# Dynamical Reduction Models

Angelo Bassi*
The Abdus Salam International Centre for Theoretical Physics, Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy.
and

GianCarlo Ghirardi†
Department of Theoretical Physics of the University of Trieste, and
the Abdus Salam International Centre for Theoretical Physics, Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy.

**Abstract**

The report presents an exhaustive review of the recent attempt to overcome the difficulties that standard quantum mechanics meets in accounting for the measurement (or macro–objectification) problem, an attempt based on the consideration of nonlinear and stochastic modifications of the Schrödinger equation. The proposed new dynamics is characterized by the feature of not contradicting any known fact about microsystems and of accounting, on the basis of a unique, universal dynamical principle, for wavepacket reduction and for the classical behavior of macroscopic systems. We recall the motivations for the new approach and we briefly review the other proposals to circumvent the above mentioned difficulties which appeared in the literature. In this way we make clear the conceptual and historical context characterizing the new approach. After having reviewed the mathematical techniques (stochastic differential calculus) which are essential for the rigorous and precise formulation of the new dynamics, we discuss in great detail its implications and we stress its relevant conceptual achievements. The new proposal requires also to work out an appropriate interpretation; a procedure which leads us to a reconsideration of many important issues about the conceptual status of theories based on a genuinely Hilbert space description of natural processes. Attention is also paid to many problems which are naturally raised by the dynamical reduction program. In particular we discuss the possibility and the problems one meets in trying to develop an analogous formalism for the relativistic case. Finally we discuss the experimental implications of the new dynamics for various physical processes which should allow, in principle, to test it against quantum mechanics. The review covers the work which has been done in the last fifteen years by various scientists and the lively debate which has accompanied the elaboration of the new proposal.

## Contents



*e-mail: bassi@ictp.trieste.it
†e-mail: ghirardi@ts.infn.it

# 1 Introduction

The twentieth century has seen the birth of what are unanimously considered the two basic pillars of modern science: relativity theory and quantum mechanics. Both these theoretical constructions have met an unprecedented predictive success in accounting for the results of the incredibly refined experiments which have been made possible by recent technological improvements. Both schemes imply radical changes concerning the (classical) views about natural phenomena.

Relativity theory has required a drastic modification of our views concerning space and time, quantum mechanics has compelled the scientific community to attribute a prominent role to chance in physics and to accept the existence of unavoidable limitations to the attainable knowledge about physical systems. However, the conceptual status of the two theories is remarkably different. If one accepts that instantaneous communication is impossible and that the velocity of light represents an upper limit to the propagation of any physical action, one must reconsider the problem of synchronizing clocks and is led to the conclusion that the space-time continuum is the correct framework for the description of natural processes. The ensuing theory, special relativity, is an example of a precisely formulated and internally consistent theory, one which, to use A. Shimony's words [1], allows to *close the circle* and to base on it a coherent worldview.

The situation is quite different with the other pillar of modern science, quantum mechanics, as is evident if one takes into account the lively debate about its interpretation which started soon after its formulation and which is still going on. This debate concerns one of the most peculiar aspects of the theory, generally known as *the measurement problem*, even though a more appropriate term to characterize it would be *the macro–objectification problem*. It stems directly from the linear nature of the theory and the way in which it connects the mathematical entities which are claimed to represent the most accurate specification which is in principle possible of the state of a system and the outcomes of prospective measurement processes. ¿From this point of view, quantum mechanics is an extremely successful and powerful mathematical device yielding the probabilities of the results of any conceivable measurement procedure. But, in contrast with this unprecedented efficiency in telling us everything about *what we find* the theory is silent, to use J.S. Bell's words [2], about *what it is*. Actually, just due to its linear nature, the quantum description of the measurement process and of all those measurement–like processes [2] *we are obliged to admit ... are going on more or less all the time, more or less everywhere* and lead to the definite perceptions which characterize our experience, contradicts the idea that all natural processes and, in particular, the micro–macro interactions occurring in the situations we have just mentioned are governed by quantum mechanics itself. In brief, the theory contains two incompatible dynamical principles, the linear Schrödinger evolution and the wavepacket reduction process which is associated to micro–macro interactions.

This serious limitation would not represent by itself a deadlock for the theory: one could simply accept that it has only a limited field of validity. However, if one takes such a position one must pretend that the theory itself allows the identification of a phenomenological area in which the transition from micro to macro, from reversible to irreversible, from deterministic to stochastic, in short, from quantum to classical, takes place. But this is not the case. The borderline between these two different regimes is by no means precisely identifiable. There are for sure many macro–systems which require a fully quantum treatment.

A significant indication of the peculiar situation we have just outlined comes from reconsidering the historical debate about the meaning and the interpretation of the formalism. Such a debate has seen the successive identification of different levels which, according to the various thinkers, should mark the place at which one has to

pose the split, which is then characterized by a certain shiftiness. For instance, in the famous Bohr–Einstein debate, Bohr, who had repeatedly claimed that the split should be associated to the micro or macroscopic nature of the physical systems under consideration, was compelled, in order to reject the pressing criticisms by Einstein, to accept that macroscopic parts of the apparatus (such as macro shutters and macro pointers) require a fully quantum treatment. An analogous situation occurred subsequently: London, Bauer and Wigner were led to identify the split with the borderline separating physical from conscious processes. But, once more, our present knowledge does not give a clear indication about what is conscious, so that a remarkable vagueness characterizes also such a proposal.

The recent years have seen a noticeable and renewed interest about the macro–objectification problem. We can quite confidently state that, nowadays, there is a large consensus among scientists interested in the foundational aspects of the theory that the so called orthodox interpretation (a rather ambiguous expression which however encompasses similar positions concerning the measurement problem) has completely failed in yielding a consistent and coherent account of natural phenomena. In connection with this renewed interest in the field, new and original attempts to overcome the difficulties have appeared. What one wants is a quantum mechanical model of measurements as dynamical processes governed by precise rules agreeing, on the one side, with the quantum description of microscopic systems, and, on the other, with the classical aspects characterizing the macroscopic world.

The present report is devoted to a detailed presentation of recent proposals (the dynamical reduction theories) aimed at overcoming the difficulties of the macro–objectification process, proposals which stick to the idea that the knowledge of the statevector represents the maximal information one can have about the state of a physical system. Accordingly, these attempts differ radically from all those, like hidden variable theories, which invoke the incompleteness of the Hilbert space description to solve the measurement problem. Within the dynamical reduction framework, the basic idea to get the desired result consists in accepting that the linear and deterministic Schrödinger equation has to be modified by the addition of non linear and stochastic terms. As we will see, this program, which Einstein himself [1] considered unviable, can be consistently followed leading to a fully satisfactory model at the non relativistic level.

Obviously, we do not think that the dynamical reduction theories we are going to discuss in this report might represent instances of the final theory of natural processes. However, the very fact that they can be consistently developed throws a new light on the subject, allows us to identify some basic features of any dynamics inducing reductions, makes precise and allows us to better understand the action at a distance of the standard theory and might also suggest interesting experiments aimed to identify possible violations of the superposition principle.

The paper is organized as follows. First of all we discuss in all details the macro–

[1] We recall that Einstein, in his *Reply to critics* [3], has explicitly considered the possibility of giving up, for micro systems, the request that they possess objectively all properties, i.e., he was prepared to accept the linear nature of their state space. But he has stressed that he could not renounce to his realistic requirements at the macro level, so that macro objects cannot be in superpositions of macroscopically different states. In accordance with this position he has contemplated the idea of abandoning the superposition principle at the macro level. His concluding remarks are of great relevance for the dynamical reduction program: *the macroscopic and the microscopic are so inter–related that it appears impracticable to give up this program* [scientific realism in the classical sense, as requiring that all systems possess objective properties] *in the microscopic alone*. The dynamical reduction program proves that the line that Einstein considered impracticable is actually consistently viable. More recently analogous remarks — which occurred repeatedly during all the history of quantum mechanics — have been put forward once more by Shimony. In contemplating the possibility of reduction at an appropriate level, he has stated that [4] *reasonable desiderata* [for the dynamical reduction program] *pull in opposite directions*.

objectification problem to make clear how it gives rise to a physically unacceptable situation which must be faced (part I). We also briefly review many other proposals to overcome the difficulties of the formalism, to make clear the differences of the dynamical reduction point of view with respect to other attempts to avoid the inconsistencies of the orthodox position. We then come to the core of the report by reviewing the development of the dynamical reduction program and by discussing all the most relevant mathematical features and physical implications of such an approach (part II). We pass to analyze the debate about the interpretation of dynamical reduction models (part III). This analysis will allow the reader to understand and to fully appreciate the crucial innovative implications of the approach. Part IV of the report is devoted to an important and to a large extent still open problem, i.e. the one of finding a fully satisfactory relativistic model inducing reductions. We will conclude our analysis with a short discussion of some experimental situations which will make clear, on the one side, how it is difficult to devise *experimenta crucis* allowing to discriminate such models from quantum mechanics, and, on the other, will give some indications about the experiments which seem most appropriate to identify violations (if they are there) of the linear nature of quantum theory (part V).

# Part I
# The Measurement Problem of Quantum Mechanics

## 2 Basic principles of Quantum Mechanics

We review here the general mathematical structure of Quantum Mechanics, with special attention to the postulate of wavepacket reduction, the one which gives rise to the measurement problem. We also introduce formal tools like the statistical operator formalism, calling attention to some subtle features which can give rise to misunderstandings concerning some fundamental questions.

### 2.1 The axioms of Quantum Mechanics

Standard Quantum Mechanics can be synthetically summarized by the following set of rules:

1. Every physical system $S$ is associated to an Hilbert space $\mathcal{H}$; the physical states of $S$ are represented by normalized vectors (called "statevectors") $|\psi\rangle$ of $\mathcal{H}$. Physical observables[2] $O$ of the system are represented by self–adjoint operators in $\mathcal{H}$: the possible outcomes of a measurement of $O$ are given by the eigenvalues $o_n$ of the corresponding operator, which we assume here, for simplicity, to have a discrete and non degenerate spectrum:

$$O\,|o_n\rangle \quad = \quad o_n\,|o_n\rangle. \tag{2.1}$$

   Since $O$ is a self–adjoint operator, its eigenvalues $o_n$ are real and the eigenvectors $|o_n\rangle$ form a complete orthonormal set in the Hilbert space $\mathcal{H}$.

2. To determine the state $|\psi(t_0)\rangle$ of the system $S$ at a given initial time $t_0$, a complete set of commuting observables for $S$ is measured: the initial statevector

[2] In the following, we will denote with $O$ both the observable and the corresponding operator in $\mathcal{H}$; when confusion arises, we will specify whether $O$ refers to the observable or to the operator.

is then the unique common eigenstate of such observables. Its subsequent time evolution is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle \quad = \quad H\,|\psi(t)\rangle, \tag{2.2}$$

which uniquely determines the state at any time once one knows it at the initial one. The operator $H$ is the Hamiltonian of the system $S$.

3. The probability of getting, in a measurement at time $t$, the eigenvalue $o_n$ in a measurement of the observable $O$ is given by:

$$P[o_n] \quad = \quad |\langle o_n|\psi(t)\rangle|^2, \tag{2.3}$$

$|\psi(t)\rangle$ being the state of the system at the time in which the measurement is performed.

4. The effect of a measurement on the system $S$ is to drastically change its statevector from $|\psi(t)\rangle$ to $|o_n\rangle$, $o_n$ being the eigenvalue obtained in the measurement:

$$|\psi(t)\rangle \text{ before measurement} \qquad \longrightarrow \qquad |o_n\rangle \text{ after measurement.}$$

This is the famous **postulate of wavepacket reduction (WPR)**.

These, in short, are the postulates of Quantum Mechanics. Of course, we have somewhat simplified the exposition; for example we have ignored the possibility of the operator $O$ having a continuous spectrum besides or in place of the discrete one; we have not discussed the case of degenerate eigenvectors, and so on. All such features, even though important, are not crucial for understanding the measurement problem.

## 2.2 Schrödinger evolution and wavepacket reduction

The Schrödinger equation (2.2) has two basic properties. First, it is a first order differential equation in the time variable; this means that once the initial state of the system $|\psi(t_0)\rangle$ is known, its future evolution is completely determined. The evolution of the statevector of any physical system is thus perfectly **deterministic**, like in classical mechanics.

The solution of equation (2.2) can be written as follows:

$$|\psi(t)\rangle \quad = \quad U(t,t_0)\,|\psi(t_0)\rangle, \tag{2.4}$$

where $U(t,t_0)$ is the evolution operator. This is a unitary operator, a necessary requirement in order that it preserves the norm of the statevector and, accordingly, to make tenable the probabilistic interpretation of the theory. Its formal expression is:

$$U(t,t_0) \quad = \quad e^{-i\,H\,(t-t_0)/\hbar}. \tag{2.5}$$

The second important feature of the Schrödinger equation is that it is **linear**: if $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ are two possible solution of (2.2), then also $\alpha\,|\psi_1(t)\rangle + \beta\,|\psi_2(t)\rangle$ is a possible solution, where $\alpha$ and $\beta$ are two arbitrary complex numbers[3]. This is the mathematical formulation of the celebrated[4] **superposition principle**.

[3] If the original vectors are normalized and the two states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ are orthogonal, their linear combination is normalized when $|\alpha|^2 + |\beta|^2 = 1$.

[4] To be precise, the superposition principle includes also the assumption that all states of the Hilbert space can actually occur and thus in particular that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states for $S$, then also $\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle$ is a possible state.

The postulate of wavepacket reduction exhibits features which are at odds with Schrödinger's evolution. First of all it describes a **nonlinear** evolution of the statevector, since it transforms the state $\alpha\,|\psi_1(t)\rangle + \beta\,|\psi_2(t)\rangle$ into either $|\psi_1(t)\rangle$ or $|\psi_2(t)\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively. The second important property of wavepacket reduction is its genuinely **probabilistic** nature: in general we cannot know to which one of the eigenstates of $O$ the statevector of $S$ will be reduced as a consequence of a measurement process; the theory determines only the *probability* of the reduction to any particular eigenstate. This is where probability and indeterminacy enter into play, making Quantum Mechanics so different from classical theories.

Summing up, we have:

| Schrödinger evolution | Wavepacket reduction |
|---|---|
| linear | nonlinear |
| deterministic | stochastic |

The above table shows clearly that standard Quantum Mechanics has some peculiar features: it contains **two dynamical evolution principles**, one governed by the Schrödinger equation and the other taking place when wavepacket reduction occurs. They are radically different, and they contradict each other. This fact gives rise to the measurement problem of the theory, which represents the starting point which has led to the elaboration of the theories which are the subject of the present report.

## 2.3 The statistical operator

The quantum mechanical rules sketched in section 2.1 refer only to **pure states**, i.e. to physical systems whose statevector is perfectly known. As already remarked, the state of a system can be determined exactly only by measuring a complete set of commuting observables.

In practice it is not always possible to perform such kinds of measurements; the real experimental situation could not even require it. It may very well happen that we have (or need) only a partial knowledge of the state of the system, pretty much like in classical statistical mechanics: the statistical operator formalism has been designed to deal with these situations.

Suppose we know that the statevector describing an individual system is one among a set $\{|\psi_i\rangle\}$ of vectors, but we do not know which one it actually is: we only know the probability $p_i$ that $|\psi_i\rangle$ is the correct statevector. Equivalently, we can suppose that we have an ensemble of $N$ systems, a fraction $N_1$ of which is described by the vector $|\psi_1\rangle$, a fraction $N_2$ by the vector $|\psi_2\rangle$ and so on. Taking a system out of the ensemble, the probability that $|\psi_i\rangle$ is its associated statevector is $p_i = N_i/N$. We call such ensembles **statistical mixtures**; they are characterized by the vectors $\{|\psi_i\rangle\}$ together with their probability distribution $p_i$.

The **statistical operator** is then defined as follows:

$$\rho \quad = \quad \sum_i p_i\,|\psi_i\rangle\langle\psi_i|. \tag{2.6}$$

The operator $\rho$ is a trace class, trace one semi–positive definite operator, and replaces completely the statevector when only a partial knowledge of the state of the system

is available. Of course, for a pure state $|\psi\rangle$, i.e. for one representing the most accurate knowledge that the theory considers as possible about a system, the statistical operator reduces to the projection operator $|\psi\rangle\langle\psi|$.

A nice feature of statistical operators is that they allow us to use a compact formalism to deal both with pure states and with genuine statistical mixtures and, at the same time, they allow us to distinguish between them. In fact the following property holds:

$$\begin{array}{lll} \rho^2 = \rho & & \text{for pure states} \\ \rho^2 \neq \rho & & \text{for statistical mixtures.} \end{array}$$

The quantum axioms 2–4 are expressed as follows in this new language. The evolution equation for $\rho$ is:

$$i\hbar \frac{d}{dt}\rho(t) \quad = \quad [H, \rho(t)]\,, \tag{2.7}$$

which is again a linear first order differential equation to be solved taking into account the initial condition $\rho(t_0)$. The probability that the outcome of a measurement of an observable $O$ is one, $o_n$, of its eigenvalues is given by[5]:

$$P[o_n] \quad = \quad \mathrm{Tr}[P_n\,\rho(t)], \tag{2.8}$$

$P_n$ being the operator which projects onto the linear manifold associated to the eigenvalue $o_n$. Finally, at the end of a measurement process giving $o_n$ as its outcome, the statistical operator changes in the following way:

$$\rho \text{ before measurement} \qquad \longrightarrow \qquad \frac{P_n\,\rho\,P_n}{\mathrm{Tr}[P_n\,\rho\,P_n]} \text{ after measurement.}$$

This is of course the appropriate expression for the wavepacket reduction postulate.

A final remark. The statistical operator formalism allows us to handle also *non selective measurements*, which we are now going to define, in a quite natural way. Suppose we perform a measurement on an ensemble of systems in the same state (pure or mixed); in general each of them will give different outcomes and, after the measurement, due to wavepacket reduction, they will be described by different statevectors. If we decide to keep all the systems, independently of the outcomes we have obtained, we perform a *non selective* measurement: we do not have a pure state, even when the state is pure before the measurement is performed. The measurement process turns it into the following statistical mixture:

$$\rho \text{ before measurement} \qquad \longrightarrow \qquad \sum_n P_n\,\rho\,P_n \text{ after measurement,}$$

$P_n$ being the projection operators associated to the eigenmanifolds of the observable which has been measured. Since the measurement is non selective, the operators $P_n$ sum up to the identity: it is easy to check that all the mathematical properties of $\rho$ are preserved.

## 2.4 Property attribution in standard quantum mechanics

As already remarked, the characteristic trait of standard Quantum Mechanics is that, in general, it allows only probabilistic predictions about the possible outcomes of measurements. Such probabilities have a truly **nonepistemic** character, i.e. they are not due to our ignorance about the precise state of the system, like in classical statistical mechanics: rather, quantum theory is such that physical systems by themselves do

[5] "Tr" denotes the trace of the operator.

not possess all properties one can think of. Another way to put it — following the Copenhagen doctrine — is to say that, in general, we are not allowed to speak of the properties of a system concerning most of its observables: the best we can do is to speak of what we can *observe* about the system.

However, it is possible to rescue, still remaining within the standard formalism, a sort of "minimal ontology" for physical systems. This was first formulated by Einstein, Podolski and Rosen [5]:

> If, without in any way disturbing the system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

Following axiom 3, an eigenvalue $o_n$ of an observable $O$ has probability 1 of being found in a measurement of $O$ if and only if the statevector $|\psi(t)\rangle$ of the system is an eigenstate of $O$ pertaining to the considered eigenvalue. We can then say that **a system possesses a property if and only if its statevector belongs to the eigenmanifold associated to the eigenvalue corresponding to that property**. Any measurement aimed to test this statement will give a positive result. Of course, we can develop a similar argument when the probability of an outcome is equal to zero and we can claim that the corresponding property is certainly not possessed by the system.

It is not difficult to understand that within this "minimal ontology" physical systems do not possess all the properties one would be inclined to attach to them (like position and momentum at any given time), because a vector cannot be a simultaneous eigenvector of too many operators, due to the non–abelian character of the algebra of the operators. Worse than that, individual systems in entangled states possess in general no properties at all [6].

In particular, one can almost never attach definite macro–properties, specifically precise locations in space, even to macroscopic systems, so that they cannot be considered as being in a definite region of space. This fact gives rise to the so called **macro–objectification problem**, i.e. to the necessity of accounting for the emergence of the properties corresponding to our definite perceptions for such systems. This crucial point, another aspect of the quantum measurement problem, will be extensively analyzed in the following section.

When dealing with statistical mixtures, property attribution is rather delicate since there is an interplay between *epistemic* (classical–like) and *nonepistemic* (quantum–like) probabilities. To be precise, let us consider an arbitrary mixture of states $\{|\psi_i\rangle\}$, with probabilities $p_i$; in analogy with classical statistical mechanics, the state of any system of the ensemble is described by a precise vector of the set $\{|\psi_i\rangle\}$, but we are ignorant about which is the correct one: we only know the probability characterizing each of them. Accordingly, the probability distribution $p_i$ has an epistemic character. On the other hand, each such vector has a probabilistic physical content which, as before, is genuinely nonepistemic.

# 3 The quantum measurement, or macro–objectification, problem

This section is devoted to a general and detailed discussion of the measurement, or macro–objectification, problem of Quantum Mechanics. The nature of the problem has already been anticipated: the linear nature of Quantum Mechanics allows the occurrence of superpositions of macroscopically different states of a macro–object, e.g. concerning their location, in spite of the fact that macroscopic systems are

always located, or at least we perceive them as being located, in a well defined region of space.

We will first analyze the measurement problem within the framework of the von Neumann scheme for an ideal measurement process (subsection 3.1): this is a very simple and elegant measurement model which goes directly to the root of the problem. Nontheless, von Neumann's argument has been repeatedly criticized for the over–simplified assumptions on which it is based. In subsection 3.2, we will show that even by adopting a very general and realistic measurement model, one can derive the same conclusions reached by von Neumann: superpositions of different macroscopic states cannot be avoided within the quantum framework.

## 3.1 The von Neumann measurement scheme

The first explicit example of the quantum description of a measurement process was presented by John von Neumann [7] and is usually referred to as the "ideal" measurement scheme. It gained great popularity since, due to its simplicity, it allows us to grasp immediately the key points of the problem; nowadays almost all textbooks on the foundations of Quantum Mechanics make reference to it. The von Neumann argument goes as follows.

Let us consider a microscopic system $S$ and one of its observables $O$. Let $o_n$ be the eigenvalues of $O$ (we assume, for simplicity, that its spectrum is purely discrete and non–degenerate) and $|o_n\rangle$ the corresponding eigenvectors. Let us call $M$ the apparatus devised to measure the observable $O$ of the microsystem $S$. $M$ has a ready–state $|A_0\rangle$, i.e. a state in which the apparatus is ready to measure the considered property, plus a set of *mutually orthogonal* states $|A_n\rangle$ corresponding to *different macroscopic configurations* of the instrument, like, e.g., different positions of a pointer along a scale.

Finally, we assume that the interaction between the microsystem $S$ and the apparatus $M$ is *linear* (since the Schrödinger equation is supposed to govern all natural processes) and that it yields a *perfect correlation* between the initial state of $S$ and the final state of the apparatus, i.e.

$$\text{Initial state:} \quad |o_n\rangle \otimes |A_0\rangle \qquad \longrightarrow \qquad \text{Final state:} \quad |o_n\rangle \otimes |A_n\rangle; \tag{3.1}$$

in this way we are sure that if the final state of the apparatus is $|A_n\rangle$ (i.e. the pointer for example is in the $n$–th position along the scale), the final state of the particle is $|o_n\rangle$ and the observable $O$ has the value $o_n$, in accordance with the property attribution discussed in section 2.4.

The measurement problem arises when the initial state of the particle, previous to the measurement, is not just one of the eigenvectors $|o_n\rangle$ like in equation (3.1), but a superposition of them, for example:

$$|m+l\rangle \quad = \quad \frac{1}{\sqrt{2}}[|o_m\rangle + |o_l\rangle],$$

which can be very easily prepared in our laboratories. In this case, if the *linear* evolution equation of the theory is assumed to govern all physical processes, the final state of the microsystem+apparatus will be:

$$\begin{aligned} |m+l\rangle \otimes |A_0\rangle \quad &= \quad \frac{1}{\sqrt{2}}\,[|o_m\rangle + |o_l\rangle] \otimes |A_0\rangle \longrightarrow \\ &\longrightarrow \quad \frac{1}{\sqrt{2}}\,[|o_m\rangle \otimes |A_m\rangle + |o_l\rangle \otimes |A_l\rangle]. \end{aligned} \tag{3.2}$$

Such a state is an entangled state of the microscopic system and of the apparatus, which is not an eigenstate of the relevant observable $M$ of the apparatus, i.e. the

position of the pointer. In situations like this, as already discussed, it is not legitimate, *even in principle*, to state that the properties associated to the states $|A_m\rangle$ or $|A_l\rangle$ are possessed by the apparatus: as a consequence the apparatus is not in any of the macroscopic definite configurations we perceive it to be. This is the first part of the quantum measurement problem.

The *standard* way out from this difficulty is given by the *wavepacket reduction postulate* (axiom 4 listed in section 2.1), which states that "at the end of the *measurement* process" the final vector in equation (3.2) reduces to one of its terms:

$$|o_m\rangle \otimes |A_m\rangle \qquad \text{or} \qquad |o_l\rangle \otimes |A_l\rangle,$$

with a probability given by the square modulus of the coefficient associated to that term (1/2 for both cases, in our example).

We have already mentioned, and we have proved now, that the postulate of wavepacket reduction contradicts the assumption of the general validity of the Schrödinger equation; this of course is a very unsatisfactory feature of standard Quantum Mechanics: it incorporates two contradictory dynamical evolutions, something we cannot accept for a physical theory. Moreover, the real physical difficulty is not only the one of the consistency of the Schrödinger evolution and wavepacket reduction[6]; the even more serious problem is that the theory does not tell us in which precise cases the linear hamiltonian evolution has to be suspended and wavepacket reduction takes place. As we will see, dynamical reduction models offer a natural, precise and unambiguous solution to both problematic aspects of the measurement problem.

Coming back to the standard theory, we mention that, in spite of the above difficulties, various authors [8, 9, 10, 11] have maintained that the measurement problem does not derive from the structure of quantum mechanics (in particular from the linear character of the quantum evolution), or from the postulate of wavepacket reduction, but from adopting the over–simplified model of measurement processes put forward by von Neumann. If one takes into account more realistic models, they argue, the measurement problem turns into a false one: the postulate of wavepacket reduction is not anymore necessary and, consequently, there is no need to modify the interpretation of the theory, or even to put forward a new theory.

