s \frac{\partial Q_1}{\partial t}(t) \right)$$

has rank at most 2.

Lemma 3.5. *Let $P' \in \Sigma_{d_1, d_2}$ and let $\alpha: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^4$ be a projection entering in a commutative diagram of projections*

$$\begin{array}{ccc} & & \mathbb{P}^4 \\ & \nearrow \alpha & \mid \\ \Sigma_{d_1, d_2} & \xrightarrow{\beta} & \mathbb{P}^3 \\ & & \mid \\ & & \downarrow \psi \end{array}$$

where the image of β is a good rational ruled surface. Suppose that α is singular at P' . Then $\alpha(P')$ lies on a torsal line of $\alpha(\Sigma_{d_1, d_2})$.

⁷ Recall that the *critical values* of a differentiable map are the images of the points at which the differential of the map does not have maximal rank.

Proof. Recalling notation introduced above and the fact that a line is torsal when the tangent planes of the ruled surface at each of its points coincide, we have to prove that the matrix

$$\left(Q_1(t_0) \mid Q_2(t_0) \mid \frac{\partial Q_1}{\partial t}(t_0) \mid \frac{\partial Q_2}{\partial t}(t_0) \right)$$

has rank 3, where (s_0, t_0) are the coordinates of P' . Since α is singular at P' , this matrix cannot have rank 4. On the other hand, if this matrix had rank 2, then the line in $\alpha(\Sigma_{d_1, d_2})$ passing through $\alpha(P')$ would be singular. However, there are no singular lines in $\alpha(\Sigma_{d_1, d_2})$, since otherwise also the image of β would have singular lines, and this is not allowed by our general assumption. \square

Lemma 3.6. *Let T be a rational ruled surface in \mathbb{P}^4 with at least $2(\deg(T) - 2)$ torsal lines. Suppose that there exists a projection $T \rightarrow S$ where $S \subset \mathbb{P}^3$ is a rational ruled surface with at most ordinary singularities. Then T is degenerate, namely it is contained in a hyperplane.*

Proof. The surface T is the image of a projection $\alpha: \Sigma_{d_1, d_2} \rightarrow T$ from a rational normal scroll. Let $Q_1, Q_2 \in \mathbb{C}[t]^5$, as above, be the two vectors of polynomials of degree d_1 and d_2 , respectively, encoding the projection α . Torsal lines in T correspond to values $t_0 \in \mathbb{C}$ such that the matrix

$$M := \left(Q_1(t) \mid Q_2(t) \mid \frac{\partial Q_1}{\partial t}(t) \mid \frac{\partial Q_2}{\partial t}(t) \right)$$

has rank 3 at t_0 . These values are precisely the common zeros of the determinants of the submatrices M_0, \dots, M_4 obtained by removing a row from the previous matrix. The degree of these determinants (as polynomials in t) is at most $2(\deg(T) - 2)$. In fact, elementary column operations transform the previous matrix into

$$\left(\frac{\partial Q_1}{\partial t}(t) \mid \frac{\partial Q_2}{\partial t}(t) \mid d_1 Q_1 - t \frac{\partial Q_1}{\partial t}(t) \mid d_2 Q_2 - t \frac{\partial Q_2}{\partial t}(t) \right)$$

and $\deg(T) = d_1 + d_2$. Hence all the five determinants $\mathcal{M}_i := \det(M_i)$ are of the form $\lambda_i \mathcal{M}$ for $\lambda_i \in \mathbb{C}$ and $\mathcal{M} \in \mathbb{C}[t]$. The kernel of M^t contains the element

$$(\mathcal{M}_0, -\mathcal{M}_1, \dots, \mathcal{M}_4) = \mathcal{M}(\lambda_0, -\lambda_1, \dots, \lambda_4),$$

thus $\lambda_0 Q_{10}(t) - \lambda_1 Q_{11}(t) + \dots + \lambda_4 Q_{14}(t) = 0$ and similarly for $Q_2(t)$. Hence all the points of the form $Q_2(t) + s Q_1(t)$ are contained in a hyperplane, namely T is degenerate. \square

We can now prove that the solution space for the polynomial F defining the map $\Sigma_{d_1, d_2} \rightarrow \mathbb{P}^3$ is exactly four-dimensional. In fact, if it were bigger, we would get a projection $\alpha: \Sigma_{d_1, d_2} \rightarrow \mathbb{P}^4$ which is singular at all points P' that are preimages of the transversal intersections of the proper silhouette and the singular image coming from pinch points of S . By (Piene, 2005, Section 6), the surface S has $2(\deg(S) - 2)$ pinch points. By Lemma 3.5, they determine $2(\deg(S) - 2)$ torsal lines in $T := \alpha(\Sigma_{d_1, d_2})$. Lemma 3.6 shows the contradiction.

The discussion so far proves the correctness of Algorithm `UsePinchPoints`.

We can sum up the findings of this section in the following algorithm, which reconstructs, up to projective automorphisms preserving the silhouette, a good rational ruled surface S starting from the singular image and the proper silhouette of a good projection $S \rightarrow \mathbb{P}^2$:

Algorithm `ReconstructRatRuledSurface`.