In particular, the following assumptions have been criticized:

- That the measuring apparatus can be prepared in a precise state $|M_0\rangle$: since the instrument is a macroscopic object with many degrees of freedom, it is impossible to know its precise state at any given time.
- That one can safely neglect the interactions between the apparatus and the surrounding environment. Such interactions with the environment (which are referred to as decoherence) produce essentially a randomization of the phases associated to the different components of the wavefunction, a process which can be seen as an *apparent* collapse of the wavefunction into one of its components.
- That the final states of the apparatus, corresponding to perceptively different macroscopic configurations of the apparatus itself, are orthogonal: actually, different states usually correspond to different *positions* of some part of the instrument, and since no wavefunction can have compact support in configuration space (because of the quantum evolution), wavefunctions corresponding to different states cannot, in general, be orthogonal.
- That the final state of the apparatus gets perfectly correlated to the initial state of the microscopic system: this is an highly idealized characteristic which is not shared by any realistic physical instrument.

[6]Such a difficulty could be circumvented by assuming that the theory has only a limited field of validity and, in particular, it does not apply to macro–systems.

In the next subsection we will consider a very general measurement scheme [12] which takes into account all the above criticisms. We will show that superpositions of states corresponding to different macroscopic configurations of macro–objects cannot be avoided within a strict quantum mechanical context. Correspondingly, the appearance of macroscopic situations which are incompatible with our definite perceptions about such objects is inescapable. This "empasse" can only be circumvented either by adopting a precise and unambiguous interpretation which differs from the orthodox one, or by modifying the theory itself[7].

## 3.2 A completely general measurement scheme

In this subsection we re–derive von Neumann's conclusions on the basis of what, in our opinion, is the most general possible description of a measurement instrument and of a measurement process. We begin first by defining the microscopic system whose properties we want to measure.

### 3.2.1 The Microscopic System

For simplicity we consider the simplest system upon which non–trivial measurements can be performed, i.e., a system $S$ whose associated Hilbert space $\mathcal{H}_S$ is two–dimensional — like the one describing the spin of an electron, or the polarization states of a photon — and we consider an observable $O$ having two different eigenvalues; let us call $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ the eigenstates associated to these eigenvalues. For definiteness, we will consider an individual such system and we will call "spin" its degree of freedom; we will say that the particle has "spin Up" when it is in state $|\mathrm{u}\rangle$, and that it has "spin Down" when it is in state $|\mathrm{d}\rangle$. Besides these two states, also their superpositions can be taken into account, in particular the following one:

$$|\mathrm{u}+\mathrm{d}\rangle \quad = \quad \frac{1}{\sqrt{2}}\,[\,|\mathrm{u}\rangle + |\mathrm{d}\rangle\,],$$

a vector describing a new state, "spin Up + spin Down", of the particle. Without any loss of generality, we will assume that, by resorting to appropriate procedures, one can "prepare" the system $S$ in any one of the three above considered states $|\mathrm{u}\rangle$, $|\mathrm{d}\rangle$ and $|\mathrm{u}+\mathrm{d}\rangle$.

We remark that we could have considered more general physical systems, like compound ones, and observables having a more complicated spectrum. However, in accordance with the generally accepted position that microsystems can be prepared in a precise quantum state and with the nowadays common experimental practice to handle single particles and to measure their discrete properties, we have chosen to work with a very simple microsystem like the one we are considering here.

Accordingly, after the preparation, the system is in a precise and known state, and it can be treated as isolated from the rest of the universe, at least until the measurement process begins[8]. We stress that if one denies these assumptions it is not even clear what he takes quantum theory to be about.

### 3.2.2 The Measuring Apparatus

A measuring apparatus is a *macroscopic* system which, interacting with the microsystem whose properties one is interested in ascertaining, ends up into a state more or

[7] An explicit proof that releasing the request of an ideal measurement does not allow us to circumvent the measurement problem can be found in the well known book by d'Espagnat [13]; however, his proof is much more complex and much less general than the one we are going to present here.

[8] In mathematical terms, we assume that, *prior to the measurement process*, the wavefunction of the universe factorizes into the wavefunction of the particle times the wavefunction of the rest of the world.

less strictly correlated with the eigenstates of the observable it is devised to measure. The different possible outcomes of the measurement are supposed to be associated to *perceptively different macroscopic configurations* of a part of the apparatus, e.g. different positions of the pointer (for analogic instruments), different numbers on a display (for digital ones), different spots on a photographic plate, different plots on a screen, and so on. For simplicity, in what follows we will assume that the apparatus has a pointer movable along a scale, whose position registers the result of the measurement.

Contrary to microsystems, the measuring apparatus, being a macroscopic object, has many degrees of freedom, most of which — in particular the microscopic ones — we cannot control at all; and of the macroscopic ones, like the position of the pointer, we can have only a very limited control. Moreover, due to its dimensions, the apparatus is always interacting with the environment, whose degrees of freedom are also essentially out of control. Following this line of reasoning, one can remark that the apparatus — or at least its constituents — existed quite a long time before the measurement is performed, so it had all the time to interact, even if only weakly, with a large part of the universe. All these interactions make to a large extent unknown and uncontrollable the state of the macroscopic system which enters into play. In spite of this difficulty, in order to keep our analysis as general as possible, we will take into account all the above mentioned facts which make the measurement non–ideal.

With reference to the above discussion, we should in general speak of different situations of the "whole universe", even though our "reading" refers only to the degrees of freedom of the pointer; accordingly, we will indicate the statevectors we will deal with in the following way:

$$|A\ \alpha\rangle.$$

These vectors belong to the Hilbert space associated to the apparatus, the environment, and, in the most general case, to the whole universe. $A$ is a label indicating that the pointer of the apparatus is in a specific macroscopic configuration, i.e. one which we perceive and we identify with a specific position along the scale. In first approximation, we could say that $A$ is essentially the value $x$ characterizing the "projection operator" $|x\rangle\langle x|$ ($|x\rangle$ being an "improper" statevector of the Hilbert space of the pointer) giving the exact position of e.g. the centre of mass of the pointer along the scale. But it is evident that no system can be prepared in such a state since it is impossible to measure a continuous variable with perfect accuracy; and even if it were possible to do so, the hamiltonian evolution would immediately change that state.

We could try to improve our description by taking into account, in place of precise positions along the scale, small intervals $\Delta(x) = [x - \delta, x + \delta]$, and to claim that "the pointer is at position $x$" when the wavefunction is an eigenstate of the projection operator onto the interval[9] $\Delta(x)$ of the center of mass position. If one makes such a choice, the label $A$ characterizing our general state $|A\ \alpha\rangle$ refers to any wavefunction having such a property, of course with the interval $\Delta(x)$ replaced by the interval $\Delta(A)$: as a consequence, for the considered state we could claim that "the pointer is at position A". However, also this approach is not viable since the hamiltonian evolution transforms any wavefunction with compact support into a wavefunction with a non–compact one; this fact gives rise to what has been called the "tail problem", a problem which cannot be avoided, and which renders rather delicate the task of making precise the idea of "an object being somewhere" within a quantum mechanical framework. More about this in what follows.

Following the above analysis, we consider a very general physical situation: we call $V_A$ the set of all (normalized) vectors $|A\ \alpha\rangle$ for which we are *allowed* to say that "the

[9] Of course, here we are considering for simplicity a one–dimensional situation; the argument can be easily generalized to the three–dimensional case.

pointer of the apparatus is at position $A$" or, stated differently, that "the universe is in a configuration which we perceive as one corresponding to the statement: the pointer is at $A$". We do not put any restriction to the vectors belonging to $V_A$: they can represent wavefunctions with or without tails, more or less localized in space, and so on; we do not even resort to projection operators to characterize these states. The only physical requirement we put forward is that, if the pointer admits two macroscopically and perceptively different "positions" along the scale (let us call them $A$ and $B$), then any two vectors corresponding to such different configurations must be "almost orthogonal". This requirement can be translated into the following mathematical relation: denoting by $V_B$ the set of all normalized vectors corresponding to the statement "the pointer is at $B$" while $V_A$, as before, contains all the vectors corresponding to the statement "the pointer is at $A$", we must have:

$$\inf_{\substack{|A\,\alpha\rangle \in V_A \\ |B\,\beta\rangle \in V_B}} \| |A\,\alpha\rangle - |B\,\beta\rangle \| \geq \sqrt{2} - \eta \qquad \eta \ll 1, \tag{3.3}$$

i.e. the minimum distance between the vectors of the two above sets cannot differ too much from $\sqrt{2}$, which is the distance between two orthogonal normalized states. We recall that the orthogonality request of the standard measurement theory is done to be sure to be dealing with strictly mutually exclusive situations. Obviously such a request can be partially released (as we are doing here) but not given up completely if one wants to be able to "read" the outcome in a fundamentally non ambiguous way. It is evident that (3.3) is a necessary requirement if one pretends that *different* macroscopic positions of the pointer (and of any other system) represent *practically mutually exclusive* configurations of the object[10] (see also the final remark of this subsection).

Let us now comment on the second parameter $\alpha$ characterizing our states: this is an index which takes into account all other degrees of freedom that are out of control[11]; thus two vectors labeled by $A$ but with different values for $\alpha$, refer to the "same" macroscopic configuration for the pointer (or, in general, of the "part of the universe we perceive"), while they describe two different states for the rest of the universe (e.g., given a certain atom of the pointer, it might be in the ground state when the state is $|A\,\alpha\rangle$, while it might be in an excited state when it is $|A\,\beta\rangle$).

Since we are interested in the two spin states of the microscopic particle, if we want to use the apparatus to distinguish them we have to assume that the pointer admits at least two macroscopically different positions ($U$ and $D$) along the scale[12]. The previous argument requires then that there exist two sets $V_U$ and $V_D$, the first one containing all the vectors corresponding to the situation in which the pointer can be said to point at "U", the second all those vectors associated to the statement "the pointer is at D". Moreover, these two sets must be almost orthogonal in the sense of (3.3):

$$\inf_{\substack{|U\,\alpha\rangle \in V_U \\ |D\,\beta\rangle \in V_D}} \| |U\,\alpha\rangle - |D\,\beta\rangle \| \geq \sqrt{2} - \eta \qquad \eta \ll 1, \tag{3.4}$$

[10] Obviously, here we are making reference to a genuinely quantum description (with the completeness assumption). In alternative interpretations or formulations of the theory, orthogonality is not necessary to guarantee macroscopic differences. Typically, in hidden variables theories one could have non orthogonal wavefunctions and different values for the hidden variables such that the associated physical situations are macroscopically different and mutually exclusive.

[11] From the mathematical point of view, $\alpha$ stands for the eigenvalues of a complete set of commuting observables for the whole universe, exception made for the "location" of the pointer.

[12] The idea is that, if we perform the measurement and we find the pointer in the position labeled by $U$, then we can claim the "the result of the measurement is that the spin of the particle is Up"; similarly, if we find the pointer in $D$, then we can say that "the spin of the particle is Down".

One interesting property of $V_U$ and $V_D$ (which is shared by any pair of sets for which (3.4) is satisfied) is that they have no vectors in common: in fact, it is easy to see that if $V_U$ and $V_D$ had such a common vector, then the left hand side of (3.4) would take the value zero, a fact which would contradict (3.4). From the physical point of view, this property is obvious since a vector belonging both to $V_U$ and to $V_D$ would be a vector for which we could claim both that "the pointer points at U" and that "the pointer points at D", a contradictory situation since "U" and "D" correspond to macroscopically and perceptively different situations.

### 3.2.3 The Preparation of the Apparatus

A measuring instrument must be prepared before one performs a measurement, i.e. one has to arrange the apparatus in such a way that it is ready to interact with the microscopic system and give a result; following the discussion of the previous subsection, it is evident that the initial statevector must carry an index $\alpha$ which takes into account the state of the rest of the universe. Accordingly, we will denote the initial statevector as $|A_0\ \alpha\rangle$, where $A_0$ indicates that the pointer "is" in the ready ($A_0$) state.

However, we note that, besides the measuring instrument, we have also to prepare the microsystem in a precise state, and moreover we have assumed that after the preparation and immediately before the measurement process, the microsystem itself is isolated from the rest of the universe; the initial statevector for the whole universe can then be written as:

$$|A_0\ \alpha\rangle \quad = \quad |\text{spin}\rangle \otimes |A_0\ \overline{\alpha}\rangle,$$

where $\overline{\alpha}$ specifies the state of the whole universe, with the exception of the initial state of the micro–particle and the initial "position" of the pointer; $|\text{spin}\rangle$ is the initial statevector of the particle.

Obviously, also in the process of preparing the apparatus we cannot control the state of the universe, so that we do not know the precise initial state $|A_0\ \overline{\alpha}\rangle$: in practice, in any specific situation any value for the index $\overline{\alpha}$ will occur with a given probability $p(\overline{\alpha})$, which in general is unknown to us — but of course it has to satisfy appropriate requirements we will discuss in what follows. Accordingly, the initial setup for the apparatus and the microscopic particle will be described as follows:

$$\text{Initial Setup} \quad = \quad \{\, |\text{spin}\rangle \otimes |A_0\ \overline{\alpha}\rangle, \quad p(\overline{\alpha})\,\},$$

where $p(\overline{\alpha})$ gives the probability distribution of the remaining, uncontrollable, degrees of freedom.

### 3.2.4 The Measurement Process

If one assumes that Quantum Mechanics governs all physical systems, the measurement process, being an interaction between two quantum systems, is governed by a unitary operator $U(t_I, t_F)$. Suppose the initial state of the microsystem is $|\text{u}\rangle$ and the one of the apparatus (plus the rest of the universe) is $|A_0\ \overline{\alpha}\rangle$; then, during the measurement, the whole universe evolves in the following way:

$$|\text{u}\rangle \otimes |A_0\ \overline{\alpha}\rangle \quad \longrightarrow \quad U(t_F, t_I)\,[\,|\text{u}\rangle \otimes |A_0\ \overline{\alpha}\rangle] \quad = \quad |\text{F u}\ \overline{\alpha}\rangle, \tag{3.5}$$

while, if the initial state of the microsystem is $|\text{d}\rangle$, one has:

$$|\text{d}\rangle \otimes |A_0\ \overline{\alpha}\rangle \quad \longrightarrow \quad U(t_F, t_I)\,[\,|\text{d}\rangle \otimes |A_0\ \overline{\alpha}\rangle] \quad = \quad |\text{F d}\ \overline{\alpha}\rangle. \tag{3.6}$$

Some comments are needed.

- Note that in the above equations (3.5) and (3.6) the index $\overline{\alpha}$ distinguishes various possible and uncontrollable situations of the measuring apparatus in its "ready" state. Once the initial state is fully specified also the final one, since the evolution is unitary, is perfectly and unambiguously determined. Accordingly, such a state is appropriately characterized by the same index $\overline{\alpha}$. Note also that, while the state $|A_0\ \overline{\alpha}\rangle$ belongs to the Hilbert space of the whole universe exception made for the micro–particle, the state $|\mathrm{F\ d}\ \overline{\alpha}\rangle$ now includes also the particle.

- Contrary to what one does in the ideal measurement scheme of von Neumann, we do not assume that the final state is factorized; thus, in general

$$|\mathrm{F\ u}\ \overline{\alpha}\rangle \quad \neq \quad |\mathrm{u}\rangle \otimes |A_U\ \overline{\alpha}\rangle.$$

- In particular, we do not suppose that the final state of the microsystem be the same as the initial one: we allow the measurement process to modify in a significant way the state of the particle; it could even destroy the particle.

The only thing we require is that **the measuring apparatus is reliable to a high degree**, i.e. that it can safely be used to measure the state of the microsystem since in most cases it gives the correct answer. This means that if the initial state of the microsystem (prior to the measurement) is $|\mathrm{u}\rangle$, then the final state $|\mathrm{F\ u}\ \overline{\alpha}\rangle$ must belong *in most of the cases* to $V_U$, while, if the initial state of the particle is $|\mathrm{d}\rangle$, the final state $|\mathrm{F\ d}\ \overline{\alpha}\rangle$ must *almost always* belong to $V_D$. Note that by not requiring full reliability, we take into account also the possibility that the measuring instrument gives the wrong results, though pretending that such mistakes occur quite seldom.

It is possible to formalize the above reliability requests in the following way. Let us consider the set $K$ of all subsets $J$ of the possible values that the index $\overline{\alpha}$ can assume and let us equip it with the following (natural) measure:

$$\mu(J) \quad = \quad \sum_{\overline{\alpha}\in J} p(\overline{\alpha}).$$

Let us also define the two following sets:

$$J_U^- \quad = \quad \{\overline{\alpha}\ \text{such that:}\ |\mathrm{F\ u}\ \overline{\alpha}\rangle \notin V_U\}\,,$$

$$J_D^- \quad = \quad \{\overline{\alpha}\ \text{such that:}\ |\mathrm{F\ d}\ \overline{\alpha}\rangle \notin V_D\}\,.$$

$J_U^-$ is the sets of all the indices $\overline{\alpha}$ such that the states $|\mathrm{F\ u}\ \overline{\alpha}\rangle$ do not correspond to the outcome *"the pointer is at position U"*, despite the fact that prior to the measurement the state of the particle was $|\mathrm{u}\rangle$. Similarly, $J_D^-$ corresponds to the states $|\mathrm{F\ d}\ \overline{\alpha}\rangle$ for which we cannot claim that *"the pointer points at D"*, even if the initial state of the system was $|\mathrm{d}\rangle$. Let also $J_U^+ = \mathcal{C}J_U^-$ be the complement of $J_U^-$, and $J_D^+ = \mathcal{C}J_D^-$ the complement of $J_D^-$.

Given this, the requirement that the instrument is reliable can be mathematically expressed in the following way:

$$\mu(J_U^-) \ \leq \ \epsilon \qquad \mu(J_D^-) \ \leq \ \epsilon, \qquad \epsilon \ll 1, \tag{3.7}$$

i.e. the probability that the final position of the pointer does not match the initial spin–value of the particle is very small, this smallness being controlled by an appropriate parameter $\epsilon$ expressing the efficiency of the measuring device and which, as such, can change (always remaining very small) with the different actual measurement procedures one can devise.

Of course, it is easy to derive also limits on the measure for the complements of the above sets:

$$\mu(J_U^+) \;\geq\; 1-\epsilon \qquad\qquad \mu(J_D^+) \;\geq\; 1-\epsilon.$$

We need to take into account also the two sets: $J^- = J_U^- \cup J_D^-$ and $J^+ = \mathcal{C}J^- = J_U^+ \cap J_D^+$; they satisfy the following relations:

$$\mu(J^-) \;\leq\; 2\epsilon \qquad\qquad \mu(J^+) \;\geq\; 1-2\epsilon.$$

Again, all these limits simply state that, since the apparatus is reliable, the probability that — at the end of the measurement process — the pointer is not in the correct position is very small, *if* the initial state of the particle is either $|\mathrm{u}\rangle$ or $|\mathrm{d}\rangle$.

It is useful to remark that, having taken into account the possibility that the measuring instrument can make mistakes, we can easily include also the possibility that it fails to interact at all with the microsystem, thus giving no result: in such a case, the pointer remains in the "ready–state", and the corresponding vector belongs to the set $J^-$. In fact, let us consider the set $V_0$ associated to the "ready–state", as we did for the two sets $V_U$ and $V_D$ referring to the "U" and "D" positions of the pointer. By the same argument as before, $V_0$ is disjoint from the two sets $V_U$ and $V_D$, since the "ready–state" is assumed to be macroscopically different from the "U" and "D" states; consequently if the vector at the end of the measuring process belongs to $V_0$, it cannot belong either to $V_U$ or to $V_D$.

We have mentioned the possibility that the apparatus misses to detect the particle because such an occurrence affects, in some case in an appreciable way, many experimental situations; for example the efficiency of photodetectors is usually quite low. This does not pose any problem to our treatment: we can easily circumvent this difficulty by simply disregarding, just as it is common practice in actual experiments, all cases in which a detector should register something but it doesn't. The previous analysis and the sets we have identified by precise mathematical criteria must then be read as referring exclusively to the cases in which the apparatus registers an outcome.

### 3.2.5 The Measurement Problem

We recall the two basic assumptions we discussed in the previous subsections:

1. The quantum evolution of any physical system is *linear* and unitary, since it is governed by the Schrödinger equation;

2. Any two sets, like $V_U$ and $V_D$, containing vectors corresponding to *macroscopically different configuration* of a macro–object are almost orthogonal:

$$\inf_{\substack{|U\,\alpha\rangle \,\in\, V_U \\ |D\,\beta\rangle \,\in\, V_D}} \| |U\,\alpha\rangle - |D\,\beta\rangle \| \;\geq\; \sqrt{2} - \eta \qquad\qquad \eta \ll 1. \tag{3.8}$$

We think that everybody would agree that *any real* measurement situation, if it has to be described in quantum mechanical terms, shares the above two properties. Starting with these very simple premises we can now easily show that quantum mechanics must face the problem of the occurrence of superpositions of macroscopically different states of the apparatus, and in general of a macro–system[13].

In our terms, the "measurement problem" arises (as usual) when the initial spin–state of the particle is not $|\mathrm{u}\rangle$ or $|\mathrm{d}\rangle$, as we have considered in the previous subsections,

[13] As already remarked, request 2) can be violated in hidden variables theories. On the other hand, request 1) is purposedly violated in dynamical reduction theories. Since both theories account for the objectification of macroscopic properties, they must necessarily violate one of the two requests.

but a superposition of them, like the state $|\mathrm{u} + \mathrm{d}\rangle$ of section 3.2.1, which can be easily prepared in the laboratory. In such a case, due to the linearity of the evolution, the final state of the particle+apparatus system will be:

$$\begin{aligned} |\mathrm{u} + \mathrm{d}\rangle \otimes |A_0\ \overline{\alpha}\rangle \quad \longrightarrow \quad & U(t_F, t_I)\,[\,|\mathrm{u} + \mathrm{d}\rangle \otimes |A_0\ \overline{\alpha}\rangle] \quad = \quad |\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle \\ & = \frac{1}{\sqrt{2}}\,[|\mathrm{F\ u}\ \overline{\alpha}\rangle + |\mathrm{F\ d}\ \overline{\alpha}\rangle]. \end{aligned}$$

It is now very simple to prove that for each $\overline{\alpha}$ belonging to $J^+$, $|\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle$ *cannot* belong either to $V_U$ or to $V_D$. In fact, let us suppose that it belongs to $V_U$; the proof in the case in which it is assumed to belong to $V_D$ is analogous. Since the distance between $|\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle$ and $|\mathrm{F\ d}\ \overline{\alpha}\rangle$ is:

$$\begin{aligned} \||\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle - |\mathrm{F\ d}\ \overline{\alpha}\rangle\| \quad &= \quad \|1/\sqrt{2}\,|\mathrm{F\ u}\ \overline{\alpha}\rangle + \left(1/\sqrt{2} - 1\right)\,|\mathrm{F\ d}\ \overline{\alpha}\rangle\| \quad \leq \\ &\leq \quad \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} \quad = \quad 1, \end{aligned}$$

we get a contradiction, because $|\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle$ is supposed to belong to $V_U$ while $|\mathrm{F\ d}\ \overline{\alpha}\rangle$ belongs to $V_D$, and relation (3.8) must hold between any two vectors of these two sets. This completes our proof. Of course, by the same argument we can also prove that, for all $\overline{\alpha} \in J^+$, the index of the apparatus cannot be in any other macroscopic position different from "U" and "D".

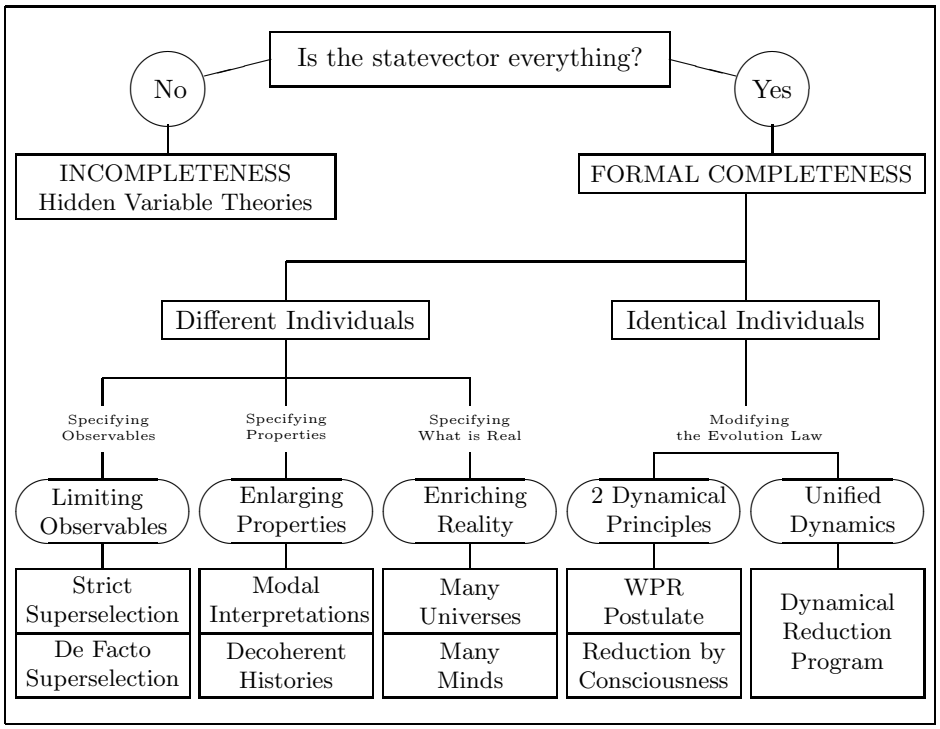
The conclusion is: for all $\overline{\alpha} \in J^+$ and for all measurements processes in which the apparatus registers an outcome, the vector $|\mathrm{F\ u{+}d}\ \overline{\alpha}\rangle$ **does not allow us to assign any macroscopic definite position to the index of the apparatus**, not even one different from "U" or "D". Stated differently, the large majority of the initial apparatus states, when they are triggered by the superposition $|\mathrm{u} + \mathrm{d}\rangle$, end up in a state which does not correspond to any definite position or, in our general language, to any definite situation of the part of the universe we perceive, i.e. one paralleling our definite perceptions.

We believe that the above argument represents the most general proof of the unavoidability of the macro–objectification problem for the absolutely minimal and physically necessary requests on which it is based: that one can prepare microscopic systems in well defined states which are eigenstates of a quantum observable and that when this is done and the considered observable is measured, one can get reliable information about the eigenvalue of the observable itself, by appropriate amplification procedures leading to perceivably different macroscopic situations of the universe.

In the next section we will analyze the various proposals which have been put forward to overcome the macro–objectification problem; we will briefly describe them and discuss their pros and cons.

# 4 Possible Ways Out of the Macro–Objectification Problem

Various ways to overcome the measurement problem have been considered in the literature: in this section we briefly describe and discuss them. It is useful to arrange the various proposals in a hierarchical tree–like structure [14], taking into account the fundamental points on which they differ: in the figure below we present a diagram which may help in following the argument. Subsequently we will comment on the various options.




## 4.1 Listing the possible ways out

A first distinction among the alternatives which have been considered in the literature derives from taking into account the role which they assign to the statevector $|\psi\rangle$ of a system. This leads to the Incompleteness versus Formal Completeness option:

**Incompleteness:** this approach rests on the assertion that the specification of the state $|\psi\rangle$ of the system is insufficient: further parameters, besides the wavefunction, must be considered, allowing us to assign definite properties to physical systems

**Formal Completeness:** it is assumed that the assignment of the statevector represents the most accurate possible specification of the state of a physical system.

When the assumption of Formal Completeness is made, two fundamentally different positions can be taken about the status of an ensemble — a pure case in the standard scheme — all individuals of which are described by the same wavefunction:

**Formal Completeness with Different Individuals:** the same wavefunction describes individuals which can have different properties, even though there is no further element in the formal theory that specifies such properties.

**Formal Completeness with Identical Individuals:** all individuals associated to the same statevector have the same properties. Pure cases correspond to genuinely homogeneous ensembles.

The two options we have just mentioned require different strategies to circumvent the difficulties related to the objectification problem. The first case can be analyzed in greater detail by considering the three following alternatives:

**Limiting the Observables:** The specification of what is actually observable in the case of a macro–system has to be reconsidered: by taking into account appropriate and unavoidable limitations of the class of observables one can legitimately consider the macro–properties to be actual.

**Enlarging the Criteria for the Attribution of Properties:** The possibility of considering a property actual is related in a more subtle way than in standard Quantum Mechanics to the statevector; in particular, an individual system can possess a property even though it is not in an eigenstate of the corresponding observable.

**Enriching Reality:** Many real happenings can occur together; all potentialities of the statevector become actual.

When the option of Formal Completeness with Identical Individuals is chosen the strategy to circumvent the difficulties consists in reconsidering the dynamics of the theory. One contemplates the possibility of a Modified Dynamics: the unitary evolution law of the theory is not always or not exactly right; the modifications which have to be taken into account make the potentialities actual. This case too leads to further alternatives:

**Two Dynamical Principles:** different physical situations require different evolution laws.

**Unified Dynamics:** the evolution equation of Quantum Mechanics has to be modified. The new dynamical principle does not lead to a violation of tested quantum predictions for microsystems but it is able to induce the dynamical objectification of macro–properties.

## 4.2 Incompleteness: the specification of the state is insufficient

This option corresponds to challenging the completeness of the quantum description of physical systems: the statevector is not all. To complete the theory, new "hidden variables" besides the statevector $|\psi\rangle$ are introduced: these are putative parameters related to properties of a physical system which are not specified by the statevector. The intended aim is that of making legitimate an epistemic interpretation of quantum probabilities.

The best known example of a hidden variable theory is Bohmian Mechanics [15, 16, 17, 18, 19, 20], where the new variables are the positions $\mathbf{x}_i$ of the particles. The basic rules are:

1. The state of a physical system $S$ at an initial time $t_0$ is given by the wavefunction $\psi(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t_0)$ together with the positions $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, $\dots \mathbf{x}_n(t_0)$ of all the particles of $S$.

2. The wavefunction evolves according to the Schrödinger equation:

$$i\hbar \frac{\partial \psi(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t)}{\partial t} \quad = \quad H \psi(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t),$$

   while the equations of motion for the positions $\mathbf{x}_i(t)$ of the particles are:

$$\frac{d\mathbf{x}_i(t)}{dt} \quad = \quad \frac{\hbar}{m_i} \operatorname{Im} \left. \frac{\psi^*(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t) \nabla_i \psi(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t)}{|\psi(\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_n; t)|^2} \right|_{\mathbf{q}_i = \mathbf{x}_i}$$

3. The Schrödinger equation can be solved with the given initial condition. Once the solution has been found, it is used to solve the equations of motion for the "hidden" variables $\mathbf{x}_i(t)$.

Bohmian Mechanics has two basic features. First of all, the theory assigns always a definite position in space to all particles; in particular, macroscopic objects have definite properties, and they are where we see them to be: this is how Bohmain mechanics solves the measurement problem of quantum mechanics. The second basic feature

is the following: let us consider an ensemble of physical systems described by the same wavefunction $\psi(\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_n; t)$, each containing $n$ particles whose positions are $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$. Let us also suppose that the probability distribution $\rho(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n; t_0)$ of the positions of the particle in the ensemble, at a given initial time $t_0$ is:

$$\rho(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n; t_0) \quad = \quad |\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n; t_0)|^2.$$

It follows that the trajectories followed by the particles of the systems in the ensemble are such that, at any later time $t$:

$$\rho(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n; t) \quad = \quad |\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n; t)|^2.$$

This means that the theory is predictively equivalent to standard Quantum Mechanics concerning the positions of all the particles of the universe.

However, one has to call attention to a peculiar aspect (shared by all hidden variable theories) of Bohmian Mechanics, i.e. to its **contextual** nature. Various authors [21, 22] have exhibited general proofs showing that the very algebraic structure of quantum formalism implies[14] that any complete specification of the state of a system can assign, in general, a definite truth value to most of the propositions concerning its properties only with reference to a specified context. This means that within such a framework, the most complete specification of the state of an individual physical system is not sufficient, by itself, to determine the outcome of a measurement process for most of the observable quantities one can consider, but that such an outcome depends from the overall factual situation. For instance within Bohmian Mechanics, a system with a precise wavefunction and a precise position, when subjected, e.g., to a measurement of its momentum, may give one or the other of the outcomes compatible with its wavefunction, depending on the specific apparatus one chooses to perform the measurement.

This situation, which at first sight might be considered as puzzling, in reality gives simply important indications about the ontology which is appropriate for the theory. The way out derives from taking the attitude that the only physical entities the theory is about are the noncontextual ones. In Bohmian Mechanics the positions of the particles play such a privileged role: they are the only non contextual, objective, *real* variables (the "local beables" [23, 24]) of the theory[15]. What about the other observables? [20] "Properties that are merely contextual are not properties at all; they do not exist, and their failure to do so is in the strongest sense possible".

A weakness, in our opinion, of the theory is that one can exhibit [25] infinitely many inequivalent hidden variable theories — whose hidden variables are the position of the particles of the universe — different from Bohmian Mechanics. They are all perfectly consistent, differing among themselves only for the trajectories they assign to the particles.

Of course, this is not a mathematical fault of Bohmian Mechanics; however, it casts some shadow over the "ontological" basic position of the theory: that particles have always definite positions and follow precise trajectories. If many inequivalent Bohmian–like theories assigning different trajectories to particles are possible, which trajectories are the correct ones? Is there a criterion to choose only one among them? Some authors [26] have tried to identify such criterion with the so called "request of compoundational invariance" of the theory. However, such a request does not seem logically necessary[16].

[14] For a Hilbert space of more than two dimensions.

[15] Of course at the formal level one can easily work out models making e.g. the momentum variables non contextual. However, the ensuing contextual nature of positions makes it quite difficult to build up a coherent description of natural phenomena based on such a scheme.

[16] S. Goldstein [private communication], has raised the objection that also within dynamical re-

In spite of this difficulty, Bohmian Mechanics is undoubtedly one of the (few) promising and consistent theories solving the measurement problem of Quantum Mechanics; of course, the great challenge is to formulate a relativistic invariant version of it.

## 4.3 Limiting the class of observables

The attempts to get objectification through a limitation of the class of observables have received great attention [27, 28, 29, 9, 10, 30]. We consider it appropriate to distinguish two different positions which have been taken when trying to implement such an approach:

**Strict Superselection Rules:** the set of the observables which can actually be measured for any macro–system does not coincide with the set of self–adjoint operators of the associated Hilbert space; it admits superselection rules. In particular, the eigenmanifolds corresponding to different macroscopic properties are superselected.

**De Facto Superselection Rules:** The impossibility of putting into evidence macroscopic coherence is not a matter of principle but derives from practical, but practically insurmountable, limitations.

Notice that both the above programs require the consideration of the dynamics of the theory; the possibility and the consistency of assuming limitations of measurability cannot be analyzed at the kinematical level only. We turn now to discuss the two cases.

### 4.3.1 Strict Superselection Rules

Suppose that, at a certain level — in the present case, the macroscopic one — the set of observables of the system admits strict superselection rules, i.e. that the set of operators associated to all physical quantities which are *actually* measurable is an abelian set. In such a case, as well known, the phase relations between components of the statevector belonging to different superselected manifolds become physically irrelevant. This amounts to saying that what actually characterizes the states of physical systems (ensembles) are not the statevectors or the statistical operators, but the *equivalence classes* $[\rho]$ of statistical operators with respect to the allowed observables $\Omega$, i.e.:

$$[\rho] \quad = \quad \{\rho^* \in T_1 \,:\, \mathrm{Tr}(\rho\,\Omega) = \mathrm{Tr}(\rho^*\Omega),\ \forall\ \Omega\} \tag{4.1}$$

where we have denoted by $T_1$ the set of positive trace–class operators of trace 1. The basic idea for circumventing the difficulties of Quantum Mechanics goes as follows. Consider, e.g., the evolution characterizing an ideal measurement process:

$$\frac{1}{\sqrt{2}}\,[|\mathrm{u}\rangle + |\mathrm{d}\rangle] \otimes |A_0\rangle \quad \longrightarrow \quad \frac{1}{\sqrt{2}}\,[|\mathrm{u}\rangle \otimes |A_u\rangle + |\mathrm{d}\rangle \otimes |A_d\rangle]\,, \tag{4.2}$$

where $|A_0\rangle$, $|A_u\rangle$ and $|A_d\rangle$ denote the "ready", "points at u" and "points at d" states of the macroscopic pointer. The states $|A_u\rangle$ and $|A_d\rangle$ are macroscopically different

duction models "there is a good deal of arbitrariness, in the choice of parameters, of smoothing functions, of basic observables, and the like", suggesting that if this arbitrariness is not a problem for collapse models, it should not be a problem also for Bohmian mechanics. We do not agree on this point, since the situation is radically different in the two theories. Within dynamical reduction models, changing the values of the parameters or of the smoothing functions one modifies the theory in a — at least in principle — *testable* way (this will become clear after the analysis of the following sections). In Bohmian mechanics, on the other hand, the different formulations are equivalent and lead exactly to the same physical predictions.

and therefore, according to our assumptions, belong to superselected eigenmanifolds; one can then legitimately assert that the final situation consists of the equal weights statistical mixture $E = (E_u) \cup (E_d)$ of the pure cases $|\mathrm{u}\rangle \otimes |A_u\rangle$ and $|\mathrm{d}\rangle \otimes |A_d\rangle$.

The program is appealing, even though to take it seriously as a candidate for a coherent worldview one should make it more precise. In particular, one should exhibit the formal elements accounting for the way in which the superselection rules emerge (the location of the split between two types of physical systems, i.e., those for which no limitation of observability occurs and those for which it does), allowing the precise identification of the superselected manifolds (the "preferred basis problem"). There is, however, a more fundamental reason which forbids us to take it seriously: it meets insurmountable difficulties when one takes into account the dynamics [31]. This is easily proved by taking into account that the initial and final system+apparatus states in a measurement process, being necessarily macroscopically distinguishable, must belong to different equivalence classes. As a consequence, the hamiltonian itself, since it connects different superselected manifolds, is not an allowed observable: this is quite peculiar.

Another related problem derives from considering the "reversibility" of the process. To discuss this let us consider, e.g., the final state (4.2) and the equivalent statistical mixture $E = (E_u) \cup (E_d)$. Suppose then one can "evolve back" or "reverse" the measurement process. According to whether one starts from the state (4.2) or from the statistical mixture, one goes back either to the state $1/\sqrt{2}\,[|\mathrm{u}\rangle + |\mathrm{d}\rangle] \otimes |A_0\rangle$ or to the equal weights mixture of the states $|\mathrm{u}\rangle \otimes |A_0\rangle$ and $|\mathrm{d}\rangle \otimes |A_0\rangle$. Should one then perform, by means of another apparatus, a measurement process to ascertain the value of the observable $\sigma_x$, he would, in the first case, get the result $+1$ with certainty, while in the second case there is a probability $1/2$ of getting the result $-1$. The combined "reversal of the process" and "measurement of $\sigma_x$" would then constitute a measurement process which allows us to distinguish the final pure state from the equivalent statistical mixture, contradicting the assumption that only superselected observables are allowed[17].

To conclude, the previous analysis should have made clear why the strict superselection program cannot be fulfilled.

### 4.3.2 The De Facto Superselection Rules option

Recall that what makes the strict superselection program not viable is the fact that the Hamiltonian connects eigenmanifolds corresponding to different macroscopic properties, and the related fact that the possibility of reversing the evolution leading to superpositions of macroscopically different states contradicts the assumption that the superposition is in the same equivalence class as the corresponding statistical mixture.

This provides the basic idea of the de facto superselection option. Since, when macroscopic objects are involved it is practically impossible to distinguish pure states from statistical mixtures or to "undo" a measurement process, one could be tempted to assert that for such systems a *de facto* limitation of observability must be recognized. Such a position has actually been taken in many interesting papers [28, 29, 9, 10, 30]. In these papers attention has been called to various features and mechanisms inducing the de facto impossibility we have just mentioned: the extreme complexity of a macroscopic object, its unavoidable and uncontrollable interactions with the environment, and so on.

In the previous section we have shown that such proposals cannot overcome, in principle, the measurement problem of Quantum Mechanics; however a deeper anal-

[17] Note that the only way out from this problem would be to deny the reversible nature of quantum evolution at least for processes involving macroscopic systems. But this would amount to accept that [32] "Scrhödinger's equation is not always right".

ysis may be helpful to clarify the matter. We begin with a digression. We have used the expression "de facto", in place of the fashionable acronym (introduced by J. S. Bell [2]) FAPP (for all practical purposes), for a precise reason. It seems to us that describing this position as FAPP suggests accusing people following this line of taking an instrumentalist position about science. We do not think that most of the proposals for a de facto superselection solution to the objectification problem require such instrumentalism. Most people taking the de facto attitude would claim that this is as legitimate as accepting the de facto validity of the second law of thermodynamics, in spite of the reversibility of the basic mechanical laws. Obviously it would be inappropriate to maintain that accepting thermodynamics involves taking an instrumentalist position.

Having stated this we would like, however, to call attention to the fundamental conceptual differences between the case of thermodynamics in relation to classical mechanics and the case of the de facto superselection assumption in relation to the unitary evolution. To do this we start by considering the two premises:

**1C** The reversible classical laws are the "correct" laws of nature;

**1Q** The superposition principle has unlimited validity;

and the legitimate classical statement:

**2C** Under appropriate circumstances the irreversible thermodynamical laws are "de facto" correct.

Taking the de facto superselection position amounts to claiming that the corresponding quantum statement:

**2Q** Under appropriate circumstances the irreversible process of wavepacket reduction and the replacement of a pure state with a statistical mixture are "de facto" correct

is equally legitimate.

Can we take such a position consistently? To answer this question let us consider the classical case. It is obviously true that the irreversible thermodynamical laws cannot describe correctly the behaviour of e.g. a gas for arbitrarily large times since it is a consequence of assumption 1C that the point representing the system in phase space will, after Poincarè recurrence times, return as close as one wants to its present value. One could then raise the question: does this fact imply that the assertion that "de facto, in an ensemble of gases almost all of them are **now** evolving irreversibly towards equilibrium" will be falsified by the **future** behaviour? Surely not.

Zurek [9, 10], in his detailed analysis seems to suggest that, since the situation in the quantum case is analogous to the classical one, statement 2Q has the same conceptual status as 2C. To this purpose he proves that, due to the unavoidable interactions with the environment, in the case of a macroscopic system in a superposition of macroscopically distinguishable states the off–diagonal elements of the reduced statistical operator (i.e. the one obtained by tracing out the environment variables) become rapidly negligibly small and remain so for times comparable to the Poincarè recurrence times for a gas. This is certainly true; but does it prove that the situation is conceptually analogous to that of thermodynamics? We think not. In fact in the quantum case the assumption 1Q that the linear laws of Quantum Mechanics are correct and have universal validity implies that the result of a prospective measurement on an ensemble *in the very far future* would falsify not only our statements about future events but also the assertion that **now** the ensemble is the union of the pure subensembles corresponding to definite macroscopic positions. Such an assertion would turn out to be in no sense, even approximately, correct. The argument we have

presented briefly has been expounded with great clarity and precision by d'Espagnat in [33], to which we refer the reader for a deeper analysis.

We can further clarify the matter by repeating the previous analysis within the context of the pilot wave theory; which, we recall, is fully equivalent to Quantum Mechanics in its physical predictions and which assigns definite positions to all particles of a system at all times. The approximation which corresponds to assuming that wavepacket reduction occurs, consists in disregarding, in the description of the evolution of an "up" ("down") pointer position (after the measurement is over), the contribution to the wave function coming from the term corresponding to the "pointer down" ("pointer up") in the statevector. Again, such an assumption will surely be proved false by events in the very far future. However, both the approximate and the "true" versions of the theory assert that presently the pointer is either up or down, or equivalently that all pointers of the ensembles are in one of the two positions, and the future happenings neither falsify this statement nor deny that the approximate description is extremely accurate for extremely long times. Therefore, within the pilot wave framework, the analog of assertion 2Q has the same conceptual status as 2C. As discussed above, this is not the case for the de facto superselection program.

Other significant differences between the thermodynamical and the de facto superselection situations deserve to be mentioned. For instance, in the classical case the following three statements are correct:

**3C** The assignment of the phase–space distribution identifies with sufficient precision the corresponding physical ensemble.

**4C** The approximation made in using thermodynamical equations to describe the behaviour of a system is under control. The split between mechanical and thermodynamical systems is not shifty. Just to give an example, while few molecules are not a gas, an Avogadro's number of them is a gas.

**5C** The exact (mechanical) and the approximate (thermodynamical) laws both make sense and both allow simple and sensible assumptions about the psycho–physical correspondence allowing us "to close the circle" for the appropriate classes of phenomena.

But the corresponding statements in the quantum case are fully inappropriate. In fact:

**3Q** The same statistical operator corresponds to completely different physical ensembles.

**4Q** The approximation made in breaking linearity is shifty: macroscopic systems exist which require a genuine quantum treatment (more on this in section 5.1).

**5Q** The correct (linear evolution) law leads to a situation which does not make sense from the point of view of our (definite) perceptions, only the approximation allows a sensible psycho–physical correspondence.

Statement 3Q further emphasizes the difficulties in relating the states of the system to our perceptions. Even ensembles corresponding to the same statistical operator can be very different in their compositions in pure subensembles [34]. This proves once more that the simple recognition that two ensembles can be de facto in the same equivalence class is not sufficient to explain why our perceptions unavoidably correspond to a specific composition, i.e. the one whose subensembles have definite macroscopic properties. As recognized by Joos and Zeh [30] who have presented one of the most interesting proposals along these line: *"perhaps (this fact) can be justified by a fundamental (underivable) assumption about the local nature of the observer"*.

Our conclusion is that one cannot consider the de facto superselection proposals as yielding a consistent way of "closing the circle". We will come back again to this point in what follows since, from the mathematical point of view, it has strict connections with Dynamical Reduction Models.

## 4.4 The Modal Interpretations of Quantum Mechanics

These approaches [35, 36, 37, 38, 39, 40, 41, 42] rest on the introduction of appropriate rules which allow us to attribute some properties to the subsystems of a composite system even when there is no observable whose outcome can be predicted with certainty. To illustrate the general lines of the program we make reference to the proposal by Dieks [37].

Consider a composite system containing several (let us say $N$) constituents and suppose that we are dealing with a pure case associated to an entangled statevector. The proposal goes as follows. Any subsystem of the whole system has at any time definite properties, identified by the following procedure. Suppose we are interested in the subsystem $S^M$ constituted by a group of particles (let us say the first $M < N$); the case of only one particle or even of a specific degree of freedom of a particle is not excluded. We also denote by $S^{N-M}$ the system of the remaining particles. One then considers the whole Hilbert space as the direct product of the Hilbert space referring to the considered group and to the rest:

$$\mathcal{H} \equiv \mathcal{H}(1) \otimes \mathcal{H}(2) \otimes ... \mathcal{H}(N) = \mathcal{H}(1, ..., M) \otimes \mathcal{H}(M+1, ..., N) \tag{4.3}$$

Accordingly, one takes into account the biorthonormal decomposition of the statevector:

$$|\psi(1, ..., N)\rangle = \sum_i \sqrt{p_i}\, |\chi_i(1, ..., M)\rangle \otimes |\Omega_i(M+1, ..., N)\rangle; \tag{4.4}$$

in the above equation, the parameters $p_i$ are positive constants summing up to 1: they are the eigenvalues of the reduced statistical operators obtained by taking the partial trace of $|\psi\rangle\langle\psi|$ either on $\mathcal{H}(1, ..., M)$ or on $\mathcal{H}(M+1, ..., N)$. The states $|\chi_i\rangle$ and $|\Omega_i\rangle$ satisfy:

$$\langle\chi_i|\chi_j\rangle = \langle\Omega_i|\Omega_j\rangle = \delta_{ij}. \tag{4.5}$$

As proved by von Neumann [7], such a decomposition is uniquely determined by $|\psi(1, ..., N)\rangle$ except in the case of degeneracy of the above eigenvalues. Ignoring the complications arising from accidental degeneracy, we can now state the rule for assigning properties to the subsystems $S^M$ and $S^{N-M}$: *when dealing with a pure case associated to the state (4.4) the subsystems $S^M$ and $S^{N-M}$ have definite properties. They are those associated to the observables having the states $|\chi_i\rangle$ and $|\Omega_i\rangle$ as eigenvectors. The probability of the i-th property to be actually possessed (or better: the fraction of systems in the ensemble which have such a property) is given by $p_i$* (in other words, the model is basically a hidden variable model [42], whose hidden variables are identified via the biorthonormal decomposition (4.4)).

As usual we indicate the way in which the proposal circumvents the problems of the theory of measurement. According to the ideal von Neumann measurement scheme the final state in a measurement process consists, typically, of a superposition of states each term of which involves an eigenvector referring to a different reading of the apparatus (compare equation (3.2)): this final state already gives the von Neumann biorthonormal decomposition for the system+apparatus so that, according to the previous criterion one can assert that the appropriate fractions of the apparata have their pointers in precise and different positions.

Such proposals are surely interesting but they meet various difficulties which have been discussed e.g. in [31, 43, 44, 45]. We refer the reader to the above papers for

details. Apart from this, we would like to call attention on the fact that the proposal raises other problems, in particular, it lacks what we might call *"structural completeness"*. The situation can be summarized in the following simple terms. Quantum mechanics, in its general formulation, allows the treatment of statistical ensembles. We are considering now theoretical models which accept that systems associated to the same statevector have different properties.

Suppose now we are dealing with an ensemble which is a pure case associated to the state (4.4). What meaning can be attached to the statement that the considered subsystems have properties? Even within very weak varieties of realism, this amounts to asserting that the ensemble is inhomogeneous, that it actually is the union $E = \cup_i E_i$ of different subensembles. One can then raise the question: can we prepare such an ensemble in the standard way, i.e. by taking (for all $i$) a fraction $p_i$ of subensembles in the state $|\chi_i\rangle|\Omega_i\rangle$? Obviously not. In fact, if we were to prepare the subsystems in the states $|\chi_j\rangle|\Omega_j\rangle$ it would be false to assert, contrary to what this interpretation holds true, that the composite system has with certainty the properties associated according to the above procedure to an ensemble described by $|\psi\rangle$. This means that the model deals with two kinds of statistical ensembles: the one associated to the state (4.4) and the one prepared by taking a fraction $p_i$ of subsystems in the state $|\chi_j\rangle|\Omega_j\rangle$. They both are the union $E = \cup_i E_i$ of the same kind of subensembles (each taken with the same probability $p_i$), but they are *structurally different* since one of them has specific hidden features to which one does not have access.

## 4.5 The Decoherent Histories approach

The main purpose of this proposal [46, 47, 48, 49, 50] is to assign definite probabilities to alternative histories of a physical system, which may also be the whole universe. The idea goes as follows. One defines histories as sequences of events yielding a sort of motion picture of the evolution of a system, and attaches appropriate probabilities to them. Let us first define the notation we will use.

We consider, for a given $k$, a set of orthogonal projection operators $P^k_{\alpha_k}$ yielding a resolution of the identity:

$$\sum_{\alpha_k} P^k_{\alpha_k} \;=\; 1; \qquad P^k_{\alpha_k} P^k_{\beta_k} \;=\; \delta_{\alpha_k \beta_k} \tag{4.6}$$

The index $k$ labels the "question" or "property" we are interested in, while the parameter $\alpha_k$, which runs over an appropriate range, labels a set of "alternative values for the considered property" which are, according to (4.6), exhaustive and mutually exclusive. It is important to have clear the meaning of the considered projection operators. For this purpose we refer, for simplicity, to the spin space of a spin 3/2 particle. In such a case, specifying $k$ could mean, e.g., to specify a spin component, so that $k = 1$ could be related to $S_z$ and $k = 2$ to $S_x$. For fixed $k$ the index $\alpha_k$ identifies a set of mutually orthogonal manifolds (each of them being either one or the direct sum of various eigenmanifolds of the $k$–th operator) whose direct sum is the whole space. So one could have, for $k = 1$, $\alpha_1$ taking e.g. 3 values:

- $P^1_{1_1}$ projects on the eigenmanifold spanned by the eigenstates belonging to $S_z = 3/2$ and $S_z = 1/2$.
- $P^1_{2_1}$ projects on the eigenmanifold spanned by the eigenstate belonging to $S_z = -1/2$.
- $P^1_{3_1}$ projects on the eigenmanifold spanned by the eigenstate belonging to $S_z = -3/2$.

Analogously $P^2_{\alpha_2}$ could represent, for $\alpha_2$ taking only two values, the two projection operators on the positive and negative parts of the spectrum of $S_x$. We will denote by $P^k_{\{\alpha_k\}}$ the set of all the projection operators associated to the "property $k$" when $\alpha_k$ runs through its range. One then considers the corresponding projectors in the Heisenberg picture at time t:

$$P^k_{\alpha_k}(t) \quad = \quad e^{iHt}\, P^k_{\alpha_k}\, e^{-iHt} \tag{4.7}$$

A **history** is defined by a succession of times $t_1 < t_2 ... < t_n$ and a sequence of projection operators. It will be denoted by $(\alpha_n, t_n)...(\alpha_2, t_2)\,(\alpha_1, t_1)$. If the initial conditions are fixed by specifying the initial statevector $|\psi(0)\rangle$, one attributes to the above history the probability:

$$\begin{aligned} &P[\alpha_n, t_n; ... \,\alpha_2, t_2;\, \alpha_1, t_1] \;= \\ &\quad \mathrm{Tr}\left[P^n_{\alpha_n}(t_n)...P^1_{\alpha_1}(t_1)\,|\psi(0)\rangle\langle\psi(0)|P^1_{\alpha_1}(t_1)...P^n_{\alpha_n}(t_n)\right] \end{aligned} \tag{4.8}$$

One then considers the set of all alternative histories, which we will denote with self–explanatory notation as $(\{\alpha_n\}, t_n)...(\{\alpha_2\}, t_2)\,(\{\alpha_1\}, t_1)$, i.e., the set of all quantum histories obtained by letting each $\alpha_j$ take all values in its range. Due to quantum interference, the probabilities of the histories of this set turn out not to satisfy, in general, the additivity conditions which are necessary in order that they could be interpreted as true probabilities. For instance one usually has:

$$P(\alpha_2, t_2) \quad \neq \quad \sum_{\alpha_1} P(\alpha_2, t_2; \alpha_1, t_1) \tag{4.9}$$

To circumvent this difficulty one introduces the idea of a decoherent set of alternative histories. This can be implemented mathematically by defining the **decoherence functional**:

$$\begin{aligned} &D[\alpha_n, t_n; ... \,\alpha_1, t_1 \,|\, \beta_n, t_n; ... \,\beta_1, t_1] \;= \\ &\quad \mathrm{Tr}\left[P^n_{\alpha_n}(t_n)...\,P^1_{\alpha_1}(t_1)\,|\psi(0)\rangle\langle\psi(0)|P^1_{\beta_1}(t_1)...\,P^n_{\beta_n}(t_n)\right] \end{aligned} \tag{4.10}$$

If such a functional vanishes whenever at least one of the $\beta_k$ differs from $\alpha_k$, one says that the considered set of alternative histories is consistent since the associated probabilities satisfy all necessary requirements. For a given set of histories one may construct coarser–grained histories by summing over the finer–grained projections.