Input: The singular image W and the proper silhouette B of a good projection $S \rightarrow \mathbb{P}^2$ of a good rational ruled surface S . A map $r: (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is B , the singular image W of a good projection $S \rightarrow \mathbb{P}^2$ where S is a good rational ruled surface, and the images in \mathbb{P}^2 of the pinch points of S .

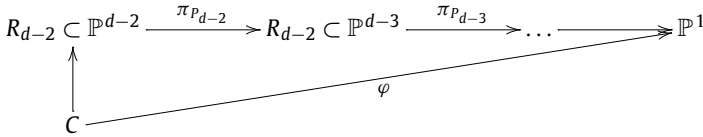
Output: A parametrization of the surface S .

- 1: **Apply** Algorithm `ReconstructRationalScroll` to the input B and obtain a map $r: (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^2$ as in Equation (2), whose branching locus is B .
 - 2: **Apply** Algorithm `CollapseMates` or `UsePinchPoints` to the map r and the curves B and W and obtain a parametrization of S , up to projective automorphisms preserving B and W .
 - 3: **Return** the parametrization of S .
-

4. A faster parametrization for the silhouette

The bottleneck of Algorithm `ReconstructRationalScroll` is the computation of the parametrization of the dual of the silhouette. Our situation is quite special: by assumption the silhouette admits only nodes and cusps. General-purpose algorithms for parametrizing curves (as, for example, the one implemented in Maple), do not have the possibility to take into account this special structure of the curve. This is why we implement an *ad hoc* procedure for the parametrization of the silhouette that uses the fact that we only have nodes and cusps. Although the methods used are all known, we believe it could be beneficial for the reader to have an overview of this algorithm.

We use the well-known technique of *adjoints* to compute the parametrization, see (Sendra et al., 2008, Section 4.7). Given a planar curve C of degree d with only nodes and cusps, the linear system of adjoints is given by those homogeneous forms of degree $d - 2$ that pass through the singularities of C . In order to get the adjoint forms, we have to take the homogeneous component of degree $d - 2$ of the radical of the Jacobian ideal of C . One way to obtain this radical ideal is the following: consider the discriminant of the curve C along a random projection; this is a bivariate homogeneous polynomial whose factors H_{nodes} of order 2 correspond to nodes of C and whose factors H_{cusps} of order 3 correspond to cusps of C . If we add the form $H_{\text{nodes}} \cdot H_{\text{cusps}}$ to the Jacobian ideal of C , then we get its radical. The image of C under the rational map induced by the linear system of adjoints is a rational normal curve R_{d-2} in \mathbb{P}^{d-2} of degree $d - 2$. Suppose now that a smooth point $P \in C$ is known. Then we get a point P_{d-2} in R_{d-2} , and the projection from P_{d-2} maps R_{d-2} to a rational normal curve R_{d-3} in \mathbb{P}^{d-3} of degree $d - 3$. Since the projection is a map between smooth curves, it can be extended also to P_{d-2} , which gets mapped to the image of the tangent line $T_{P_{d-2}}R_{d-2}$ under the projection. In this way we obtain a point $P_{d-3} \in R_{d-3}$, so we can repeat the procedure until we land on \mathbb{P}^1 :



In this way, we get a map $\varphi: C \rightarrow \mathbb{P}^1$, whose inverse is the desired parametrization. Notice that the map φ can be computed by selecting those adjoint forms that vanish with multiplicity $d - 3$ at P .

This discussion leads to the following algorithm:

Algorithm ParametrizeSilhouette.

Input: A rational curve $C \subset \mathbb{P}^2$ with only nodes and cusps and a smooth point $P \in C$.

Output: A parametrization $\psi: \mathbb{P}^1 \rightarrow C$ of C .

- 1: **Compute** the radical J of the Jacobian ideal of C : for example, consider the discriminant H of a general projection of C on a line, factor H and add to the ideal of derivatives of C the factors of H of order 2 and 3.
 - 2: **Let** L be the saturation of the ideal generated by the equation of C and by the $(d - 3)^{\text{rd}}$ power of the ideal of the point P .
 - 3: **Let** $K := J \cap L$.
 - 4: **Compute** a basis \mathcal{B} of the homogeneous component of degree $d - 2$ of K .
 - 5: **Compute** the inverse ψ of the map $C \rightarrow \mathbb{P}^1$ induced by \mathcal{B} .
 - 6: **Return** ψ .
-

We implemented the algorithms in Maple and tested it on a computer with an Intel I7-5600 processor (1400 MHz). We report the timings in Table 1.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Table 1

The table shows the degree d of the surface S , the degree of the proper silhouette B , its number of nodes n and of cusps c in the ruled case, and the degree of the nodal curve N , of the cuspidal curve C , and the number i of inflection lines in the tangent developable case, and the computing time in CPU seconds. Notice that the general algorithm developed in Gallet et al. (2019) takes 4s and 130s in the cases of ruled surfaces of degree 4 and 5, respectively.

d	B	n	c	N	C	i	Time	Type
4	-	-	-	6	4	6	2s	developable
5	-	-	-	16	5	9	28s	developable
6	-	-	-	30	6	12	145s	developable
4	6	4	6	-	-	-	< 1s	ruled
5	8	12	9	-	-	-	90s	ruled

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