We do not want to be more specific about this program. We refer the reader to the references quoted above, particularly to the book [49] by Omnès for a thorough analysis. We prefer to comment about the relations between the Decoherent Histories approach and some of the approaches we have already discussed.

Let us look at the Decoherent Histories approach from the point of view of the strict superselection option. If the conditions presupposed by the strict superselection rules were satisfied, i.e. if there were a level at which macroscopically different eigenmanifolds are strictly superselected and are not connected by the Hamiltonian, all histories attributing macroscopic properties to the physical system at the considered level would decohere: one could then truly describe the unfolding of the evolution by a consistent snapshot–like motion picture. This comparison with the strict superselection case immediately reveals an interesting advantage of the Decoherent Histories approach. Namely, within the strict superselection scheme the assignment of the statevector at various times tells us which fractions of systems have various macro–properties but it does not attach probabilities to time sequences of events; the snapshots at different times cannot be organized in motion pictures as in the case of Decoherent Histories.

Since, as already remarked, the assumptions of strict superselection rules cannot hold consistently, it becomes quite natural to look at Decoherent Histories from the point of view of the de facto superselection rules. In particular, by taking advantage of the many proofs that the environment induces "de facto" superselection rules associated to macroscopic position variables, one could limit one's considerations, within the Decoherent Histories approach, to alternative histories specifying e.g. the intervals in which the macroscopic pointer lies at various times. Again this point of view represents an interesting improvement with respect to the simple de facto superselection program since it allows the consideration of a time–chain of events. Moreover, it gives precise criteria to select decoherent histories from non decoherent ones.

Decoherent Histories supporters maintain that the decoherent sets of histories can be considered completely in general, i.e. with reference both to macroscopic and microscopic systems, that one can assign probabilities to them if the consistency conditions are met, and that within the scheme decoherence replaces the notion of measurement. This seems to suggest that some objective meaning is given to consistent alternative histories. Here a serious problem arises. It can easily be seen that alternative histories involving only one time always decohere. One can then consider, at a given time, different sets of incompatible alternative histories. If the probabilities are related to possessed properties, then one should assign objective meaning to the different possible incompatible sets of decoherent histories. In references [51, 52][18] we have proved that this cannot be done in a consistent way. The problem is the same as the one of contextuality in hidden variable theories: not too many properties can be assigned to quantum systems. As a consequence it is not clear the meaning which should be given to decoherent histories.

A last remark: decoherent histories can also be considered as strictly referring only to the universe as a whole. When one takes such an attitude one invokes the natural de facto decoherence of histories about the universe. As appropriately remarked in [55] one can then raise the question: what is the status of two histories belonging to incompatible sets of alternative histories? References [55] and [56] have also called attention to peculiar difficulties that the Decoherent Histories approach meets concerning the future–past relation.

We do not pursue the analysis further. Concluding, it seems to us that even Decoherent Histories do not allow us to attribute consistently an objective meaning to statements about possessed properties.

## 4.6 Enriching reality

Such proposals [57, 58, 59, 60, 61, 62] maintain that the unitary evolution holds in all circumstances and dispose of the embarrassment arising from the occurrence of superpositions of perceptively different states by assuming that in a sense, all potentialities of the wavefunction become actual. The most widely known proposal of this type is usually referred to as "The Many Universes Interpretation of Quantum Mechanics" [60, 61, 62].

According to this proposal, each time an interaction leading to superpositions of macroscopically distinguishable situations occurs, the universe literally splits into (in general infinitely many) replicas of itself: each replica corresponds to one of the terms in the superposition and occurs with the appropriate probability. So, in a situation like the one of equation (4.2) one would state that, after the measurement is over, there are actually two types of universes; in those of the first type there is an apparatus whose pointer "points at u" and in the second an apparatus whose pointer "points at d". Needless to say, if one wants to describe the situation at later times one has to go on with the unitary evolution taking into account all interactions which take

[18] See also [53, 54] for further discussion on this issue.

place and then, having expressed the final statevector as a superposition of states in each of which all macro—objects have definite macroscopic properties, associate each term of the superposition to universes of a different type.

A serious limitation for the proposal comes from the fact that it leaves largely undefined how and when the multifurcation of the universe takes place. This ambiguity reflects the basic difficulty that Quantum Mechanics meets in locating the "shifty split" between micro and macroscopic phenomena.

Detailed analyses of the many universes theories have been presented [60, 61, 62]. Here we want to stress that, since to close the circle one needs also some assumptions about the process of perception, there are at least two choices for this, which give rise to two quite different alternatives. If one makes simple assumptions about the psycho–physical correspondence one has the "genuine" many world interpretation: in the process of replicating the universe also the perceiving subject is replicated, so that in the above example there will be universes in which we perceive that the pointer points up and universes with replicas of ourselves having the other perception. Within each universe the perception is strictly correlated to statevectors corresponding to different macroscopic situations.

On the other hand, one can take the attitude that it is the perception mechanism which is more complex than we usually assume; this leads to what has been referred to as "the many minds interpretation" of the theory. Such a formulation has the advantage of allowing us to circumvent the ambiguities about the branching of the universe; there is only one universe and there are many minds (i.e. each mind exhibits some sort of a full spectrum of perceptions reflecting the macroscopically different states in the superposition). Many interesting problems arise when one takes this attitude, the most relevant ones having to do with the intersubjective agreement and with the reliability of our beliefs. We will not discuss, for lack of space, the details of these approaches.

We conclude this subsection by stating that, even though we consider these "enriching reality" proposals interesting, they seem to require a too radical change in our views about reality and the adoption of a rather strange ontology. For, according to them science does not deal any longer with the one world we live in or the perceptual processes we experience, but at the same time with the totality of all possible worlds and all possible perceptions.

## 4.7 Modifying the dynamics

When one assumes that the theory is complete and that pure cases describe genuinely homogeneous ensembles, the only way to dispose of the embarrassing superpositions is to say that in one way or another the dynamical equations of the theory are not always or not exactly right. At this point, two completely different positions can be and have actually been taken. We will briefly discuss them in the two following subsections.

### 4.7.1 Two dynamical principles

This line of thought plainly accepts that there are two dynamical principles which must be used for describing different physical situations. The best known example of this attitude consists in accepting wavepacket Reduction: the evolution of microscopic quantum systems is governed by the unitary and reversible linear Schrödinger equation; the measurement process is governed by the nonlinear process of wavepacket reduction transforming, in general, pure states into statistical mixtures. The reasons for this different treatment of physical systems are traced back to the recognition that there are two classes of phenomena in nature, the quantum and the classical, the reversible and the irreversible, the microscopic and the macroscopic ones. Further

support to this attitude is given by saying that, in a sense, classical concepts are a prerequisite for the very formulation of quantum formalism.

One could find many reasons for considering legitimate such a position; after all, all physical theories have a limited range of applicability. In this respect Quantum Mechanics would be claimed to find its limit in the description of the micro–macro interactions taking place in the measurement process. However, as repeatedly stressed by J. Bell and as already discussed in the previous section, the real difficulty which this line meets does not stem from its dualistic attitude about our understanding of natural phenomena, but derives from the fact that there is nothing in the theory which allows us to locate the "split" between the two considered classes of phenomena. When trying to follow this line, as J. Bell has stated [2], *are we not obliged to admit that measurement like processes are going more or less all the time, more or less everywhere?*

The remark is so appropriate that E.P. Wigner [63], having recognized the unavoidability of accepting two dynamical principles, felt the necessity of following von Neumann's proposal: to solve the problem one has to go to the extreme end of the chain of observation and to assume that reduction does not take place until somebody knows that it must, i.e. up to when conscious observers are involved. This position leads to a quite peculiar conclusion, i.e. that the world as we know it, is very much a product of conscious mind. In spite of this, one could say that such a position represents a simple and effective solution (reduction actually takes place) to the problems we are debating except that it suffers once again from an intrinsic ambiguity. For the question: "what is conscious?" does not admit any unambiguous answer on the basis of our present knowledge about nature and human beings.

The impossibility of locating the split between the two types of physical systems (quantum–classical) which should be governed by different laws as well as the impossibility of clearly identifying the processes involving consciousness clearly show that also the program outlined here does not allow one to "close the circle". We pass then to consider the other option: the dynamical equation of the theory is not right.

### 4.7.2 Unified dynamics: Dynamical Reduction Models

The program seeks a modification of the evolution law in such a way that measurement–like processes have definite outcomes as a consequence of the unified dynamics governing all physical processes [64, 65, 66, 67, 68]. In this search some guidance is of course given by the fact that the modified dynamics should imply wavepacket reduction as a consequence of the interaction of the microsystem and the macro–apparatus and, more generally, forbid the persistence of linear superpositions of macroscopically different states. With this in mind, one remarks that the characteristic features distinguishing quantum evolution from wavepacket reduction are that, while Schrödinger equation is linear and deterministic, wavepacket reduction is nonlinear and stochastic. It is then natural to entertain the idea of nonlinear and stochastic modifications of the standard Hamiltonian dynamics.

Obviously such a program must respect strict constraints, in particular, it must not contradict any known fact about micro–phenomena. Secondly, to meet the requests we have repeatedly mentioned in this paper, it must allow a clearcut identification of the split between phenomena for which standard Quantum Mechanics holds (obviously this has now to be read: for which the approximation consisting in disregarding the nonlinear terms of the "exact" theory is legitimate and under control) and those for which the new dynamics leads to relevant differences with respect to the standard theory, more specifically to a "classical behaviour". The analysis of proposals of this type, of what they have accomplished and of the difficulties they meet will be the subject of the rest of the report.

# Part II
# Non Relativistic Dynamical Reduction Models

## 5 Preliminary considerations

The aim of Dynamical Reduction Models is to account for the process of wavepacket reduction and the Schrödinger evolution in terms of a unique dynamical equation leading to the spontaneous suppression of the superpositions of different macroscopic states of a macro–system; at the same time, the new dynamics must not change in any appreciable way all the known properties of microscopic quantum systems. As already stated, one tries to achieve this goal by introducing nonlinear and stochastic modifications of the standard Hamiltonian dynamics.

In this section we want to prove that both non linearity and stochasticity are necessary ingredients in order to account for an acceptable spontaneous reduction mechanism. More specifically, we will show that neither a nonlinear but deterministic modification nor a stochastic but linear one, can lead to a consistent theory of dynamical reductions. A linear and stochastic modification induces at most an *apparent* collapse of the wavefunction; a nonlinear but deterministic modification, on the other hand, unavoidably violates basic relativistic constraints. Before discussing these issues, we will answer the following question: should the localization mechanism act at the wavefunction level, or is it sufficient, as suggested by some authors [9, 10, 11], that it suppresses the the off–diagonal elements of the statistical operator? The answer will be clear: a consistent dynamical reduction theory must induce localizations directly at the wavefunction level (we speak in this case of individual or **Heisemberg reductions** [69]), and not only at the statistical operator level (which we refer to as ensemble or **von Neumann** reductions).

### 5.1 Individual and ensemble reductions

We have widely discussed in the previous section the fact that the macro–objectification problem arises when a superposition of macroscopically different states of a macroscopic object — for example the superposition with equal weights of two such states $|\text{Here}\rangle$ and $|\text{There}\rangle$ — occurs, e.g. as the result of a measurement process:

$$|\psi\rangle \quad = \quad \frac{1}{\sqrt{2}} \left[ |\text{Here}\rangle \; + \; |\text{There}\rangle \right]. \tag{5.1}$$

In the language of the statistical operator, state (5.1) is represented by:

$$\begin{aligned} \rho \equiv |\psi\rangle\langle\psi| \quad = \quad & \frac{1}{2} \left[ |\text{Here}\rangle\langle\text{Here}| \; + \; |\text{There}\rangle\langle\text{There}| \; + \right. \\ & \left. |\text{Here}\rangle\langle\text{There}| \; + \; |\text{There}\rangle\langle\text{Here}| \right], \end{aligned} \tag{5.2}$$

whose matrix representation with respect to the basis[19] $|\text{Here}\rangle$ and $|\text{There}\rangle$ is:

$$\rho \quad = \quad \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.3}$$

Let us consider now $N$ identical macroscopic systems whose state is (5.1). Our declared goal is to find a universal mechanism which transforms such an ensemble into

[19] Of course, here we make a gross simplification, treating a macroscopic object like a simple two–dimensional system; however, this does not invalidate the basic conclusions of our analysis.

the statistical mixture in which half of the systems are in the state $|\text{Here}\rangle$, and the other half in the state $|\text{There}\rangle$, in accordance with the wavepacket reduction postulate:

$$\frac{1}{2} \text{ systems in state } |\text{Here}\rangle \text{ and } \frac{1}{2} \text{ systems in state } |\text{There}\rangle. \tag{5.4}$$

The ensemble (5.4), in which all systems have definite macro–properties, can be easily described within the statistical operator formalism:

$$\rho' \quad = \quad \frac{1}{2} \left[ |\text{Here}\rangle\langle\text{Here}| \; + \; |\text{There}\rangle\langle\text{There}| \right]; \tag{5.5}$$

the corresponding density matrix is:

$$\rho' \quad = \quad \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

Thus we see that, in order to eliminate the embarrassing superposition of different macroscopic states, the dynamics we are looking for must induce the following change of the statistical operator:

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \stackrel{\text{evolution}}{\longrightarrow} \quad \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.7}$$

i.e. it must suppress the off–diagonal terms of the density matrix, corresponding to matrix elements connecting different macroscopic states.

Here comes the crucial point: does a dynamical evolution like (5.7) really guarantee by itself the suppression of superpositions of different macroscopic states? The answer is negative. The reason for this lies in the fact that **the statistical operator describing a statistical mixture, describes at the same time infinitely many inequivalent statistical mixtures** — this is the weak point (for the problem we are interested in) of the statistical operator formalism. In fact, let us consider the following statistical mixture:

$$\text{Half systems in state } \frac{1}{\sqrt{2}} \left[ |\text{Here}\rangle + |\text{There}\rangle \right] \tag{5.8}$$

$$\text{and half systems in state } \frac{1}{\sqrt{2}} \left[ |\text{There}\rangle - |\text{There}\rangle \right]. \tag{5.9}$$

Of course, (5.8) describes a statistical ensemble which is completely different from the one defined in (5.4); however, it is easy to check that the statistical operator describing it is (5.5), i.e. the same one associated to the mixture (5.4).

The root of the problem should be clear: working only at the statistical operator level, we cannot be sure that a dynamical evolution like (5.7) transforms the pure state (5.1) into the statistical mixture (5.4) — a mixture whose elements have definite macroscopic properties — instead of transforming it into a mixture like (5.8), whose elements are still superpositions of different macro–states. This means that, in order to work out a fully consistent and unambiguous theory of dynamical reductions, we have to assume that the localizations affect directly the wavefunction, not only the statistical operator.

## 5.2 Linear and stochastic modifications of the Schrödinger equation

The easiest way to implement a dynamical evolution like (5.7), which suppresses the interference terms arising from superpositions of different macro–states of a macroscopic system, is to add a white noise stochastic potential $V(t)$ to the standard

Schrödinger equation for the wavefunction [70]. Here we give a simplified description of how this can be achieved, considering the case of one particle in one dimension. We disregard the Hamiltonian evolution and discretize the real axis $\mathbf{R}$ into intervals $\Delta_i$ of appropriate length. We define the projection operators

$$P_i\,\psi(x) \quad = \quad \chi_i(x)\,\psi(x), \tag{5.10}$$

where $\chi_i(x)$ is the characteristic function of the $i$–th interval.

Let us now consider the following equation:

$$i\hbar\,\frac{d}{dt}|\psi(t)\rangle \quad = \quad \sum_i P_i\,V_i(t)\,|\psi(t)\rangle. \tag{5.11}$$

$V_i(t)$ are white noise processes characterized by the expectation values[20]:

$$\langle\!\langle V_i(t)\rangle\!\rangle \;=\; 0 \qquad\qquad \langle\!\langle V_i(t)\,V_j(t')\rangle\!\rangle \;=\; \gamma\,\delta_{ij}\,\delta(t-t'). \tag{5.12}$$

The formal solution of equation (5.11) is:

$$|\psi(t)\rangle \;=\; e^{-\frac{i}{\hbar}\int_0^t d\tau \sum_i P_i\,V_i(\tau)}\,|\psi(0)\rangle \;=\; \sum_i e^{-\frac{i}{\hbar}\int_0^t d\tau V_i(\tau)}\,P_i\,|\psi(0)\rangle. \tag{5.13}$$

If one defines the statistical operator

$$\rho(t) \quad = \quad \langle\!\langle |\psi(t)\rangle\langle\psi(t)|\rangle\!\rangle, \tag{5.14}$$

the stochastic average is easily evaluated and one discovers that $\rho(t)$ obeys the evolution equation:

$$\frac{d}{dt}\,\rho(t) \quad = \quad \gamma\sum_i P_i\rho(t)P_i \;-\; \frac{\gamma}{2}\sum_i \left\{P_i^2,\rho(t)\right\}, \tag{5.15}$$

which, as we shall see, is basically the same equation as the one characterizing the dynamical reduction models we will discuss in great detail in the following sections.

Let us now consider the vectors $|\Delta_i\rangle$ whose position representation is $\langle x|\Delta_i\rangle = \chi_i(x)$; then, equation (5.15) leads to the following equation for the matrix elements $\langle\Delta_i|\rho(t)|\Delta_j\rangle$ of the statistical operator:

$$\frac{d}{dt}\,\langle\Delta_i|\rho(t)|\Delta_j\rangle \quad = \quad \gamma\,(\delta_{ij} \;-\; 1)\,\langle\Delta_i|\rho(t)|\Delta_j\rangle. \tag{5.16}$$

We see that the diagonal elements do not change in time, while the off–diagonal elements are exponentially damped, with a rate given by $\gamma$; this means that equation (5.16) embodies precisely an evolution like (5.7). Does equation (5.11), then, lead to the reduction of the statevector into one of the states $|\Delta_i\rangle$, as it seems to follow from equation (5.16)? The answer is no, for the following reason.

Let us consider the average values $\langle\psi(t)|P_i|\psi(t)\rangle$, measuring the portion of the wavefunction $\psi(x,t) = \langle x|\psi(t)\rangle$ which is contained within the interval $\Delta_i$. If equation (5.11) induces, as one could naively believe due to (5.16), the reduction of $\psi(x,t)$ into one interval, let us say $\Delta_k$, then the following relation would necessarily hold:

$$\langle\psi(t)|P_i|\psi(t)\rangle \;\longrightarrow\; \delta_{ik} \qquad \text{for } t\to\infty. \tag{5.17}$$

[20] We indicate with $\langle\!\langle\cdot\rangle\!\rangle$ the average value of the quantity contained within the "brackets".

On the contrary, for any realization of the stochastic potential and for any time $t$, since $P_i$ commutes with the evolution operator:

$$\langle\psi(t)|P_i|\psi(t)\rangle \quad = \quad \langle\psi(0)|P_i|\psi(0)\rangle. \tag{5.18}$$

This means that, if $|\psi(0)\rangle$ corresponds to a non localized function, the individual members of the ensemble are always associated to non localized functions. The diagonalization of $\rho$ arises from the random phases acquired by the states in different intervals $\Delta_i$, not from a real reduction of the statevector.

Stapp [69] has considered this problem and has suggested to accept a stochastic mechanism of this type, relating it to fluctuations associated to the background radiation. We do not like this proposal. At the macro–level we want that each individual has actualities: those of being in a given region. If this does not happen, how can one avoid the problem arising from the many to one relation of ensembles with statistical operators, discussed in the previous subsection? This shows that the fact that equation (5.11) leads to a statistical operator of type (5.4) is a necessary but not a sufficient condition in order that the ensemble (5.14) can be considered a union of pure cases corresponding to localized states. This is why, in the following sections, we will confine our considerations to models yielding Heisenberg reductions for the statevector.

To conclude: we have presented a stochastic equation for the statevector, equation (5.11), leading to equation (5.15) for the statistical operator, which induces precisely an evolution like the one of equation (5.7), i.e. an evolution which would be considered as transforming ensembles of non localized states into ensembles of localized ones. Equation (5.11) is characterized by a hermitian coupling of the stochastic noise to the operators $P_i$, and is linear; however, it gives rise only to the diagonalization of the statistical operator leaving the individual wavefunctions spatially extended. This strongly suggests that the introduction of stochasticity into the evolution equation (i.e. the possibility that a given state evolves in different states according to its own story) combined with the requirement that in the long run the statevector ends up in one of the eigenmanifolds characterizing the preferred basis (actual individual reductions) implies that the dynamics must be nonlinear. As a matter of fact, the evolution laws of the dynamical reduction models are stochastic and nonlinear.

As we will see in the next section, in a certain sense also the converse is true, i.e. the consideration of nonlinear modifications of the evolution equation requires, when some basic relativistic constrains are added, the introduction of stochasticity into the equation.

## 5.3 Nonlinear and deterministic modifications of the Schrödinger equation

Within standard quantum formalism the postulate of wavepacket reduction is chosen in such a way that, even though the state of the system can be instantaneously changed by a distant measurement, such change cannot be used to send faster than light signals between distant observers [71, 72, 73, 74]. This is a nice feature in absence of which an unacceptable violation of relativistic requirements would occur. In fact, even though quantum mechanics as considered here, and in particular the process of wavepacket reduction, does not pretend to be a relativistically invariant theoretical scheme, the fact that the instantaneous changes induced by wavepacket reduction itself do not depend in any way whatsoever from the distance between two constituents one of which is subjected to a measurement, forbids us to think that wavepacket reduction itself might represent some non relativistic approximation of a relativistic process. If the considered changes permit one observer to become instantaneously aware of the fact that the other (far away) constituent has been subjected to a measurement, an

explicit clash with basic relativistic requirements would emerge, making the process unacceptable.

Obviously the problem of possible instantaneous and detectable effects at a distance must be faced when formulating a dynamical reduction model, since, a priori, one cannot be sure that they do not occur. In this respect it is appropriate to take into account a quite general and interesting result obtained by N. Gisin [75]. Let us consider a map from statistical ensembles to statistical ensembles (we remember that we characterize a statistical ensemble by specifying the pure states appearing in it and the associated statistical weights):

$$M_t\, E(0) \quad \longrightarrow \quad E(t). \tag{5.19}$$

In accordance with the analysis of section 4.3.1, we say that two statistical ensembles $E$ and $E'$ are *equivalent* if the corresponding statistical operators belong to the same *equivalence class*:

$$E \;\sim\; E' \quad \text{iff: } \rho(E),\, \rho(E') \,\in\, [\rho]. \tag{5.20}$$

We can prove the following **theorem**: a necessary condition in order that the map $M_t$ describes an evolution which does not conflict with relativity in the sense specified above (i.e. it does not permit faster than light signaling), is that the equivalence relation be preserved by $M_t$, i.e.

$$E(0) \;\sim\; E'(0) \quad \Longrightarrow \quad E(t) \;\sim\; E'(t) \tag{5.21}$$

Technically one expresses this requirement by stating that the evolution equation for the statistical operator is closed.

Let us sketch the proof: assume $E_1(0) \sim E_2(0)$ but $E_1(t) \not\sim E_2(t)$, i.e. $\rho_1(t) \neq \rho_2(t)$; this means that $E_1(t)$ and $E_2(t)$ do not belong to the same equivalence class. $E_1(0)$ and $E_2(0)$ are ensembles which are union of pure cases $|\psi_i\rangle$ with associated weights $x_i$ and of pure cases $|\chi_i\rangle$ with weights $y_j$, respectively. Then one can show[21] that it is possible to choose an Hilbert space $\mathcal{K}$ and two orthonormal sets $|\alpha_i\rangle$ and $|\beta_j\rangle$ in it such that, in $\mathcal{H} \otimes \mathcal{K}$ one has

$$\sum_i \sqrt{x_i}\, |\psi_i\rangle \otimes |\alpha_i\rangle \;=\; \sum_j \sqrt{y_j}\, |\chi_j\rangle \otimes |\beta_j\rangle \;=\; |S+K\rangle. \tag{5.22}$$

Moreover one can make the state $|S+K\rangle$ correspond to the system $S$ and the auxiliary system $K$ (associated to the Hilbert space $\mathcal{K}$) being located in two distant regions $R_1$ and $R_2$ respectively. The idea is then very simple: one prepares an ensemble of systems $S+K$, all of which are in the pure "composite state" $|S+K\rangle$. Then, in region $R_2$ one measures either the observable $A$ whose eigenstates are $|\alpha_i\rangle$ or the observable $B$ whose eigenstates are $|\beta_j\rangle$. Because of wavepacket reduction, the ensemble of systems $S$ (in region $R_1$) becomes equivalent either to the ensemble $E_1(0)$ or to the ensemble $E_2(0)$, according to the measurement which has been performed on the auxiliary system at the time $t = 0$. These two ensembles are equivalent. However, by hypothesis, they will no longer be equivalent at a later time $t$, and consequently the corresponding statistical operators will be different: $\rho_1(t) \neq \rho_2(t)$. The evolution of the two ensembles then yields, at subsequent times, physically distinguishable situations in $R_1$. In this way the observer in $R_2$ can let another observer in $R_1$ know what measurement he has decided to perform on his (distant from $R_1$) auxiliary system, and this allows faster than light signaling. It is important to remark that the dynamics for the statevector given by the dynamical reduction models which are the subject of the present report

[21] See, e.g., ref.[76] and references therein.

actually leads to a closed evolution equation for the statistical operator, a necessary condition, according to Gisin's theorem, in order that they satisfy the no faster than light constraint[22].

The most interesting aspect of Gisin's result, from the point of view we are interested in here is that, in a sense, it proves "that nonlinearity requires stochasticity". In fact, suppose we consider a deterministic map of pure states into pure states

$$S_t \, |\psi(0)\rangle \quad \longrightarrow \quad |\psi(t)\rangle \tag{5.23}$$

Then a mixture of states $|\psi_i\rangle$ with weights $x_i$ evolves into a mixture of states $S_t \, |\psi_i\rangle$ with the same weights. In particular

$$\sum_i x_i \, |\psi_i\rangle\langle\psi_i| \quad \longrightarrow \quad \sum_i x_i \, S_t|\psi_i\rangle\langle\psi_i|S_t^\dagger. \tag{5.24}$$

Let us consider now, at the initial time $t = 0$, two physically different ensembles $E(0)$ with states $|\psi_i\rangle$ and weights $x_i$ and $E'(0)$ with states $|\chi_i\rangle$ and weights $y_j$, which are equivalent, i.e. $\rho(0) = \sum_i x_i \, |\psi_i\rangle\langle\psi_i| = \sum_j y_j \, |\chi_i\rangle\langle\chi_i|$. Two cases are then possible:

1. In at least one such case the evolved ensembles are inequivalent. Then an unacceptable conflict with relativity arises, as implied by Gisin's theorem.

2. The evolved ensembles are always equivalent. Then, by a general theorem of Davies [78], one can conclude that the evolution given by $S_t$ must be linear and unitary.

It is interesting to note that the above argument [79] shows that the attempt by Weinberg [80] of introducing nonlinear deterministic modifications of quantum mechanics turns out to be unacceptable.

Taking the risk of being pedantic, we stress once more that from our point of view the interest of Gisin's theorem lies in the fact that it proves that if one wants to consider nonlinear modifications of quantum mechanics one is forced to introduce stochasticity and thus, in particular, the dynamics must allow the transformation of ensembles corresponding to pure cases into statistical mixtures.

## 5.4 Brief history of dynamical reduction models

We conclude the section with a brief review of the historical development of dynamical reduction models. The history goes back to the years 1970–1973, when G.C. Ghirardi, L. Fonda, A. Rimini and T. Weber were working on quantum decay processes and in particular on the possibility of deriving, within a quantum context, the exponential decay law [81, 82]. Some features of their approach have been extremely relevant for the subsequent elaboration of the dynamical reduction program:

1. One deals with individual physical systems.

2. The statevector is supposed to suffer random processes occurring at random times, leading to appropriate sudden changes of it:

$$|\psi\rangle \quad \longrightarrow \quad \frac{P_u|\psi\rangle}{\|P_u|\psi\rangle\|};$$

when $P_u$ is identified with the projection operator on the unstable state manifold, one gets the desired result.

[22] A. Kent [77] has proposed a dynamical reduction model which allows a simple treatment of systems with identical constituents; however, this can be easily shown to imply that equivalent ensembles can evolve into inequivalent ones, with the possibility of faster than light signaling, so that the proposal has to be disregarded.

3. To make the treatment quite general (the apparatus does not know which kind of unstable system it is testing) the authors have been led to identify the random processes with localization processes of the relative coordinates of the decay fragments. Such an assumption, combined with the peculiar "resonant dynamics" of an unstable system, yield completely in general the desired result. The "relative position basis" is the preferred basis of this theory.

4. The authors have also applied their ideas to measurement processes [83].

5. The final equation for the evolution at the ensemble level is of the quantum dynamical semigroup type [84, 85] and has a structure extremely similar to the final one of the GRW theory.

In 1973 P. Pearle was the first to suggest to account for the reduction process in terms of stochastic differential equations. He pursued this line for various years. However, he did not succeed in identifying the appropriate states to which the dynamical equation should lead and consequently a mechanism whose effectiveness could have been negligible for microsystems but extremely relevant for the macroscopic ones. The lack of the identification of the preferred basis, i.e. of what "is out there", was the main obstacle for the success of the program.

The breakthrough is dated 1984. In that year the research program suggested by Ghirardi, Rimini and Weber [64, 65, 86] started to be developed[23]. In these papers, the first consistent and satisfactory model (QMSL) of dynamical reductions, the one on which all subsequent attempts are based, was presented and discussed in detail. The key assumption is that each elementary constituent of any system is subjected, at random times according to a Poisson distribution with mean frequency $\lambda = 10^{-16}$ $\text{sec}^{-1}$, to random and spontaneous localization processes around appropriate positions. One assumes that the probability distribution of these processes is such that hittings, i.e. spontaneous localizations around specific points in space, occur with a higher probability at those places where, in the standard quantum description, there is a higher probability of finding the particle. As will be clear in the following section, the above prescriptions can be satisfied by introducing precise non–linear and stochastic elements in the dynamics.

It is extremely easy also to convince oneself that the model embodies the so–called **trigger mechanism**, i.e. that the spontaneous reductions become more and more frequent with increasing the number of particles of the system under consideration. The models thus "has the property required ... of having little impact for small systems but nevertheless suppressing macroscopic superpositions" [87].

In the years 1989–90 the efforts of Ghirardi, Rimini, Weber on the one side and of P. Pearle on the other, were joined together and CSL (the Continuous Spontaneous Localizations model) was developed [88, 66]. CSL is based on a modified Schrödinger equation containing new stochastic and nonlinear terms besides the standard hamiltonian. These new terms induce a diffusion process for the statevector which acts like a continuous hitting: it is precisely this diffusion process which is responsible for the reduction of the statevector.

The next obvious step was to generalize dynamical reduction models to relativistic quantum field theories. Various attempts have been made [89, 67], typically by considering models in which quantum fields are locally coupled to scalar white noises. All the desired properties of the non relativistic theory, the most important one being the localization in space of macroscopic objects, hold also in the relativistic case. However, the reduction mechanism induces an infinite increase of the energy of physical systems per unit time and unit volume: such models are then physically unacceptable.

[23] Actually, even though ref.[64] has been published in 1985, it appears among the proceedings of a Conference held at Heidelberg in 1984.

The work on relativistic dynamical reduction models is still in progress; there are some hints that the difficulties so far encountered can be solved by generalizing the coupling of the fields to the stochastic noises [90, 91, 92]; such generalizations however have still to be studied in detail.

# 6 Quantum Mechanics with Spontaneous Localizations (QMSL)

The guiding lines which led Ghirardi, Rimini and Weber to formulate the first consistent dynamical reduction model, called Quantum Mechanics with Spontaneous Localizations (QMSL) [64, 65], are basically two:

1. The "preferred basis" — the basis on which reductions take place — must be chosen in such a way to guarantee a definite position in space to macroscopic objects.

2. The modified dynamics must have little impact on microscopic objects, but at the same time must reduce the superposition of different macroscopic states of macro–systems. There must then be an "amplification" mechanism when moving from the micro to the macro level.

The section is devoted to the analysis of how these ideas have been successfully implemented. We first list the axioms defining the universal (i.e. valid both at the microscopic and at the macroscopic level) dynamics of QMSL and we show, by resorting to a simple example, how the reduction mechanism works; the general discussion of statevector collapse is more easily accomplished within the framework of the continuous model (CSL) analyzed in the following sections, so it will be postponed. In subsection 6.2 we derive the dynamical evolution law of the statistical operator, and we discuss the effect of the modified dynamics on a free particle.

Subsections 6.3 and 6.4 are the core of the section. In 6.3 we specialize our analysis to the case of a simple macroscopic system, a free macroscopic particle, proving that the reducing terms of QMSL yield a classical description for the macro–particle: this is how the macro–objectification problem of Quantum Mechanics finds a natural solution within QMSL. In subsection 6.4 we deepen our analysis of macroscopic systems, showing how their classical properties emerge from the quantum properties of their constituents. This means that QMSL embodies a single universal dynamics which takes into account both the quantum properties of microscopic systems and the classical properties of macroscopic objects.

In 6.5 we discuss the possible numerical choices of the two parameters appearing in the modified dynamics. They must be chosen in such a way that all known properties of microscopic quantum systems are not altered in any significant way, while the classical properties of macroscopic systems must be guaranteed. The final subsection is devoted to a mathematical review of quantum dynamical semigroups, a class of evolution dynamics to which the QMSL basic equation belongs.

## 6.1 The assumptions of the model

Quantum Mechanics with Spontaneous Localizations [64, 65] is based on the following assumptions:

1. Each particle of a system of $n$ distinguishable particles experiences, with a mean rate $\lambda_i$, a sudden spontaneous localization process.

2. In the time interval between two successive spontaneous processes the system evolves according to the usual Schrödinger equation.

3. The sudden spontaneous process is a localization described by:

$$|\psi\rangle \quad \stackrel{\text{localization}}{\longrightarrow} \quad \frac{|\psi^i_{\mathbf{x}}\rangle}{\||\psi^i_{\mathbf{x}}\rangle\|}, \tag{6.1}$$

where $|\psi^i_{\mathbf{x}}\rangle = L^i_{\mathbf{x}}\,|\psi\rangle$. $L^i_{\mathbf{x}}$ is a norm–reducing, positive, self–adjoint, linear operator in the $n$–particle Hilbert space $\mathcal{H}$, representing the localization of particle $i$ around the point $\mathbf{x}$.

4. The probability density for the occurrence of a localization at point $\mathbf{x}$ is assumed to be

$$P_i(\mathbf{x}) \quad = \quad \||\psi^i_{\mathbf{x}}\rangle\|^2 : \tag{6.2}$$

This requires that

$$\int d^3x \, \left[L^i_{\mathbf{x}}\right]^2 \quad = \quad 1 \tag{6.3}$$

5. The localization operators $L^i_{\mathbf{x}}$ have been chosen to have the form:

$$L^i_{\mathbf{x}} \quad = \quad \left(\frac{\alpha}{\pi}\right)^{3/4} \, e^{-\frac{\alpha}{2}\,(\mathbf{q}_i - \mathbf{x})^2}, \tag{6.4}$$

$\mathbf{q}_i$ being the position operator for particle $i$.

Before going on, we want to make clear a fundamental conceptual point. Here, and in the following, when we speak of "particles" we are are simply using the standard, somehow inappropriate, quantum mechanical language. Within dynamical reduction models particles are not point–like objects which move in space following appropriate trajectories according to the forces they are subjected to (as it is the case of, e.g., Bohmian mechanics). In dynamical reduction models, like in standard quantum mechanics, particles are represented just by the wavefunction which, in general, is spread all over the space. As we will see, the basic property of the models analyzed here is that, when a large number of "particles" interact with each other in appropriate ways, they end up being always extremely well localized in space, leading in this way to a situation which is perfectly adequate for characterizing what we call a "macroscopic object". Thus, strictly speaking [32] the are no particles in dynamical reduction models at the fundamental level; there is simply a microscopic, quantum, wave–like realm which gives rise to the usual classical realm at the macroscopic level.

It is easy to see how the reduction mechanism works with the help of the following simple example. Let us consider the superposition of two gaussian functions, one centered around position $-a$ and the other around position $a$ (for simplicity we deal with the one–dimensional case):

$$\psi(z) \;=\; \frac{1}{\mathcal{N}} \left[ e^{-\frac{\gamma}{2}\,(z+a)^2} \;+\; e^{-\frac{\gamma}{2}\,(z-a)^2} \right]; \tag{6.5}$$

$\mathcal{N}$ is a normalization constant. Let us assume that $1/\sqrt{\gamma} \ll 1/\sqrt{\alpha}$ and $a \gg 1/\sqrt{\alpha}$: the distance between the two gaussians is much greater than the localization amplitude, while their width is much smaller than it.

Let us now consider a hitting centered around $a$; the wavefunction changes as follows:

$$\psi(z) \;\longrightarrow\; \psi_a(z) = \frac{1}{\mathcal{N}_a} \left[ e^{-2\alpha a^2} \, e^{-\frac{\gamma}{2}\,(z+a)^2} \;+\; e^{-\frac{\gamma+\alpha}{2}\,(z-a)^2} \right]. \tag{6.6}$$

We see that the gaussian function centered around position $-a$ has been exponentially suppressed with respect to the other term, whose width is practically left unchanged: the new wavefunction describes a particle very well localized around position $a$. Moreover, it is easy to check that the probability for such a hitting to occur, as given by (6.2), is almost equal to $1/2$ i.e. the quantum mechanical probability to find the particle in $a$. Of course, a similar argument holds if the hitting takes place around position $-a$.

Finally, let us consider the case in which a hitting takes place in a region far from both gaussians, e.g. around the origin 0. In such a case it is easy to verify that the wavefunction does not change in any appreciable way. The reduction to a localized state does not occur. However, the probability for such an hitting to occur is extremely small, about $e^{-\alpha a^2}$. Concluding, reductions are more likely to occur where the probability to find a particle, according to the standard interpretation of the wavefunction, is greater.

## 6.2 The equation for the statistical operator

Within QMSL, the reduction mechanism transforms pure states into statistical mixtures; we can then resort to the statistical operator formalism to investigate specific physical consequences of the theory, such as the time evolution of the mean values of dynamical variables. However, it is important to stress once more that the reduction mechanism must be effective at the wavefunction level, as it should be clear according to the analysis of section 5.1, and as we have already proved with the help of the very simple example of the previous section (the general situation will be discussed in the next section).

Let us consider a single particle. Suppose it suffers a hitting process: its wavefunction $|\psi\rangle$ changes it into the new wavefunction $|\psi_{\mathbf{x}}\rangle$. We do not know where the hitting occurs, but only the probability for it to occur around position $\mathbf{x}$. Accordingly, the pure state is transformed into the following statistical mixture:

$$\begin{aligned} |\psi\rangle\langle\psi| \quad &\longrightarrow \quad \int d^3x\, P(\mathbf{x})\, \frac{|\psi_{\mathbf{x}}\rangle\langle\psi_{\mathbf{x}}|}{\||\psi_{\mathbf{x}}\rangle\|^2} \;= \\ &= \quad \int d^3x\, L^i_{\mathbf{x}}|\psi\rangle\langle\psi|L^i_{\mathbf{x}} \;\equiv\; T[|\psi\rangle\langle\psi|]. \end{aligned} \tag{6.7}$$

Of course, if the initial state of the particle is not pure but a statistical mixture given by the operator $\rho$, the effect of a hitting process is the same as the one described above: $\rho$ changes into $T[\rho]$.

We derive now the evolution equation for $\rho(t)$. In a time interval $dt$, the statistical operator evolves in the following way: since the localization mechanism is Poissonian, there is a probability $\lambda\, dt$ for a hitting to occur during that time interval, in which case $\rho$ changes to $T[\rho]$, and a probability $1 - \lambda\, dt$ for no hittings to occur so that the statistical operator evolves according to the usual Schrödinger equation:

$$\rho(t+dt) \quad = \quad (1-\lambda\, dt)\left[\rho(t) - \frac{i}{\hbar}\,[H,\rho(t)]dt\right] \;+\; \lambda\, dt\, T[\rho(t)],$$

i.e.:

$$\frac{d}{dt}\,\rho(t) \quad = \quad -\frac{i}{\hbar}\,[H,\rho(t)] \;-\; \lambda\,(\rho(t) \;-\; T[\rho(t)])\,. \tag{6.8}$$

This is the **master equation** of QMSL, describing the quantum evolution of a single particle which undergoes random localization processes; it has a quantum–dynamical–semigroup structure, which we will discuss at the end of the section.

In the coordinate representation one has, according to assumption 5:

$$\langle \mathbf{q}'|T[\rho]|\mathbf{q}''\rangle \;=\; e^{-\frac{\alpha}{4}(\mathbf{q}'-\mathbf{q}'')^2}\,\langle \mathbf{q}'|\rho|\mathbf{q}''\rangle. \tag{6.9}$$

Since, owing to equation (6.9),

$$\langle \mathbf{q}|T[\rho]|\mathbf{q}\rangle \;=\; \langle \mathbf{q}|\rho|\mathbf{q}\rangle, \tag{6.10}$$

equation (6.8) is obviously trace preserving. Moreover, using equation (6.8), it can be easily proved that

$$\frac{d}{dt}\,\mathrm{Tr}[\rho^2] \;<\; 0. \tag{6.11}$$

This implies that the dynamical evolution transforms pure states into statistical mixtures.

Let us now consider equation (6.8) in the case in which $H$ is the Hamiltonian for a free particle; for simplicity we work in one dimension. In the coordinate representation we get

$$\begin{aligned}\frac{\partial}{\partial t}\,\langle q'|\rho(t)|q''\rangle \;&=\; \frac{i\hbar}{2m}\left[\frac{\partial^2}{\partial q'^2}-\frac{\partial^2}{\partial q''^2}\right]\langle q'|\rho(t)|q''\rangle \\ &\quad -\lambda\left[1-e^{-\frac{\alpha}{4}(q'-q'')^2}\right]\langle q'|\rho(t)|q''\rangle.\end{aligned} \tag{6.12}$$

One can express the solution of the above equation satisfying given initial conditions in terms of the solution $\langle q'|\rho_{\mathrm{Sch}}(t)|q''\rangle$ of the pure Schrödinger equation ($\lambda = 0$) satisfying the same initial conditions, according to [65]:

$$\langle q'|\rho(t)|q''\rangle \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}dy\,e^{-\frac{i}{\hbar}ky}\,F(k,q'-q'',t)\,\langle q'+y|\rho_{\mathrm{Sch}}(t)|q''+y\rangle \tag{6.13}$$

where

$$F(k,q,t) \;=\; e^{-\lambda t+\lambda\int_0^t d\tau\, e^{-\frac{\alpha}{4}\left(q-\frac{k\tau}{m}\right)^2}}. \tag{6.14}$$

The Hermitian symmetry of $\rho(t)$ follows from the property $F(k,q,t) = F(-k,-q,t)$.

To understand the dynamical evolution described by equation (6.12) we can evaluate the mean values, spreads, and correlations for the position and momentum operators for all times. In the considered case, it turns out [65] that these variables are related to those of the pure Schrödinger evolution by:

$$\langle\!\langle q\rangle\!\rangle \;=\; \langle\!\langle q\rangle\!\rangle_{\mathrm{Sch}} \tag{6.15}$$

$$\langle\!\langle p\rangle\!\rangle \;=\; \langle\!\langle p\rangle\!\rangle_{\mathrm{Sch}} \tag{6.16}$$

$$\{q\} \;\equiv\; \langle\!\langle[q-\langle\!\langle q\rangle\!\rangle]^2\rangle\!\rangle \;=\; \{q\}_{\mathrm{Sch}} \;+\; \frac{\alpha\lambda\hbar^2}{6m^2}\,t^3 \tag{6.17}$$

$$\{qp\} \;\equiv\; \langle\!\langle[(q-\langle\!\langle q\rangle\!\rangle)(p-\langle\!\langle p\rangle\!\rangle)]_{\mathrm{sym}}\rangle\!\rangle \;=\; \{qp\}_{\mathrm{Sch}} \;+\; \frac{\alpha\lambda\hbar^2}{4m}\,t^2 \tag{6.18}$$

$$\{p\} \;\equiv\; \langle\!\langle[p-\langle\!\langle p\rangle\!\rangle]^2\rangle\!\rangle \;=\; \{p\}_{\mathrm{Sch}} \;+\; \frac{\alpha\lambda\hbar^2}{2}\,t. \tag{6.19}$$

In equation (6.18) we have denoted by $[\,\cdot\,]_{\mathrm{sym}}$ the Hermitian part of the quantity in square brackets. The shorthands $\{q\}$, $\{qp\}$, $\{p\}$ have been introduced to simplify

the notation of the formal developments of the following sections. We note that the mean values are not affected by the non Hermitian term in equation (6.8). For what concerns spreads and correlation, the corrections depend only on the combination $\alpha\lambda$ of the parameters $\alpha$ and $\lambda$.

## 6.3 Discussion of the non Hamiltonian dynamics for a free macroscopic particle

In this subsection we begin the analysis of the effects of the modified dynamics on macroscopic systems; for simplicity, we first consider the case of a free macroscopic particle. Its time evolution is embodied into equation (6.12), where the mass $m$ now has the order of magnitude of the mass of a macroscopic object.

We remark that the standard quantum dynamics, in the case of a free particle, induces for the mean values $\langle\!\langle q \rangle\!\rangle_{\rm Sch}$ and $\langle\!\langle p \rangle\!\rangle_{\rm Sch}$ exactly the classical evolution. Moreover, for any reasonable choice of the initial spreads of the position $\Delta q = \sqrt{\{q\}}$ and of the momentum $\Delta p = \sqrt{\{p\}}$, the increase of $\Delta q$ when time elapses, in virtue of the smallness of the Planck constant and of the large value of the mass of a macroscopic object, can be completely disregarded for all interesting times. However, the recognition of this fact does not exhaust the problem of the derivation of the classical behaviour of a macroscopic object from quantum principles, since problems remain open when linear superpositions of macroscopically distinguishable states can occur. In such cases, as already discussed, a satisfactory classical description would require that the statistical ensemble decomposes into a statistical mixture of macroscopically distinguishable states. Let us discuss the above points within the framework of the non Hamiltonian dynamics introduced in the previous subsections.

First of all we can observe that Ehrenfest's theorem holds true also for the modified dynamics. In fact, for any dynamical variable $X$, which is a function of the operator $q$ only, it is easily shown that

$$\mathrm{Tr}\{X(q)\,T[\rho]\} \quad = \quad \mathrm{Tr}[X(q)\,\rho]. \tag{6.20}$$

This in turns implies

$$\frac{d}{dt}\,\langle\!\langle X(q) \rangle\!\rangle \;=\; \mathrm{Tr}\left[X(q)\,\frac{d\,\rho}{dt}\right] \;=\; -\frac{i}{\hbar}\,\mathrm{Tr}\{X(q)\,[H,\,\rho]\}, \tag{6.21}$$

as it happens for the Schrödinger evolution. It follows that

$$\frac{d}{dt}\,\langle\!\langle q \rangle\!\rangle \quad = \quad \frac{1}{m}\,\langle\!\langle p \rangle\!\rangle. \tag{6.22}$$

For the operator $p$ one finds

$$\mathrm{Tr}\{p\,T[\rho]\} \quad = \quad \mathrm{Tr}\{p\,\rho\}. \tag{6.23}$$

Then, if $H = p^2/2m + V(q)$, we have

$$\frac{d}{dt}\,\langle\!\langle p \rangle\!\rangle \quad = \quad -\left\langle\!\!\left\langle \frac{\partial V}{\partial q} \right\rangle\!\!\right\rangle. \tag{6.24}$$

In accordance with this property, equations (6.15) and (6.16) show that in the case of a free macroscopic particle[24] the mean values of position and momentum are not affected by the non–Hamiltonian terms. On the contrary, in the expression for the spreads additional terms appear. These terms increase with time, so that one can

[24] Actually, of any free particle.

identify a characteristic time interval $T$ during which they remain small with respect to those expressing the Schrödinger evolution. $T$ turns out to be of the order of the smaller of the two times $T_1$ and $T_2$ given by

$$T_1 \;=\; \left[\frac{6m^2(\Delta q_{\rm Sch})^2}{\alpha\lambda\hbar^2}\right]^{1/3} \qquad\qquad T_2 \;=\; \frac{2(\Delta p_{\rm Sch})^2}{\alpha\lambda\hbar^2}. \tag{6.25}$$

For the time interval $T$ the spreads given by equations (6.17)–(6.19) coincide practically with the Schrödinger values, which in turn are negligible for any reasonable choice of their initial values. We shall discuss below the values taken by $T$ when the parameters of the model are appropriately chosen.

The fact that $\Delta\, q^2$ and $\Delta\, p^2$ are very close to the Schrödinger values for an appropriate time interval is strictly related to the small influence of the non–Hamiltonian term on the matrix elements of the statistical operator $\langle q'|\rho|q''\rangle$ when $|q'-q''| \ll 1/\sqrt{\alpha}$. On the contrary, the non–Hamiltonian dynamics has a drastic effect on the off–diagonal elements when $|q'-q''| \geq 1/\sqrt{\alpha}$. This can be easily understood by observing that the properties of the function $F(k,q,t)$ are remarkably different in the two cases $q=0$ and $q\neq 0$. In fact, when $q=0$ the integral in the exponent in equation (6.14) for sufficiently small $t$ behaves like $t$, yielding the cancellation of the factor $e^{-\lambda t}$ and making $F(k,0,t)$ very near to 1. Since $F=1$ implies $\langle q'|\rho(t)|q''\rangle = \langle q'|\rho_{\rm Sch}(t)|q''\rangle$, this shows that the (almost) diagonal matrix elements of the statistical operator in the coordinate representation are practically unaffected for an appropriate time interval by the non–Hamiltonian term in the evolution equation. On the contrary, for $q\neq 0$, the integral in equation (6.14) cannot cancel, even for small times, the damping factor $e^{-\lambda t}$, so that the off–diagonal elements are rapidly suppressed.

To make these statements more precise we derive two inequalities for the function $F(k,q,t)$ for the two cases $q=0$ and $q>0$.

**a)** $q=0$ Since

$$\frac{1}{t}\int_0^t d\tau\, e^{-\dfrac{\alpha k^2\tau^2}{4m^2}} \;\geq\; e^{-\dfrac{\alpha k^2 t^2}{4m^2}}, \tag{6.26}$$

it follows that

$$F(k,0,t) \;>\; e^{-\lambda t\left(1-e^{-\alpha k^2t^2/4m^2}\right)} \;\geq\; 1-\frac{\alpha\lambda k^2t^3}{4m^2}, \tag{6.27}$$

the last inequality being useful for $\alpha\lambda k^2t^2/4m^2<1$. We then have

$$1-F(k,0,t) \quad\leq\quad \frac{\alpha\lambda k^2t^3}{4m^2}. \tag{6.28}$$

**b)** $q>0$ The function $F$ can be written as

$$F(k,q,t) \;=\; e^{-\lambda t\left[1-h\left((\sqrt{\alpha}/2)\frac{kt}{m},(\sqrt{\alpha}/2)q\right)\right]}, \tag{6.29}$$

where

$$h(x,y) \quad=\quad \frac{1}{x}\int_{-y}^{x-y} dz\, e^{-z^2}. \tag{6.30}$$

The function $h(x,y)$ is the mean value of $e^{-z^2}$ on the interval $(-y,x-y)$. Clearly one has

$$h(x,y) \;<\; h(y,y) \;=\; h(2y,y) \tag{6.31}$$

for $x < y$, and

$$h(x,y) \;<\; h(2y,y) \tag{6.32}$$

for $x > 2y$. For $y < x < 2y$ one finds

$$h(x,y) \;<\; \frac{1}{x}\int_{-y}^{y} dz\, e^{-z^2} \;<\; \frac{1}{y}\int_{-y}^{y} dz\, e^{-z^2} \;=\; 2\,h(2y,y), \tag{6.33}$$

so that, on the whole,

$$h(x,y) \;<\; 2\,h(2y,y) \;=\; \frac{\sqrt{\pi}}{y}\,\mathrm{erf}(y). \tag{6.34}$$

In turn the function $F$ obeys the inequality

$$F(k,q,t) \quad < \quad e^{-\lambda\beta t}, \tag{6.35}$$

where

$$\beta \;=\; 1 \;-\; \frac{\sqrt{\pi}}{(\sqrt{\alpha}/2)q}\,\mathrm{erf}\,[(\sqrt{\alpha}/2)q]. \tag{6.36}$$

Inequalities (6.27) and (6.35) prove the correctness of our previous statements about the behaviour of $F(k,q,t)$ and, consequently, about the features of the dynamics concerning the diagonal and off–diagonal elements of the density matrix in configuration space.

We come back to the discussion of the diagonal elements of the statistical operator, using the just derived inequalities (6.27) and (6.35) for the function $F(k,q,t)$. From equation (6.13) we see that

$$\begin{aligned}\langle q|\rho_{\mathrm{Sch}}(t)|q\rangle - \langle q|\rho(t)|q\rangle \quad &= \quad \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dk\,[1-F(k,0,t)]\int_{-\infty}^{+\infty} dy\, e^{-\frac{i}{\hbar}ky} \\ &\quad \langle q+y|\rho_{\mathrm{Sch}}(t)|q+y\rangle.\end{aligned} \tag{6.37}$$

To illustrate the implications of this equation, we discuss a simple example. Suppose that $\langle q|\rho_{\mathrm{Sch}}(t)|q\rangle$ is a mixture of Gaussian terms whose spreads are $\Delta_i$ with minimum $\Delta_0$. Then the Fourier transform appearing in equation (6.37) yields terms containing Gaussian factors $e^{-\Delta_i^2 k^2/2\hbar^2}$, whose maximum width is $\hbar/\Delta_0$, so that the integral in $k$ is concentrated in a region $|k| < \hbar/\Delta_0$. Inequality (6.28) shows then that the integrand in equation (6.37) contains a factor smaller than $(\alpha\lambda\hbar^2/4m\Delta_0^2)t^3$. The condition $t \ll T_1$, the time $T_1$ being given by equation (6.25), implies

$$\langle q|\rho(t)|q\rangle \quad \simeq \quad \langle q|\rho_{\mathrm{Sch}}(t)|q\rangle. \tag{6.38}$$

Obviously this result holds for those matrix elements which are appreciably different from zero.

For the off–diagonal elements we consider the case $q' > q''$. Obviously the same results are valid for $q' < q''$ due to the Hermitian symmetry of $\rho(t)$. Inequality (6.35) gives, for $q' - q'' > 2\sqrt{\pi/\alpha}$, a significant bound on $F$ independent of $k$. This shows that expression (6.13) for $\langle q'|\rho(t)|q''\rangle$ contains an exponentially damping factor whose lifetime is $\tau = 1/\lambda\beta$, a consequence of the fact that (in a time interval of the order of $\tau$) linear superpositions of states separated by distances larger than the characteristic localization distance $1/\sqrt{\alpha}$ are transformed into one or the other of their terms.

As we shall see, one can choose the parameters $\lambda$ and $\alpha$ in such a way that the time $T = \min(T_1, T_2)$ is very large and $\tau$ extremely small so that we can conclude that the modified dynamics leads to an evolution agreeing with the classical one in the case of a macroscopic object and overcomes the problems arising from linear superpositions of states localized in far apart regions.

## 6.4 Macroscopic dynamics from the microscopic one

In the previous subsection we have introduced a non–purely–Hamiltonian dynamics to describe the motion of a macroscopic particle and we have outlined how this modification can be used to overcome some of the difficulties in the description of such objects. However, macroscopic objects are composite systems and standard quantum mechanics gives definite prescriptions for their description. It is an important feature of quantum mechanics that, under suitable conditions, the internal and the center–of–mass motions of the composite systems decouple and, moreover, that the equation of motion for the center of mass is formally identical to the equation prescribed by the theory for the description of a single particle. Here we want to investigate whether it is possible to obtain the non–purely–Hamiltonian dynamics for macroscopic particles described in the previous subsections from a modification of the standard quantum dynamics for their microscopic constituents. If such a modification leaves practically unaltered the behaviour of microscopic systems as accounted for by quantum mechanics we can say we have laid the foundations of a possible unified description able to account for both the quantum and the classical behaviours of microscopic and macroscopic systems, respectively.

In section 6.1 we have assumed that the localization process $T[\,\cdot\,]$ occurs individually for each constituent of a many–particle system. We consider now a system of $N$ particles in one dimension. Assuming that the accuracy of the localizations is the same for all constituents, the evolution equation for the composite system is

$$\frac{d}{dt}\,\rho(t) \;\;=\;\; -\frac{i}{\hbar}\,[H,\rho(t)] \;-\; \sum_{i=1}^{N}\lambda_i\,(\rho(t) \;-\; T_i[\rho(t)])\,, \tag{6.39}$$

where

$$T_i[\rho(t)] \;=\; \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty} dx\, e^{-\frac{\alpha}{2}\,(q_i-x)^2}\,\rho\, e^{-\frac{\alpha}{2}\,(q_i-x)^2}\,, \tag{6.40}$$

$q_i$ being the position operator for the $i$th particle of the system (throughout the subsection, we will keep working in 1 dimension).

It is worthwhile to illustrate the physical consequences of the above equation for the important conceptual problem of the possible occurrence of linear superpositions of states corresponding to different locations of a macroscopic object. Such a situation occurs, for instance, in the quantum theory of measurement, in connection with possible macroscopically different pointer positions. With reference to such a case we consider the linear superposition $\psi = \psi_1 + \psi_2$ of two states corresponding to two different pointer positions. We remark that in the case under discussion there is a macroscopic number $N$ of particles which are located in macroscopically different positions when the state is $\psi_1$ or $\psi_2$ (to be precise, in our model this means located at a distance larger than $1/\sqrt{\alpha}$). If a spontaneous localization process takes place for one of such particles, this particle is constrained to be either in the spatial region which it occupies when the state is $\psi_1$, or in the one corresponding to $\psi_2$. The linear superposition is consequently transformed into a statistical mixture of states $\psi_1$ and $\psi_2$. Since the number of differently located particles is $N$, the reduction of states $\psi_1$ and $\psi_2$ occurs with a rate which is amplified by a factor $N$ with respect to the one, $\lambda_i$, which characterizes the elementary spontaneous localizations.

The model yields therefore a natural solution to the puzzling situation originating from the occurrence of linear superpositions of differently located states. These considerations, however, do not exhaust the problems to be discussed. In fact, we must still check that the modification of the dynamics for the microscopic constituents does not imply physically unacceptable consequences for the dynamics of the system as a whole. Actually, according to the previous discussions, we would like to have for the

macroscopic object a dynamical equation of the type considered in section 6.3. To discuss this point, let us introduce the center of mass and relative motion position operators $Q$ and $r_j$ $(j = 1, 2, ..., N-1)$, related to the operators $q_i$ by

$$q_i \quad = \quad Q \; + \; \sum_{j=1}^{N-1} c_{ij}\, r_j . \tag{6.41}$$

Equation (6.39), when the Hamiltonian $H$ can be split into the sum of the center of mass and internal motion parts $H_Q$ and $H_r$ acting in the respective state spaces, reads

$$\frac{d}{dt}\,\rho(t) \quad = \quad -\frac{i}{\hbar}\,[H_Q, \rho(t)] \; - \frac{i}{\hbar}\,[H_r, \rho(t)] \; - \; \sum_i \lambda_i \left(\rho(t) \; - \; T_i[\rho(t)]\right), \tag{6.42}$$

where the operator $T_i[\rho]$ can now be written as

$$T_i[\rho] \; = \; \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\, e^{\; -\frac{\alpha}{2}\left[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x\right]^2} \rho\, e^{\; -\frac{\alpha}{2}\left[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x\right]^2} . \tag{6.43}$$

The dynamical evolution of the **center of mass** of the system is described by the statistical operator

$$\rho_Q \quad = \quad \mathrm{Tr}^{(r)}[\rho], \tag{6.44}$$

obtained by taking the partial trace on the internal degrees of freedom of the statistical operator $\rho$ for the complete $N$–particle system. Taking the $r$ trace of the operation $T_i[\rho]$ one gets

$$\int dr_1 \; \ldots \, dr_{N-1} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\; e^{\; -\frac{\alpha}{2}\left[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x\right]^2} \cdot \\ \cdot \, \langle r_1 \ldots r_{N-1} | \rho | r_1 \ldots r_{N-1} \rangle \; e^{\; -\frac{\alpha}{2}\left[Q + \sum_{j=1}^{N-1} c_{ij} r_j - x\right]^2} , \tag{6.45}$$

so that, by shifting the integration variable $x$ by the amount $\sum_j c_{ij} r_j$, one finds

$$\mathrm{Tr}^{(r)}(T_i[\rho]) \; = \; T_Q[\mathrm{Tr}^{(r)}(\rho)], \tag{6.46}$$

where

$$T_Q[\,\cdot\,] \; = \; \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\, e^{\; -\frac{\alpha}{2}\,(Q-x)^2} [\,\cdot\,]\, e^{\; -\frac{\alpha}{2}\,(Q-x)^2} . \tag{6.47}$$

If one takes the $r$ trace of equation (6.42) one then gets

$$\frac{d}{dt}\,\rho_Q(t) \quad = \quad -\frac{i}{\hbar}\,[H_Q, \rho_Q] \; - \; \sum_i \lambda_i \left(\rho_Q \; - \; T[\rho_Q]\right). \tag{6.48}$$

We have thus shown that the equation describing the reduced dynamics of the center of mass has exactly the same form of equation (6.8), the parameter $\lambda$ being replaced by the sum of the $\lambda_i$'s for the individual constituents of the many–body system. This is a direct consequence of the formal property (6.46).

It is worthwhile stressing that the non–Hamiltonian term in equation (6.48) is directly generated by the analogous terms of equation (6.39) and is not due to the elimination of the internal degrees of freedom. In fact, if within the standard formalism one considers a composite system with an Hamiltonian $H = H_Q + H_r$, the reduced dynamics for the center of mass motion is necessarily Hamiltonian, and therefore it allows for the occurrence of linear superpositions of widely separated states of the center of mass. To avoid this, one could couple the system to some other system whose dynamics is then eliminated [93]. This, however, gives rise to a chain process when larger and larger external parts are included. If one wants to reach a point where linear superpositions of far–away states cannot actually occur, one has to break this chain in an arbitrary way. In our approach the non–Hamiltonian dynamics for a macroscopic object is induced by a basic non–Hami1tonian dynamics for its microscopic constituents.

Let us now investigate briefly the effect of the modified dynamics on the **relative variables**. From a physical point of view it is particularly simple and interesting to consider the case in which the internal motion Hamiltonian gives rise to a sharp (with respect to $1/\sqrt{\alpha}$) localization of the internal coordinates, as it happens, for an appropriate choice of $\alpha$, in an insulating solid. In such a case it is evident that localizing with an accuracy $1/\sqrt{\alpha}$ any one of the points of the almost rigid structure of the solid induces a corresponding localization of the center of mass. In this situation something more can be proved, i.e., that the internal and the center of mass motion decouple almost exactly and the internal motion is not affected by the non–Hamiltonian terms in (6.39). To be precise, we assume that the matrix elements $\langle Q', r'|\rho|Q'', r''\rangle$ are non–negligible only when the conditions

$$|\sum_{j=1}^{N-1} c_{ij}\, r'_j - a_i| \ll \frac{1}{\sqrt{\alpha}} \qquad |\sum_{j=1}^{N-1} c_{ij}\, r''_j - a_i| \ll \frac{1}{\sqrt{\alpha}}, \qquad i = 1, \ldots, N \quad (6.49)$$

are satisfied, $a_i$ being the equilibrium position of constituent $i$ relative to the center of mass. Since conditions (6.49) imply

$$|\sum_{j=1}^{N-1} c_{ij}(r'_j - r''_j)| \ll \frac{1}{\sqrt{\alpha}} \qquad i = 1, \ldots, N, \quad (6.50)$$

$\langle Q', r'|\rho|Q'', r''\rangle$ is negligibly small unless condition (6.50) is satisfied. From the definition (6.40) one gets

$$\begin{aligned}\langle Q', r'|T_i[\rho]|Q'', r''\rangle &= \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{\alpha}{2}\left[Q' + \sum_{j=1}^{N-1} c_{ij} r'_j - x\right]^2} \cdot \\ &\cdot \langle Q', r'|\rho|Q'', r''\rangle\, e^{-\frac{\alpha}{2}\left[Q'' + \sum_{j=1}^{N-1} c_{ij} r''_j - x\right]^2} = \\ &= e^{-\frac{\alpha}{4}\left[Q' - Q'' + \sum_{j=1}^{N-1} c_{ij}(r'_j - r''_j)\right]^2} \langle Q', r'|\rho|Q'', r''\rangle. \end{aligned} \quad (6.51)$$

The exponential factor appearing in the last line of equation (6.51) is a Gaussian in the variable $Q' - Q''$ displaced by the amount $\sum_j c_{ij}(r'_j - r''_j)$. Because of equation

(6.50), the displacement of the Gaussian can be neglected with respect to its width, so that in this approximation

$$T_i[\rho] \quad = \quad T_Q[\rho]. \tag{6.52}$$

The physical meaning of equation (6.52) is that, as foreseen, a localization of a single constituent of a rigid system is equivalent to a localization of the center of mass. Equation (6.42) shows that, if the initial statistical operator has the form of a direct product $\rho_Q \, \rho_r$, it remains of the same type, and the statistical operators $\rho_r$ and $\rho_Q$ obey the equations

$$\frac{d}{dt} \, \rho_r \quad = \quad -\frac{i}{\hbar} \, [H_r, \rho_r] \tag{6.53}$$

and (6.48) respectively. We conclude that in the considered case the internal and the center of mass motions decouple, the internal motion of the solid being unaffected by the localization process introduced in equation (6.39) and the center of mass motion being affected by such a process with a characteristic rate equal to the sum of the rates for all single constituents.

The considerations which can be done in the case of an almost rigid body ensure that the density operator retains the form $\rho = \rho_Q \, \rho_r$, when it is initially of this form. Therefore, the non Hamiltonian terms of equation (6.48), which appear as a consequence of the localization mechanism, expresses meaningfully the destruction of the long–distance coherence, as they entail the suppression of the off–diagonal elements of $\rho_Q$. The situation we have just discussed can be considered, with some idealization, typical of the case in which one is dealing with a macroscopic body.

To conclude this subsection we observe that if one assumes for simplicity that the localization rates $\lambda_i$ of all microscopic (e.g., atomic) constituents of a macroscopic body are of the same magnitude ($\lambda_i = \lambda_{\text{micro}}$), the center of mass is affected by the same process with a rate $\lambda_{\text{macro}} = N\lambda_{\text{micro}}$, where $N$ is of the order of Avogadro's number. As we shall see in the next subsection, this will allow us to choose the parameters $\lambda_{\text{micro}}$ and $\alpha$ in such a way that standard quantum mechanics holds exactly for extremely long times for microscopic systems, while for a macroscopic body possible linear superpositions of far–away states are rapidly suppressed, the dynamical evolution of the center–of–mass position is the classical one and the internal structure remains unaffected.

## 6.5 Choice of the parameters and its consequences

A crucial feature of the point of view which has been adopted in QMSL, i.e., that of considering all elementary constituents of any system as subjected to localizations, consists in the fact that one can choose the parameters of the elementary processes in such a way that (i) the quantum–mechanical predictions for microscopic systems are valid for extremely long times, (ii) the dynamics of a macroscopic object, when it is consistently derived from that of its microscopic constituents, turns out to coincide with the classical one for a sufficiently long time interval, (iii) the suppression of long–distance coherence for macroscopic objects be effective enough to imply that, after a microscopic system has triggered a measuring apparatus, the dynamical evolution leads to the reduction of the wavepacket with well–defined pointer positions.

To give orientative indications on the numerical values of the parameters appearing in our model, we start by remarking that, as it is clear from the formulas of the previous subsections, all physically significant effects of the modified dynamics for a macroscopic object are governed (for a remarkably large range of variability of these parameters) by the product $\alpha\lambda_{\text{macro}}$. For the choice of the parameter $\alpha\lambda_{\text{macro}}$ we have some important criteria which must be taken into account. First of all we want the mean time $1/\lambda_{\text{macro}}$ elapsing between two successive localizations to be such that the

transition to statistical mixtures for states spreading over distances larger than the localization distance $1/\sqrt{\alpha}$ takes place in a very small fraction of a second. A further requirement which has to be taken into account is that, when one is trying to identify particle trajectories for a macroscopic system using the selective form of our equation, the disagreement with the classical predictions which, as shown in [65], unavoidably arises for large times, be unimportant for times which are long with respect to those during which one can keep the macroscopic system isolated. Finally, and more important, we want the modification of the dynamics for microscopic systems with respect to the standard one to be totally irrelevant. The simplest way to obtain this is to assume that the mean rate $\lambda_{\rm micro} = \lambda_{\rm macro}/N$ of the spontaneous localization processes for a microscopic system be extremely small.

For what concerns the parameter $\alpha$ it is necessary to choose the localization distance $1/\sqrt{\alpha}$ large with respect to the atomic dimensions and to the mean spreads around the equilibrium positions of the lattice points of a crystal. In this way, even when one of the extremely infrequent localization processes takes place for a constituent of an atomic system, the localization itself does not modify the internal structure of that system and the decoupling of the center of mass and relative motions discussed in section 6.4 still holds. On the other hand $1/\sqrt{\alpha}$ represents the distance after which a linear superposition is transformed into a statistical mixture. This parameter must then be chosen in accordance with the requirement of avoiding the embarrassing occurrence of linear superpositions of appreciably different locations of a macroscopic object.

These considerations lead us to discuss the following choice for the order of magnitude of the parameters. For the localization rate of the microscopic constituents of any system we choose

$$\lambda_{\rm micro} \simeq 10^{-16}\,{\rm sec}^{-1}. \tag{6.54}$$

This means that such systems are localized once every $10^8$–$10^9$ years. For the parameter $1/\sqrt{\alpha}$ we choose:

$$1/\sqrt{\alpha} \simeq 10^{-5}\,{\rm cm}. \tag{6.55}$$

The fact that a microscopic system is practically never localized, entails that standard quantum mechanics remains fully valid for this type of system. Moreover, for a composite system for which the relative coordinates are confined within a spatial range much smaller than the localization distance $1/\sqrt{\alpha}$, as it happens for atoms and molecules, the process $T[\,\cdot\,]$ is almost ineffective even when it takes place, a fact that strengthens the above conclusion.

For what concerns macroscopic objects (containing a number of constituents of the order of Avogadro's number), according to the considerations of section 6.4 showing that the individual tests on the constituents add for the center–of–mass dynamics, we get as characteristic localization rate:

$$\lambda_{\rm macro} \simeq 10^{7}\,{\rm sec}^{-1}. \tag{6.56}$$

If we take, for the sake of definiteness, the mass of such an object to be of the order of 1 g, and the initial spread of the position $\Delta\, q_0$ again of the order of $10^{-5}$ cm, we know that the quantum increase of the spread in the position is negligible for extremely long times ($\sim 10^{10}$ yr), so that the quantum evolution is practically the same as the classical one. In such a case [compare equation (6.25)], the additional term appearing in $\Delta\, q^2$ equals $\Delta\, q_0^2$ at the time $T_1$, which is of the order of 100 yr. This is a very long time for keeping isolated a macroscopic object. A much longer time $T_2$ is required in order that the additional term in $\Delta\, p^2$ has an appreciable effect for any reasonably chosen initial spread of the momentum. As far as the occurrence of linear superpositions of far away states is concerned, as we have seen, the off–diagonal elements of the statistical operator are exponentially suppressed with the

lifetime $\tau = 1/\lambda\beta$. For $|q - q'| = 4 \times 10^{-5}$ cm we have $\tau = 10^{-6}$ sec [see equation (6.36)]. Therefore after times of this order linear superpositions of states separated by distances larger than $10^{-4}$ cm are transformed into statistical mixtures.

Considerations of this type are important for the quantum theory of measurement. In fact, at least in the case in which the interaction leading to the triggering of the apparatus takes place in a very short time, we can apply our treatment to the macroscopic parts of the apparatus itself, obtaining in this way a consistent solution of the difficulties related to the quantum theory of measurement for what concerns wavepacket reduction and the definite final position of the pointer.

It has to be remarked that the basic evolution equation (6.8), due to the appearance of the non–Hamiltonian terms, implies a nonconservation of energy. Let us give an estimate of this effect in the case of the free particle on the basis of a choice for the parameters we have just made. From equations (6.19) we see that, in our case

$$\langle\!\langle E \rangle\!\rangle \quad = \quad \langle\!\langle E \rangle\!\rangle_{\rm Sch} \; + \; \frac{\lambda\alpha\hbar^2}{4m}\, t. \tag{6.57}$$

where $\langle\!\langle E \rangle\!\rangle_{\rm Sch}$ is the conserved energy for free Schrödinger evolution[25]. Energy nonconservation is then expressed by the term

$$\delta E \quad = \quad \frac{\lambda\alpha\hbar^2}{4m}\, t. \tag{6.58}$$

Let us evaluate this term for the case of a microscopic system. Since $\lambda_{\rm micro} = 10^{-16}$ sec$^{-1}$, $m \simeq 10^{-23}$ g,

$$\frac{\delta E}{t} \quad \simeq \quad 10^{-25} \text{ eV sec}^{-1}, \tag{6.59}$$

which means that to have an increase of 1 eV il takes a time of $10^{18}$ yr. In the case of the center–of–mass equation for a macroscopic system, since both the rate $\lambda$ and the mass increase proportionally to the number of constituents, the energy non conservation is of the same amount. However this argument applies only to the increase of the energy of the center of mass. There is also an increase of energy in the internal motion which, as can be easily understood considering a system of free particles, is the same for all constituents. When this fact is taken into account one can conclude that the estimated energy increase for a system of $N$ [$\simeq$ Avogadro's number] atoms is

$$\frac{\delta E}{t} \quad \simeq \quad 10^{-14} \text{ erg sec}^{-1}, \tag{6.60}$$

Referring to an ideal monoatomic gas the increase in temperature with time is then of the order of $10^{-15}$ K per year.

We conclude that QMSL reproduces in a consistent way quantum mechanics for microscopic objects and classical mechanics for macroscopic objects, and provides the basis for a conceptually appealing description of quantum measurement process[26], and of the behavior of macroscopic systems.

## 6.6 Quantum dynamical Semigroups

Among the non–Hamiltonian evolution equations which have been considered in the literature, there is a class which has been studied in great detail and has proved to

[25] It is easy to prove that the relation $d\langle\!\langle H \rangle\!\rangle/dt = \lambda\alpha\hbar^2/4m$, from which (6.57) can be derived, holds in general even when a potential term $V(q)$ is present in the Hamiltonian.

[26] Actually, this conclusion, to be taken seriously, requires also the proof, which will be presented in what follows, that the macroscopic outcomes which emerge in a measurement process, occur with the probabilities attached to them by the standard formalism.

be useful in the description of various physical processes, which is particularly simple. This class of equations is usually referred to as **quantum dynamical semi–group** (QDS) equations [84]. Let as give a precise definition of a QDS.

Consider the Banach space $T_S(\mathcal{H})$ of the self–adjoint trace–class operators (equipped with the trace norm, denoted as usual by $\| \cdot \|_{\rm Tr}$) on the Hilbert space $\mathcal{H}$ of the considered physical system. A QDS is a one parameter family of linear operators:

$$\Sigma_t : \; T_S(\mathcal{H}) \; \longrightarrow \; T_S(\mathcal{H}) \qquad \text{defined for } t \geq 0,$$

satisfying:

$$\begin{aligned}
&1) \quad \rho \geq 0 \quad \Longrightarrow \quad \Sigma_t(\rho) \geq 0 \quad \forall\, t \geq 0 \\
&2) \quad {\rm Tr}[\Sigma_t(\rho)] = {\rm Tr}[\rho] \qquad\qquad \forall\, \rho \in T_S(\mathcal{H}), \quad t \geq 0 \\
&3) \quad \Sigma_t \Sigma_s(\rho) = \Sigma_{t+s}(\rho), \qquad\qquad \forall\, \rho \in T_S(\mathcal{H}), \quad t, s \geq 0 \\
&4) \quad \lim_{t \to 0} \|\Sigma_t(\rho) - \rho\|_{\rm Tr} = 0 \qquad\qquad \forall\, \rho \in T_S(\mathcal{H}).
\end{aligned}$$

We note that, when $\rho$ is the statistical operator describing the state of a quantum system, the first two conditions correspond to the requirements which are necessary for probability conservation and the third one expresses the Markovian nature of the process (which implies the independence of the evolution law from the time origin).

In terms of the map $\Sigma_t$ one can define the infinitesimal generator $Z$ of the QDS by the equation

$$Z[\rho] \quad = \quad \lim_{t \to 0} \left[ \frac{\Sigma_t(\rho) - \rho}{t} \right], \qquad \text{in the norm } \| \cdot \|_{\rm Tr}. \tag{6.61}$$

Obviously, the simplest case of a QDS is represented by the standard Hamiltonian evolution equation

$$\Sigma_t(\rho) \quad = \quad e^{-\frac{i}{\hbar} H\, t} \, \rho \, e^{\frac{i}{\hbar} H\, t}, \tag{6.62}$$

where $H$ is self–adjoint; in such case

$$Z[\rho] \quad = \quad -\frac{i}{\hbar} \, [H, \, \rho]. \tag{6.63}$$

As we have already stated, QDS equations have been studied in great detail [78] and many general results have been obtained. Lindblad [84] has been able to identify the most general form for the infinitesimal generator of a QDS when two more conditions are added to those previously considered, i.e.

$$\begin{aligned}
&5) \quad Z \text{ is bounded,} \\
&6) \quad \Sigma_t \text{ is completely positive definite.}
\end{aligned}$$

Complete positiveness has to be understood in the sense of Stinespring [94]. In such a case, the evolution equation for the statistical operator $\rho$ can be written as

$$\frac{d}{dt} \, \rho(t) \quad = \quad -\frac{i}{\hbar} [H, \rho(t)] \; + \; \lambda \left\{ T[\rho(t)] \; - \; \frac{1}{2} \, \rho(t) J \; - \; \frac{1}{2} \, J \rho(t) \right\}. \tag{6.64}$$

where

$$T[\rho] \; = \; \sum_{i \in K} A_i \, \rho A_i^\dagger, \qquad\qquad J \; = \; \sum_{i \in K} A_i^\dagger A_i. \tag{6.65}$$

Here $K$ is a finite or countable set, $H$ is a bounded self–adjoint operator, and $A_i$ are operators satisfying

$$\sum_{i \in K_0} A_i^\dagger A_i \leq 1 \qquad \forall\, K_0 \subset K.$$

The series in equation (6.65) converges in the trace norm topology. Davies [78] has proved that equation (6.64) generates a QDS even when $H$ is not bounded. The basic QMSL equation (6.8) is a particular type of QDS equation, where $J$ is the identity operator:

$$\frac{d}{dt}\,\rho(t) \quad = \quad -\frac{i}{\hbar}[H, \rho(t)] \;+\; \lambda\,\{T[\rho(t)] \;-\; \rho(t)\}\,. \tag{6.66}$$

The map $T[\,\cdot\,]$ appearing in equations (6.64) and (6.65) is a particular case of what is usually called an operation. An operation $T[\,\cdot\,]$ is, in general, a map

$$T: \;\; T(\mathcal{H}) \;\longrightarrow\; T(\mathcal{H}),$$

of the set of trace class operators into itself which is linear, positive and bounded with respect to the trace norm, with bound less or equal to one.

Non–Hamiltonian equations of type (6.64) have been proved useful for the description of many interesting physical processes. We recall here, in particular, the successful use of such equations in the description of the Wigner–Weisskopf atom and of beam foil spectroscopy (see, e.g., reference [78], section 7 and references therein). The quantum description of decay processes, and in particular the exponential nature of the decay law, have obtained an important clarification by the use of such an equation as describing the evolution of an unstable quantum system in the presence of apparatuses devised to detect the decay [81]. A very interesting investigation [30], aimed to find a solution to the problems raised by the quantum theory of measurement, has led Joos and Zeh to derive an equation of the above type starting from the Hamiltonian dynamics describing the unavoidable coupling of macroscopic systems to their environment.

# 7 Stochastic processes in Hilbert space

QMSL is the first consistent proposal to overcome the measurement problem of Quantum Mechanics in which wavefunction collapse is naturally induced by the unique dynamical principle governing the evolution of all physical systems. QMSL exhibits all the desired features one seeks in a theory of spontaneous reductions; nontheless, it has to face two problems, one "aesthetic" and one physical.

The aesthetic drawback is that the modified dynamical evolution of QMSL, though perfectly definite, is not expressed in terms of a compact mathematical equation for the statevector leading to equation (6.8) for the statistical operator. The physical problem is that the dynamics does not preserve the symmetry character of wavefunctions describing systems of identical particles. Both problems have been solved by CSL, the Continuous Spontaneous Localization model [88, 66].

In this section we review the formalism of stochastic processes in Hilbert spaces, which is the mathematical background of CSL. In subsection 7.1 we resort to Itô's formalism to derive a modified Schrödinger equation for the evolution of the statevector. This equation is linear, but it does not conserve the norm, so it needs to be supplemented by further formal prescriptions which we will analyze. In 7.2 we derive the corresponding norm–preserving equation, which is non linear.

Subsection 7.3 is devoted to the general discussion of statevector reduction: we show that the modified Schrödinger equation introduced previously leads to the spontaneous reduction of the statevector into one, among a set, of appropriate manifolds characterized by the equation itself.

In subsection 7.4 we re–derive the results of subsections 7.1–7.3, resorting to the Stratonovich in place of the Itô formalism. This is an alternative, and physically more intuitive formalism to deal with stochastic differential equations.

In the final subsection we show that the modified Schrödinger equation of CSL yields an evolution equation for the statistical operator of the quantum dynamical semigroup type, analogous to the equation of QMSL.

## 7.1 Raw and physical processes: Itô linear equation

Within the Hilbert space, let us consider the Markov process $|\psi_B(t)\rangle$ satisfying the Itô stochastic differential equation [95]:

$$d\,|\psi\rangle \;\;=\;\; [C\,dt \;+\; \mathbf{A}\cdot d\mathbf{B}]\,|\psi\rangle \tag{7.1}$$

where $C$ in an operator, $\mathbf{A} = \{A_i\}$ is a set of operators, and $\mathbf{B} = \{B_i\}$ is a set of real Wiener processes such that

$$\langle\!\langle dB_i \rangle\!\rangle \;=\; 0, \qquad\qquad \langle\!\langle dB_i\, dB_j \rangle\!\rangle \;=\; \gamma\,\delta_{ij}\,dt, \tag{7.2}$$

$\gamma$ being a real constant. The index $i$ can be continuous, in which case the sum becomes an integral and the Kronecker $\delta$ becomes a Dirac $\delta$. Given an initial state $|\psi(0)\rangle$, equation (7.1) generates at time $t$ an ensemble of statevectors $|\psi_B(t)\rangle$, where $B$ denotes a particular realization $B_i(t)$ of the Wiener processes. To simplify the notation, the dependence of $|\psi(t)\rangle$ on $t$ and $B_i$ will be often dropped, as in equation (7.1). The process (7.1) and the ensemble generated by it will be called the *raw* process and ensemble. In the raw ensemble, each statevector $|\psi_B(t)\rangle$ has the same probability as the particular realizations $B_i(t)$ that originates it through equation (7.1).

The raw process (7.1) does not conserve the norm of vectors, in general. In fact, using Itô calculus, one finds

$$\begin{aligned} d\,\||\psi(t)\rangle\|^2 \;&=\; \langle d\,\psi|\psi\rangle \;+ \langle\psi|d\,\psi\rangle \;+ \langle\!\langle \langle d\,\psi|d\,\psi\rangle \rangle\!\rangle \\ &=\; \langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle\cdot d\mathbf{B} \;+\; \langle\psi|(C+C^\dagger)|\psi\rangle\,dt \;+ \\ &\quad\; \langle\psi|\mathbf{A}^\dagger\cdot\mathbf{A}|\psi\rangle\,\gamma\,dt, \end{aligned} \tag{7.3}$$

where we have used the notation $|d\,\psi\rangle = d\,|\psi\rangle$. If the statevectors $|\psi_B(t)\rangle$ were of norm 1, the probabilities of occurrence for them, which are characteristic of the raw ensemble, could naturally be interpreted as the physical probabilities. Since this is not the case, we consider the ensemble of the normalized vectors

$$|\chi_B(t)\rangle \;\;=\;\; \frac{|\psi_B(t)\rangle}{\||\psi_B(t)\rangle\|}, \tag{7.4}$$

having the same probabilities as the corresponding vectors $|\psi_B(t)\rangle$ [i.e., as the realizations $B_i(t)$ of the Wiener processes] and the ensemble of the normalized vectors

$$|\phi_B(t)\rangle \;\;=\;\; \frac{|\psi_B(t)\rangle}{\||\psi_B(t)\rangle\|}, \tag{7.5}$$

whose probabilities are those of the vectors $|\psi_B(t)\rangle$ times their squared norms $\||\psi_B(t)\rangle\|^2$. We use different symbols for the vector functions $|\chi_B(t)\rangle$ and $|\phi_B(t)\rangle$, in spite of the fact that the right–hand sides of equations (7.4) and (7.5) coincide, because the associated probabilities are different, so that as random vector functions they are different. In fact, as we shall see and as it is obvious, they obey different stochastic differential equations. We choose as the **physical probabilities** (which we shall often call

"cooked" probabilities) those of the vectors (7.5) rather than those of the vectors (7.4). The ensemble of vectors $|\phi_B(t)\rangle$ and the stochastic process in the Hilbert space that generates it will be called the *physical* ensemble and process. The prescription leading to the physical ensemble is the counterpart of the assumption 4 of QMSL and of the postulate of standard quantum mechanics on the probabilities of the outcomes of measurement processes.

Let us now investigate the relation between the raw and the physical processes. Indicating by $P_{\rm Raw}[B_i(t,t_0)]$ the probability of the realizations $B_i(t,t_0)$ of the Wiener processes (or, equivalently, of the statevector $|\psi_B(t)\rangle$) and by $P_{\rm Cook}[B_i(t,t_0)]$ the probability of the statevector $|\phi_B(t)\rangle$, one has by definition

$$P_{\rm Cook}[B_i(t,t_0)] \quad = \quad P_{\rm Raw}[B_i(t,t_0)]\,\||\psi_B(t,t_0)\rangle\|^2. \tag{7.6}$$

It is easily shown that, because of linearity of equation (7.1) together with the Markov nature of the Wiener process $B_i$, the procedure leading from the raw to the physical ensemble can be performed just at the considered final time or, in addition, any number of times between the initial and the final times. It follows that equation (7.6) can be replaced by its specialization to the infinitesimal time interval $(t_0, t_0+dt)$, i.e.,

$$P_{\rm Cook}[dB_i] \quad = \quad P_{\rm Raw}[dB_i]\,\left[1+d\,\||\psi_B\rangle\|^2\right]. \tag{7.7}$$

The possibility of considering the physical ensemble depends on the fulfillment of the condition that the total probability associated with the distribution $P_{\rm Cook}$ is 1. This amounts to requiring that, for any $|\psi\rangle$, the average relative to the distribution $P_{\rm Raw}$ of the weighting factor $\||\psi\rangle\|^2$ is 1, i.e. $d\,\langle\!\langle\||\psi\rangle\|^2\rangle\!\rangle = \langle\!\langle d\,\||\psi\rangle\|^2\rangle\!\rangle = 0$. From equation (7.3), one finds

$$C \;+\; C^\dagger \quad = \quad -\,\gamma\mathbf{A}^\dagger\cdot\mathbf{A}. \tag{7.8}$$

When this condition is taken into account, denoting by $-(i/\hbar)H$ the anti–Hermitian part of $C$, equation (7.1) becomes:

$$d\,|\psi(t)\rangle \quad = \quad \left[-\frac{i}{\hbar}\,H\,dt \;+\; \mathbf{A}\cdot d\mathbf{B} \;-\; \frac{\gamma}{2}\mathbf{A}^\dagger\cdot\mathbf{A}\,dt\right]|\psi(t)\rangle. \tag{7.9}$$

## 7.2 Itô non linear equation

The linear Itô equation (7.9) and the cooking prescription (7.6) can be joined into a single *non linear* stochastic differential equation for the physical vectors $|\phi(t)\rangle$. Let us see how this can be accomplished.

Because of relation (7.8), equation (7.3) simplifies to

$$d\,\||\psi(t)\rangle\|^2 \quad = \quad \langle\psi(t)|(\mathbf{A}+\mathbf{A}^\dagger)|\psi(t)\rangle\cdot d\mathbf{B}. \tag{7.10}$$

Then equation (7.7) becomes

$$P_{\rm Cook}[dB_i] \quad = \quad [1 \;+\; 2\,\mathbf{R}\cdot d\mathbf{B}]\;P_{\rm Raw}[dB_i], \tag{7.11}$$

where

$$\mathbf{R} \quad = \quad \frac{1}{2}\,\langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle \tag{7.12}$$

and the probability distribution $P_{\rm Cook}$ is normalized. Indicating by $dB_i'$ the random variable whose distribution is $P_{\rm Cook}$, one has

$$\langle\!\langle dB_i'\rangle\!\rangle \;=\; 2\,\gamma\,R_i\,dt, \qquad\qquad \langle\!\langle dB_i'\,dB_j'\rangle\!\rangle \;=\; \gamma\,\delta_{ij}\,dt, \tag{7.13}$$

so that

$$d\mathbf{B}' \quad = \quad d\mathbf{B} + 2\gamma\mathbf{R}dt \tag{7.14}$$

and $B'_i$ is a diffusion process having the same diffusion as $B_i$ and drift $2R_i\gamma$. The meaning of the process $B'_i$ and of its differential $dB'_i$ follows from the one of the probability distribution $P_{\rm Cook}$ which defines them. The set of all realizations $B'_i(t)$ coincides with that of all realizations $B_i(t)$ (in fact both sets coincide with the set of all functions satisfying a given initial condition), but their probabilities, according to the definition (7.6) of $P_{\rm Cook}$, are those of the physical ensemble instead of those of the raw ensemble. The stochastic differential equation for the physical process can now easily be written. We first write down the equation for the process generating the normalized vectors $|\chi\rangle$. From equation (7.9) and (7.10), by direct evaluation, one gets:

$$\begin{aligned} d\,|\chi(t)\rangle &= \left[-\frac{i}{\hbar}H\,dt \;+\; \left(-\frac{1}{2}\gamma\mathbf{A}^\dagger\cdot\mathbf{A} \;-\; \gamma\mathbf{A}\cdot\mathbf{R} \;+\; \frac{3}{2}\gamma\mathbf{R}\cdot\mathbf{R}\right)dt \;+\right. \\ &\quad \left.(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\chi(t)\rangle, \qquad (7.15) \\ \mathbf{R} &= \frac{1}{2}\,\langle\chi|(\mathbf{A}^\dagger+\mathbf{A})|\chi\rangle. \end{aligned}$$

It is easily checked that equation (7.15) conserves the norm and that this feature does not depend on $B_i$ having drift zero. The physical process is obtained by replacing each realization $B_i(t)$ of the random function $B_i(t)$ by an equivalent realization having the appropriate different probability, i.e. an equivalent realization $B'_i(t)$ of the random function $B'_i(t)$. This amounts to replace $dB_i$ by $dB'_i$ in equation (7.15), so that we get

$$\begin{aligned} d\,|\phi(t)\rangle &= \left[-\frac{i}{\hbar}H\,dt \;+\; \left(-\frac{1}{2}\gamma\mathbf{A}^\dagger\cdot\mathbf{A} \;-\; \gamma\mathbf{A}\cdot\mathbf{R} \;+\; \frac{3}{2}\gamma\mathbf{R}\cdot\mathbf{R}\right)dt \;+\right. \\ &\quad \left.(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}'\right]|\phi(t)\rangle, \qquad (7.16) \\ \mathbf{R} &= \frac{1}{2}\,\langle\phi|(\mathbf{A}^\dagger+\mathbf{A})|\phi\rangle. \end{aligned}$$

It is convenient to rewrite the above equation in terms of the original Wiener processes $B_i(t)$. One gets the final equation:

$$\begin{aligned} d\,|\phi(t)\rangle &= \left[-\frac{i}{\hbar}H\,dt \;+\; \left(-\frac{1}{2}\gamma(\mathbf{A}^\dagger-\mathbf{R})\cdot\mathbf{A} \;+\; \frac{1}{2}\gamma(\mathbf{A}-\mathbf{R})\cdot\mathbf{R}\right)dt \;+\right. \\ &\quad \left.(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\phi(t)\rangle, \qquad (7.17) \\ \mathbf{R} &= \frac{1}{2}\,\langle\phi|(\mathbf{A}^\dagger+\mathbf{A})|\phi\rangle. \end{aligned}$$

We note that the equations for the norm conserving processes (7.15) and (7.16) or (7.17), contrary to equations (7.1) or (7.9), are nonlinear.

The case in which $A_i$ is a set of self–adjoint operators is of particular interest. In this case equation (7.17) becomes[27]

$$\begin{aligned} d\,|\phi(t)\rangle &= \left[-\frac{i}{\hbar}H\,dt \;-\; \frac{1}{2}\gamma(\mathbf{A}-\mathbf{R})^2dt \;+\; (\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\phi(t)\rangle \\ \mathbf{R} &= \langle\phi|\mathbf{A}|\phi\rangle. \qquad (7.18) \end{aligned}$$

[27] Stochastic equations having a formal structure of the type (7.17) have been considered in previous works [96, 97], but there the random terms appearing at the right–hand side had a specific form devised to describe a specific measurement that was supposed to be performed. The considered equations, therefore, did not have the universal character of the CSL equations. Other investigations [75, 98, 99] deal with dynamical reduction models similar to the one considered in reference [88] and here. In reference [98] an equation very close to equation (7.17) is introduced (without deriving it from a linear process), but it is not specialized to the use of densities around space points to discriminate among different configurations. The idea of using densities is considered in reference [99], where, however, the dynamical equation has a more complicated structure than CSL.

The analysis of this and of the previous subsection have shown that one can take two different attitudes to describe the diffusion process: either one solves equation (7.9), taking as physical vectors the normalized ones and taking $P_{\rm Cook}$ as the physical probability distribution; or one considers equation (7.17), without the need to normalize vectors and without the cooking prescription. At the non relativistic level, these two attitudes are equivalent. However, relativistic considerations we will discuss in section 14.3 will indicate that the first attitude — based on the linear equation + the cooking prescription — is more suited to describe the physics of the stochastic process.

## 7.3 Reduction of the statevector

We shall now show that, when $\{A_i\}$ is a set of commuting self–adjoint operators, the new terms in equation (7.18) induce, for large times, the reduction of the statevector on the common eigenspaces of the operators $A_i$ [66].

Since here we are interested in discussing the physical effects of the new terms, we disregard for the moment the Schrödinger part of the dynamical equation. Then equation (7.18) becomes simply

$$\begin{aligned} d\,|\phi(t)\rangle &= \left[-\frac{1}{2}\gamma(\mathbf{A}\,-\,\mathbf{R})^2\,dt\,+\,(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\right]|\phi(t)\rangle \qquad (7.19)\\ \mathbf{R} &= \langle\phi|\mathbf{A}|\phi\rangle. \end{aligned}$$

Let us write

$$\mathbf{A} \;=\; \sum_\sigma \mathbf{a}_\sigma\,P_\sigma, \qquad (7.20)$$

where the orthogonal projection operators $P_\sigma$ sum up to the identity and it is understood that $\sigma\neq\tau\Rightarrow\mathbf{a}_\sigma\neq\mathbf{a}_\tau$ (i.e. $a_{i\sigma}\neq a_{i\tau}$ for at least one value of $i$). We consider the real non–negative variables

$$\langle\phi|P_\sigma|\phi\rangle \;=\; z_\sigma, \qquad (7.21)$$

having the property

$$\sum_\sigma z_\sigma \;=\; 1. \qquad (7.22)$$

In terms of such variables, one finds:

$$\begin{aligned} \mathbf{R} &= \sum_\sigma \mathbf{a}_\sigma\,z_\sigma, \qquad (7.23)\\ (\mathbf{A}-\mathbf{R})|\phi\rangle &= \sum_\sigma\sum_\tau z_\tau\,(\mathbf{a}_\sigma-\mathbf{a}_\tau)\,P_\sigma\,|\phi\rangle, \qquad (7.24)\\ (\mathbf{A}-\mathbf{R})^2|\phi\rangle &= \sum_\sigma\left[\sum_\tau z_\tau\,(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right]^2 P_\sigma\,|\phi\rangle. \qquad (7.25) \end{aligned}$$

It follows that the stochastic differential equation (7.19) can be written

$$\begin{aligned} dP_\sigma|\phi(t)\rangle &= \left[-\frac{\gamma}{2}\left(\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right)^2 dt+\right.\\ &\qquad\left.\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\cdot d\mathbf{B}\right]P_\sigma|\phi(t)\rangle. \qquad (7.26) \end{aligned}$$

Using this equation in the relation:

$$\begin{aligned} d\,\langle\phi|P_\sigma|\phi\rangle \;\; &= \;\; [d\,\langle\phi|P_\sigma]\,P_\sigma|\phi\rangle \,+ \\ &\quad \langle\phi|P_\sigma\,[d\,P_\sigma|\phi\rangle] \;+\; \langle\!\langle[\langle\phi|d\,P_\sigma]\,[d\,P_\sigma|\phi\rangle]\rangle\!\rangle, \end{aligned}$$

gives for the variables $z_\sigma$ the set of stochastic differential equations

$$d\,z_\sigma \;\; = \;\; 2\,z_\sigma \sum_\tau z_\tau\,(\mathbf{a}_\sigma \,-\, \mathbf{a}_\tau)\cdot d\mathbf{B}. \tag{7.27}$$

Qualitatively, equations (7.27) shows that the diffusion of the $\{z_\sigma\}$ vanishes when they approach the solution of the set of equations:

$$z_\sigma \sum_\tau z_\tau\,(\mathbf{a}_\sigma \,-\, \mathbf{a}_\tau) \;\; = \;\; 0, \tag{7.28}$$

so that the values of $\{z_\sigma\}$ eventually accumulate towards such solutions. A formal proof of the fact that $\{z_\sigma\}$ asymptotically reduce to one of the solutions of equation (7.28) is easily obtained. From equation (7.27) one finds:

$$d\,z_\sigma^2 \;=\; 2\,z_\sigma\,dz_\sigma \;+\; \left[2\,z_\sigma \sum_\sigma z_\tau\,(\mathbf{a}_\sigma \,-\, \mathbf{a}_\tau)\right]^2 \gamma\,dt \tag{7.29}$$

and in turn

$$d\,\langle\!\langle z_\sigma^2\rangle\!\rangle \;=\; \langle\!\langle d\,z_\sigma^2\rangle\!\rangle \;=\; \gamma\left[2\,z_\sigma \sum_\tau z_\tau\,(\mathbf{a}_\sigma \,-\, \mathbf{a}_\tau)\right]^2 dt. \tag{7.30}$$

It follows that

$$\frac{d}{dt}\,\langle\!\langle z_\sigma^2(t)\rangle\!\rangle \;\geq\; 0. \tag{7.31}$$

This result, together with the boundedness property

$$\langle\!\langle z_\sigma^2(t)\rangle\!\rangle \;\leq\; 1, \tag{7.32}$$

entails that for $t \to \infty$

$$\frac{d}{dt}\,\langle\!\langle z_\sigma^2(t)\rangle\!\rangle \;\longrightarrow\; 0. \tag{7.33}$$

Using again equation (7.30), we get

$$z_\sigma \sum_\tau z_\tau\,(\mathbf{a}_\sigma \,-\, \mathbf{a}_\tau) \;\; \longrightarrow \;\; 0. \tag{7.34}$$

In reference [66] it is shown that the only solutions to the set of equations (7.28) are of the form

$$z_1 = 0 \qquad z_2 = 0 \quad \ldots \quad z_\sigma = 1 \quad \ldots,$$

corresponding to $|\phi\rangle$ lying in one of the common eigenspaces of the operators $A_i$. Since equations (7.26) do not change the Hilbert space ray to which each component $P_\sigma|\phi(t)\rangle$ belongs, we conclude that $|\phi(t)\rangle$ asymptotically reduces to one of its initial components $P_\sigma|\phi(0)\rangle$ times a normalization factor.

The probabilities for the various possible issues are also easily calculated. In fact, since $d\,\langle\!\langle z_\sigma\rangle\!\rangle = \langle\!\langle d\,z_\sigma\rangle\!\rangle = 0$, one has

$$\langle\!\langle z_\sigma\rangle\!\rangle = z_\sigma(0). \tag{7.35}$$

On the other hand,

$$\langle\!\langle z_\sigma\rangle\!\rangle \;\longrightarrow\; \mathrm{Prob}[z_\sigma(\infty) = 1], \tag{7.36}$$

so that one finds

$$\mathrm{Prob}[z_\sigma(\infty) = 1] \;=\; z_\sigma(0), \tag{7.37}$$

i.e.,

$$\mathrm{Prob}[|\phi(\infty)\rangle \propto P_\sigma|\phi(0)\rangle] \quad = \quad \langle\phi(0)|P_\sigma|\phi(0)\rangle. \tag{7.38}$$

As one can see, this result is a direct consequence of the martingale property $\langle\!\langle d\, z_\sigma \rangle\!\rangle = 0$ [100, 101, 88]. Physically, the above equation plays a fundamental role since it guarantees that the dynamical reduction models reproduce the quantum predictions about measurement outcomes.

## 7.4 Linear and non linear equations: the Stratonovich formalism

The analysis of the previous subsections was based on the Itô formalism for stochastic differential equations. It is not difficult to re–write all the above equations, resorting to the Stratonovich formalism [95], which is easier to handle. The linear equation, corresponding to (7.9), is:

$$\frac{d}{dt}\,|\psi(t)\rangle \quad = \quad \left[-\frac{i}{\hbar}\,H \;+\; \mathbf{A}\cdot\mathbf{V}(t) \;-\; \frac{\gamma}{2}\left(\mathbf{A}^\dagger\cdot\mathbf{A} \;+\; \mathbf{A}^2\right)\right]|\psi(t)\rangle, \tag{7.39}$$

where $V_i(t)$ are $c$–number stochastic processes with probability of occurrence given by:

$$P_{\mathrm{Cook}}[V_i(t)] \quad = \quad P_{\mathrm{Raw}}[V_i(t)]\,\||\psi(t)\rangle\|^2. \tag{7.40}$$

$P_{\mathrm{Raw}}[V_i(t)]$ is the probability distribution of gaussian white noises satisfying:

$$\langle\!\langle V_i(t)\rangle\!\rangle \;=\; 0, \qquad\qquad \langle\!\langle V_i(t_1)\,V_j(t_2)\rangle\!\rangle \;=\; \gamma\,\delta_{ij}\,\delta(t_1 - t_2). \tag{7.41}$$

In the particular but very important case in which the operators $A_i$ are self–adjoint, (7.39) reduces to:

$$\frac{d}{dt}\,|\psi(t)\rangle \quad = \quad \left[-\frac{i}{\hbar}\,H \;+\; \mathbf{A}\cdot\mathbf{V}(t) \;-\; \gamma\mathbf{A}^2\right]|\psi(t)\rangle. \tag{7.42}$$

The physical meaning of equations (7.39) is the same of the corresponding Itô equation: if a homogeneous ensemble (pure case) at the initial time $t_0$ is associated with the statevector $|\psi(t_0)\rangle$, then the ensemble at a subsequent time $t$ is the union of homogeneous ensembles associated with the normalized vectors $|\psi(t)\rangle/\||\psi(t)\rangle\|$, where $|\psi(t)\rangle$ is the solution of equation (7.39) with the assigned initial conditions and for a specific stochastic process $V(t) = \{V_i(t)\}$ which has occurred in the interval $(0, t)$. The probability associated to any such homogeneous ensemble is given by (7.40).

It is not difficult to write down the non linear equation for the *physical* vectors $|\phi(t)\rangle = |\psi(t)\rangle/\||\psi(t)\rangle\|$; here we limit ourselves only to the case in which $A_i$ are self–adjoint:

$$\begin{aligned}
\frac{d}{dt}\,|\phi(t)\rangle \quad &= \quad \Big[-\frac{i}{\hbar}\,H \;+\; (\mathbf{A} - \mathbf{R})\cdot\mathbf{V}(t) \;-\; \gamma\,(\mathbf{A} - \mathbf{R})^2 \\
&+ \quad \gamma\left(\mathbf{Q}^2 - \mathbf{R}^2\right)\Big]\,|\phi(t)\rangle, \\
\mathbf{R} \quad &= \quad \langle\phi|\mathbf{A}|\phi\rangle \qquad \mathbf{Q}^2 \;=\; \langle\phi|\mathbf{A}^2|\phi\rangle.
\end{aligned} \tag{7.43}$$

It is instructive to reconsider the reduction mechanism discussed in the previous subsection, using the Stratonovich equation (7.39); for simplicity we will limit ourselves to study the case of a single self–adjoint operator $A$, so that only one stochastic field $V(t)$ appears. Suppose the initial statevector $|\psi(0)\rangle$ (for simplicity

we take $t_0 = 0$) has non vanishing projections on two distinct eigenmanifolds of $A$, corresponding to the eigenvalues $\alpha$ and $\beta$ respectively:

$$|\psi(0)\rangle \quad = \quad P_\alpha\,|\psi(0)\rangle \;+\; P_\beta\,|\psi(0)\rangle. \tag{7.44}$$

When the hamiltonian is disregarded, the solution of equation (7.39) is[28]:

$$|\psi_B(t)\rangle \quad = \quad e^{\,\alpha B(t) - \alpha^2\gamma\,t} P_\alpha\,|\psi(0)\rangle \;+\; e^{\,\beta B(t) - \beta^2\gamma\,t} P_\beta\,|\psi(0)\rangle. \tag{7.45}$$

Here $B(t)$ is the Brownian process:

$$B(t) \quad = \quad \int_0^t d\tau\, V(\tau). \tag{7.46}$$

Taking into account equation (7.45) and the cooking prescription, one gets the cooked probability density for the value $B(t)$ of the Brownian process at time $t$:

$$\begin{aligned} P_{\rm Cook}[w(t)] \quad = \quad & \|P_\alpha\,|\psi(0)\rangle\|^2 \frac{1}{\sqrt{2\pi\gamma t}}\; e^{\;-\dfrac{1}{2\gamma t}\,[B(t) - 2\gamma\alpha t]^2} \quad + \\ & \|P_\beta\,|\psi(0)\rangle\|^2 \frac{1}{\sqrt{2\pi\gamma t}}\; e^{\;-\dfrac{1}{2\gamma t}\,[B(t) - 2\gamma\beta t]^2}. \end{aligned} \tag{7.47}$$

¿From the above equation it is clear that, for $t \to \infty$, the stochastic process $B(t)$ can assume only values belonging to an interval of width $\sqrt{\gamma t}$ around[29] either the value $2\gamma\alpha t$ or the value $2\gamma\beta t$. The corresponding probabilities are $\|P_\alpha\,|\psi(0)\rangle\|^2$ and $\|P_\beta\,|\psi(0)\rangle\|^2$ respectively. The occurrence of a value "near" to $2\alpha\gamma t$ for the random variable $B(t)$ leads, according to (7.45), to a statevector which for $t \to \infty$ is driven into the eigenmanifold corresponding to the eigenvalue $\alpha$ of $A$. In fact, in such a case one gets:

$$\frac{\|P_\beta\,|\psi_B(t)\rangle\|^2}{\|P_\alpha\,|\psi_B(t)\rangle\|^2} \quad \simeq \quad e^{-2\gamma(\alpha\,-\,\beta)^2 t}\; \frac{\|P_\beta\,|\psi(0)\rangle\|^2}{\|P_\alpha\,|\psi(0)\rangle\|^2} \quad \stackrel{t\to\infty}{\longrightarrow} \quad 0. \tag{7.48}$$

Analogously, when the random variable $B(t)$ takes a value "near" $2\beta\gamma t$, for $t \to \infty$ the statevector is driven into the eigenmanifold corresponding to the eigenvalue $\beta$ of $A$.

It is then clear that the model establishes a one–to–one correspondence between the "outcome" (the final "preferred" eigenmanifold into which an individual statevector is driven) and the specific value (among the only ones having an appreciable probability) taken by $B(t)$ for $t \to \infty$, a correspondence irrespective of what $|\psi(0)\rangle$ is[30]. In the general case of several operators $A_i$, a similar conclusion holds for the "outcomes" $\alpha_i$ of $A_i$ and the corresponding noises $B_i(t)$.

## 7.5 The statistical operator

The statistical operator corresponding to the physical ensemble and its evolution equation are easily obtained from the definition:

$$\rho \;=\; \int \mathcal{D}[B_i(t)]\; \frac{|\psi\rangle}{\||\psi\rangle\|}\; \frac{\langle\psi|}{\|\langle\psi|\|}\; P_{\rm Raw}[B_i(t)]\||\psi\rangle\|^2 \;=\; \langle\!\langle |\psi\rangle\langle\psi| \rangle\!\rangle_{P_{\rm Raw}} \tag{7.49}$$

[28] In equation (7.45) and following the statevector is labeled by the Brownian motion symbol $B$, to stress the fact that, under our assumptions, the state at time $t$ does not depend on the specific sample function $V(t)$ in the interval $(0, t)$ but only on its integral given by equation (7.46).

[29] Note that even though the spread $\sqrt{\gamma t}$ tends to $\infty$ for $t \to \infty$, its ratio to the distance $2(\alpha - \beta)\gamma t$ between the two considered peaks of the distribution tends to zero.

[30] Obviously, $|\psi(0)\rangle$ enters in a crucial way in determining the probability of occurrence of the Brownian processes $B(t)$.

and equation (7.9), or from

$$\rho \;=\; \int \mathcal{D}[B_i(t)]\,|\phi\rangle\langle\phi|\,P_{\rm Cook}[B_i(t)] \tag{7.50}$$

and equation (7.17). $\int \mathcal{D}[B_i(t)]$ is the functional integral with respect to all the possible realizations of the stochastic processes $B_i(t)$.

Using once more Itô calculus in evaluating $d\rho$, one gets[31]:

$$\frac{d}{dt}\,\rho(t) \quad = \quad -\frac{i}{\hbar}\,[H,\rho(t)] \;+\; \gamma\mathbf{A}\rho(t)\cdot\mathbf{A}^\dagger \;-\; \frac{\gamma}{2}\left\{\mathbf{A}^\dagger\cdot\mathbf{A},\rho(t)\right\}. \tag{7.51}$$

where $\{\,\cdot\,,\cdot\,\}$ denotes the the anticommutator. This is the Lindblad form for the generator of a quantum dynamical semigroup, as already discussed. It is remarkable that the general Lindblad generator can be obtained from a stochastic process in Hilbert space. Note that the way we have followed to get equation (7.51) describing an ensemble associated to the statistical operator $\rho(t)$ makes clear that each member of the ensemble has a definite statevector at any time, a statevector which, eventually, ends up in one of the eigenmanifolds of the preferred basis.

# 8 Continuous Spontaneous Localizations (CSL)

All the necessary mathematical tools to work with stochastic differential equations in Hilbert space have been developed; we can now apply this formalism to work out a model of dynamical reductions which has all the desired features of QMSL, but, at the same time, overcomes the difficulties we have mentioned at the beginning of the previous section.

It should be clear from the above analysis that what we have to do is to choose the "preferred basis", i.e. the operators $A_i$ whose common eigenmanifolds are the manifolds in which the statevector is driven by the diffusion process. These operators have to be chosen is such a way that:

1. Macroscopic objects are always localized in *space*.
2. Microscopic dynamics is not altered in an appreciable way with respect to the standard quantum evolution.
3. In particular, the energy increase of the system — due to space localizations — must not be detectable.
4. The symmetry properties of systems containing identical particles must be preserved.

We now see how all these requirements are met by CSL. In the first subsection we set out the preferred basis: the operators $A_i$ will be chosen to be appropriate functions of the creation and annihilation operators of particles in space. In subsection 8.2 we discuss the implications of such a choice in the case of macroscopic systems, showing how their classical properties stem from the quantum properties of their microscopic constituents; this subsection parallels the analysis of subsection 6.4, which was performed only at the statistical operator level, not at the wavefunction level like in the present section.

In subsection 8.3 we determine the reduction rates induced by CSL, proving that, with an appropriate choices of the parameters, they are compatible with those of

[31] Of course, the same equation is obtained starting from the Stratonovich equations (7.39) or (7.43), which correspond to the Itô equation (7.9) and (7.17), respectively.

QMSL. In subsection 8.4 we discuss how the average value of physical observables are affected by the new non–Hamiltonian terms, showing once more that there are no appreciable differences with respect to QMSL. In subsection 8.5 we put forward a simple, pedagogical, CSL model, which is illuminating in order to understand how the reduction process amplifies when moving from the micro to the macro level.

In the last two subsections we discuss two new CSL models; in the first one the collapse mechanism is related to the mass density distribution of the object, rather than to the density–number operator, while in the second one the reduction mechanism is related to gravity.

## 8.1 The choice of the "preferred basis"

Let us consider the creation and annihilation operators $a^\dagger(\mathbf{y}, s)$ and $a(\mathbf{y}, s)$ of a particle at point $\mathbf{x}$ with spin component $s$ satisfying canonical commutation or anticommutation relations. We define a locally averaged density operator

$$N(\mathbf{x}) \quad = \quad \sum_s \int d^3y\, g(\mathbf{y}-\mathbf{x})\, a^\dagger(\mathbf{y}, s) a(\mathbf{y}, s), \tag{8.1}$$

where $g(\mathbf{x})$ is a spherically symmetric, positive real function peaked around $\mathbf{x} = 0$, normalized in such a way that:

$$\int d^3x\, g(\mathbf{x}) \quad = \quad 1,$$

so that

$$\int d^3x\, N(\mathbf{x}) \quad = \quad N,$$

$N$ being the total number operator. The operators $N(\mathbf{x})$ are self–adjoint and commute with each other. In what follows we choose

$$g(\mathbf{x}) \quad = \quad \left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}(\mathbf{x})^2}, \tag{8.2}$$

where $\alpha$ is a parameter such that $\alpha^{-3/2}$ represents essentially the volume over which the average is taken in the definition of $N(\mathbf{x})$. The improper vectors

$$|q, s\rangle \quad = \quad \mathcal{N}\, a^\dagger(\mathbf{q}_1, s_1) a^\dagger(\mathbf{q}_2, s_2) \ldots a^\dagger(\mathbf{q}_n, s_n)|0\rangle \tag{8.3}$$

are the normalized common eigenstates of the operators $N(\mathbf{x})$ belonging to the eigenvalues

$$n(\mathbf{x}) \quad = \quad \sum_{i=1}^{n} g(\mathbf{q}_i - \mathbf{x}).$$

We identify now, with reference to the previous section, the index $i$ which labels the operators $A_i$ with the space point $\mathbf{x}$ and the operators $A_i$ with the density operators $N(\mathbf{x})$. **Itô equation** (7.9) then becomes:

$$d|\psi(t)\rangle = \left[-\frac{i}{\hbar} H\, dt \; + \; \int d^3x\, N(\mathbf{x}) dB(\mathbf{x}) - \frac{\gamma}{2} \int d^3x\, N^2(\mathbf{x}) dt\right] |\psi(t)\rangle, \tag{8.4}$$

where:

$$\langle\!\langle dB(\mathbf{x}) \rangle\!\rangle \; = \; 0 \qquad\qquad \langle\!\langle dB(\mathbf{x})\, dB(\mathbf{y}) \rangle\!\rangle \; = \; \gamma\, \delta^3(\mathbf{x}-\mathbf{y}) dt. \tag{8.5}$$

This is, in a different notation, the process considered in references [88, 66] for identical particles. The generalization to several kinds of particles is immediate. For completeness, we write down also the corresponding **Stratonovich equation** (7.39):

$$\frac{d}{dt}|\psi(t)\rangle = \left[-\frac{i}{\hbar} H \; + \; \int d^3x\, N(\mathbf{x}) V(\mathbf{x}, t) \; - \; \gamma \int d^3x\, N^2(\mathbf{x}) dt\right] |\psi(t)\rangle; \tag{8.6}$$

the first two moments of the white noise $V(\mathbf{x},t)$ are:

$$\langle\!\langle V(\mathbf{x},t)\rangle\!\rangle \;=\; 0 \qquad\qquad \langle\!\langle V(\mathbf{x},t_1)\,V(\mathbf{y},t_2)\rangle\!\rangle \;=\; \gamma\,\delta^3(\mathbf{x}-\mathbf{y})\,\delta(t_1-t_2). \tag{8.7}$$

Note that equation (8.6) can be rewritten in the following way:

$$\begin{aligned}\frac{d|\psi(t)\rangle}{dt} \;&=\; \Big[-\frac{i}{\hbar}H \;+\; \int d^3x\,\mathcal{N}(\mathbf{x})V'(\mathbf{x},t) \\ &\quad -\;\gamma\int d^3x d^3y\,\mathcal{N}(\mathbf{x})D(\mathbf{x}-\mathbf{y})\mathcal{N}(\mathbf{y})\Big]\,|\psi(t)\rangle,\end{aligned} \tag{8.8}$$

where:

$$\mathcal{N}(\mathbf{x}) \;=\; \sum_s a^\dagger(\mathbf{x},s)\,a(\mathbf{x},s) \tag{8.9}$$

is the number–density operator, and $V'(\mathbf{x},t)$ is a new Gaussian stochastic process defined as:

$$V'(\mathbf{x},t) \;=\; \Big(\frac{\alpha}{2\pi}\Big)^{\frac{3}{2}}\int d^3y\,e^{-\frac{\alpha}{2}(\mathbf{x}-\mathbf{y})^2}V(\mathbf{y},t). \tag{8.10}$$

It is easy to check that its average value is zero, while the correlation function is:

$$\begin{aligned}\langle\!\langle V'(\mathbf{x},t_1)\,V'(\mathbf{y},t_2)\rangle\!\rangle \;&=\; \gamma\,D(\mathbf{x}-\mathbf{y})\,\delta(t_1-t_2) \;= \\ &=\; \gamma\,\Big(\frac{\alpha}{4\pi}\Big)^{\frac{3}{2}}e^{-\frac{\alpha}{4}(\mathbf{x}-\mathbf{y})^2}\delta(t_1-t_2).\end{aligned} \tag{8.11}$$

The equation for the **statistical operator** (7.51) reads:

$$\frac{d}{dt}\,\rho(t) \;=\; -\frac{i}{\hbar}\,[H,\rho(t)] \;+\; \gamma\int d^3x\,N(\mathbf{x})\rho(t)N(\mathbf{x}) \;-\; \frac{\gamma}{2}\int d^3x\,\{N^2(\mathbf{x}),\rho(t)\}. \tag{8.12}$$

In the representation given by the improper vectors (8.3), equation (8.12) becomes:

$$\begin{aligned}\frac{\partial}{\partial t}\langle q',s'|\rho(t)|q'',s''\rangle \;&=\; -\frac{i}{\hbar}\langle q',s'|\,[H,\rho(t)]\,|q'',s''\rangle + \frac{\gamma}{2}\sum_{ij}\big[2\,G(\mathbf{q}_i'-\mathbf{q}_j'')- \\ &\quad G(\mathbf{q}_i'-\mathbf{q}_j')-G(\mathbf{q}_i''-\mathbf{q}_j'')\big]\,\langle q',s'|\rho(t)|q'',s''\rangle,\end{aligned} \tag{8.13}$$

where

$$\begin{aligned}G(\mathbf{y}'-\mathbf{y}'') \;&=\; \int d^3x\,g(\mathbf{y}'-\mathbf{x})\,g(\mathbf{y}''-\mathbf{x}) \\ &=\; \Big(\frac{\alpha}{4\pi}\Big)^{3/2}\;e^{-\frac{\alpha}{4}\,(\mathbf{y}'-\mathbf{y}'')^2}.\end{aligned} \tag{8.14}$$

For a single particle, equation (8.13) reduces to:

$$\begin{aligned}\frac{\partial}{\partial t}\,\langle\mathbf{q}'|\rho(t)|\mathbf{q}''\rangle \;&=\; -\frac{i}{\hbar}\,\langle\mathbf{q}'|[H,\rho(t)]|\mathbf{q}''\rangle \;- \\ &\quad \gamma\,\Big(\frac{\alpha}{4\pi}\Big)^{\frac{3}{2}}\left[1-e^{-\frac{\alpha}{4}\,(\mathbf{q}'-\mathbf{q}'')^2}\right]\langle\mathbf{q}'|\rho(t)|\mathbf{q}''\rangle.\end{aligned} \tag{8.15}$$

We note that, taking

$$\lambda \;=\; \gamma\,\Big(\frac{\alpha}{4\pi}\Big)^{3/2}, \tag{8.16}$$

equation (8.15) coincides with the QMSL equation (6.12).

## 8.2 Dynamical reductions for macroscopic rigid bodies

We now discuss the physical implications of the modified dynamical equation (8.4) for a macroscopic system, under the assumption that the order of magnitude of the length parameter $1/\sqrt{\alpha}$ is such that it can reasonably be admitted that the wavefunction of the internal variables of a macroscopic body is sharply localized with respect to $1/\sqrt{\alpha}$: the conclusions we will reach will be analogous to those derived in section 6.4, when macroscopic objects where analyzed within QMSL.

Let $\mathbf{Q}$ be the center of mass coordinate of the system of identical particles which constitutes the considered macroscopic body,

$$\mathbf{Q} \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\mathbf{q}_i,$$

and write

$$\mathbf{q}_i \quad = \quad \mathbf{Q} + \tilde{\mathbf{q}}_i.$$

The coordinates $\tilde{\mathbf{q}}_i$ with respect to the center of mass sum up to zero, so that they are functions of $3N-3$ independent internal variables[32], which we indicate by $r$. The internal variables $r$, together with the center of mass coordinates $\mathbf{Q}$, are functions of the coordinates $\mathbf{q}_i$. So, we consider the wavefunction

$$\psi(q,s) \;=\; \Psi(\mathbf{Q})\,\chi(r,s), \qquad\qquad \chi(r,s) \;=\; \left[\begin{array}{c} \mathrm{S} \\ \mathrm{A} \end{array}\right]\Delta(r,s), \tag{8.17}$$

where "S" and "A" mean symmetrization or antisymmetrization with respect to interchanges of the arguments $(\mathbf{q}_i, s_i)$. The wavefunctions $\Psi$ and $\chi$ are understood to be separately normalized. The function $\Delta(r,s)$ is assumed to be sharply (with respect to $1/\sqrt{\alpha}$) peaked around the value $r_0$ of $r$.

The action of the operator $N(\mathbf{x})$ on the wavefunction (8.17) is easily worked out. One finds:

$$N(\mathbf{x})\,\Psi(\mathbf{Q})\,\chi(r,s) \;=\; \Psi(\mathbf{Q})\left[\begin{array}{c} \mathrm{S} \\ \mathrm{A} \end{array}\right]\sum_i\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}[\mathbf{Q}+\tilde{\mathbf{q}}_i(r)-\mathbf{x}]^2}\,\Delta(r,s). \tag{8.18}$$

According to our assumptions, the factor in front of the function $\Delta$ varies much more slowly than $\Delta$ itself, so that we can take $r = r_0$ in the factor. In other words, we treat the factor as if $\Delta(r,s)$ were of the form $\delta^{3n-3}(r-r_0)\xi(s)$. Then:

$$N(\mathbf{x})\,\Psi(\mathbf{Q})\,\chi(r,s) \quad = \quad F(\mathbf{Q}-\mathbf{x})\,\Psi(\mathbf{Q})\,\chi(r,s), \tag{8.19}$$

where:

$$F(\mathbf{Q}-\mathbf{x}) \;=\; \sum_i\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}[\mathbf{Q}+\tilde{\mathbf{q}}_i(r_0)-\mathbf{x}]^2}. \tag{8.20}$$

According to equation (8.19) the operator $N(\mathbf{x})$ acts only on the factor $\Psi$ of $\psi$. As a consequence, under the assumption that:

$$H \quad = \quad H_Q \;+\; H_r,$$

[32]The internal variables, as defined here, describe also rotations of the $N$–particle system. We assume that the the wavefunction is such that the orientation of the system (and consequently of its internal structure) is sharply defined. In the general case, one could consider 3 orientation variables, to be treated along the same lines as the center of mass coordinate, and $3N-6$ truly internal variables, to be assumed sharply localized in the wave function. However, in this case the problem would be considerably more complicated without gaining very much as regards to physical insight.

if $\Psi$ and $\chi$ satisfy the equations

$$d|\Psi\rangle \;=\; \left[-\frac{i}{\hbar}H_Q\,dt \;+\; \int d^3x\,F(\mathbf{Q}-\mathbf{x})\,dB(\mathbf{x}) \;-\; \frac{\gamma}{2}\int d^3x\,F^2(\mathbf{Q}-\mathbf{x})\,dt\right]|\Psi\rangle \tag{8.21}$$

$$d|\chi\rangle \;=\; \left[-\frac{i}{\hbar}\,H_r\,dt\right]|\chi\rangle \tag{8.22}$$

respectively, the wavefunction (8.17) satisfies equation (8.4). We can conclude that, under our assumptions, the center of mass and the internal motions decouple as in the absence of the stochastic terms in equation (8.4). Furthermore, the stochastic terms do not affect the internal structure, while the center of mass wavefunction obeys a stochastic differential equation, again of the type (7.9), whose consequences will be discussed below.

Note that equations (8.21) and (8.22) are exactly the counterpart of equations (6.48) and (6.53) of QMSL, respectively. This proves that the separation of the center of mass and internal motion takes place also at the wavefunction level, not only at the statistical operator level, as seen in section 6.

## 8.3 Reduction rates

The operators $F(\mathbf{Q}-\mathbf{x})$ appearing in equation (8.21), which correspond to the operators $A_i$ of equation (7.9), are real functions of the center of mass position operator $\mathbf{Q}$. They are a set of commuting self–adjoint operators, so that, as we know from the results of section 7, the non–Schrödinger terms in equation (8.21) induce the reduction of the statevector on the eigenvectors of the position $\mathbf{Q}$. Of course, such a process requires an infinitely long time, while, in finite times, only the reduction on approximate eigenstates of $\mathbf{Q}$ takes place[33]. We discuss here the time rate of the localization process by studying the time dependence of the off–diagonal elements of the statistical matrix $\langle\mathbf{Q}'|\rho|\mathbf{Q}''\rangle$. Again, we disregard the effect of the Schrödinger term, this approximation being justified by the fact that, for the values of $|\mathbf{Q}'-\mathbf{Q}''|$ in which we are interested, the reduction process will turn out to be very fast.

Equation (7.51) becomes in the present case:

$$\frac{\partial}{\partial t}\langle\mathbf{Q}'|\rho|\mathbf{Q}''\rangle \;=\; -\Gamma(\mathbf{Q}',\mathbf{Q}'')\,\langle\mathbf{Q}'|\rho|\mathbf{Q}''\rangle, \tag{8.23}$$

where:

$$\Gamma(\mathbf{Q}',\mathbf{Q}'') \;=\; \gamma\int d^3x\left[\frac{1}{2}F^2(\mathbf{Q}'-\mathbf{x})+\frac{1}{2}F^2(\mathbf{Q}''-\mathbf{x})-F(\mathbf{Q}'-\mathbf{x})F(\mathbf{Q}''-\mathbf{x})\right]. \tag{8.24}$$

Equation (8.23) gives:

$$\langle\mathbf{Q}'|\rho(t)|\mathbf{Q}''\rangle \;=\; e^{-\Gamma t}\,\langle\mathbf{Q}'|\rho(0)|\mathbf{Q}''\rangle. \tag{8.25}$$

It is easily found that $\Gamma$ is an even function of $\mathbf{Q}'-\mathbf{Q}''$. Since it is assumed that very many constituents of the considered body are contained in a volume $\alpha^{-3/2}$, we can use the macroscopic density approximation, consisting in replacing the sum by an integral in equation (8.20). Then one writes:

$$F(\mathbf{Q}-\mathbf{x}) \;=\; \int d^3\tilde{y}\,D(\tilde{\mathbf{y}})\left(\frac{\alpha}{2\pi}\right)^{3/2} e^{-\frac{\alpha}{2}\,(\mathbf{Q}+\tilde{\mathbf{y}}-\mathbf{x})^2}, \tag{8.26}$$

[33] Of course, for very large times the Hamiltonian $H$ cannot be any more ignored: a sort of balance between the Hamiltonian spreading of the wavefunction and the reduction mechanism is established, which keeps constant the spread of the wavefunction.

where $D(\tilde{\mathbf{y}})$ is the number of particles per unit volume in the neighborhood of the point $\mathbf{y} = \mathbf{Q} + \tilde{\mathbf{y}}$.

A further approximation, which we call the sharp scanning approximation, can be used, since we are not interested here in the details of the function $\Gamma$ for $\mathbf{Q}' - \mathbf{Q}'' \to 0$. The sharp scanning approximation consists in replacing the normalized Gaussian function appearing in equation (8.26) by the corresponding delta function. Then one has:

$$F(\mathbf{Q} - \mathbf{x}) \quad = \quad D(\mathbf{x} - \mathbf{Q}), \tag{8.27}$$

so that one gets:

$$\Gamma(\mathbf{Q}' - \mathbf{Q}'') \; = \; \gamma \int d^3x \, \left[ D^2(\mathbf{x}) - D(\mathbf{x}) D(\mathbf{x} + \mathbf{Q}' - \mathbf{Q}'') \right], \tag{8.28}$$

where suitable changes of the integration variable have also been made. The physical meaning of $\Gamma$ is easily understood by making reference to a homogeneous macroscopic body of density $D_0$. Then:

$$\Gamma \quad = \quad \gamma \, D_0 \, n_{\text{out}}, \tag{8.29}$$

$n_{\text{out}}$ being the number of particles of the body when the center of mass position is $\mathbf{Q}'$, which do not lie in the volume occupied by the body when the center of mass position is $\mathbf{Q}''$. The ratio between the macroscopic rate (8.29) and the microscopic rate (8.16) is $n_{\text{out}} \, D_0 (4\pi/\alpha)^{3/2}$.

The results (8.25) and (8.29) have to be compared with the result:

$$\langle \mathbf{Q}' | \rho(t) | \mathbf{Q}'' \rangle \quad = \quad e^{-\lambda_{\text{macro}} t} \, \langle \mathbf{Q}' | \rho(0) | \mathbf{Q}'' \rangle, \tag{8.30}$$

$$\lambda_{\text{macro}} \quad = \quad N \, \lambda, \tag{8.31}$$

valid for $|\mathbf{Q}' - \mathbf{Q}''| \gg 1/\sqrt{\alpha}$, obtained in section 6.5 for the case of distinguishable particles. We note that in the present case an additional factor $D_0 (4\pi/\alpha)^{3/2}$ appears in the macro–to–micro ratio, but such a factor is multiplied by the number of uncovered particles $n_{\text{out}}$ rather than by the total number $N$. Clearly, this is a consequence of the indistinguishability of particles and of the choice of the density as the dynamical variable governing the process. In section 6.5, the length parameter $1/\sqrt{\alpha}$ was chosen to be of the order of $10^{-5}$ cm and the microscopic rate $\lambda$ was suggested to be of the order of $10^{-16}$ sec$^{-1}$ with the aim of obtaining $\lambda_{\text{macro}} \approx 10^7$ sec$^{-1}$ for a typical macroscopic number $N \approx 10^{23}$. We repeat here the same choice,

$$\frac{1}{\sqrt{\alpha}} \quad \approx \quad 10^{-5} \text{ cm} \tag{8.32}$$

and look for a value of $\gamma$ such that the macroscopic rate $\Gamma$ is again of the order of $10^7$ sec$^{-1}$ for $n_{\text{out}} \approx 10^{13}$. Since $D_0 \approx 10^{24}$ cm$^{-3}$, we get

$$\gamma \quad \approx \quad 10^{-30} \text{ cm}^3 \text{ sec}^{-1}, \tag{8.33}$$

corresponding (according to the relation (8.16) between $\lambda$, $\gamma$ and $\alpha$) to $\lambda \approx 10^{-17}$ sec$^{-1}$. This value is such that nothing changes in the dynamics of a microscopic particle even in the case in which it has an extended wavefunction[34].

[34]Note that, in the case of a macroscopic object, QMSL requires a displacement of $\simeq 10^{24}$ particles for the reduction rate to be equal to $10^7$ sec$^{-1}$. CSL, on the other hand – in the case of normal density – requires a displacement only of about $10^{13}$ particles in order to have the same localization rate. This improvement of CSL with respect to QMSL is due to the indistinguishability of identical particles, whose effects are explicitly taken into account in CSL.

## 8.4 Position and momentum spreads

According to equation (8.28) or to the original expression (8.24), the diagonal elements $\langle \mathbf{Q}|\rho|\mathbf{Q}\rangle$ of the statistical operator in the position representation are not affected by the reduction process, as a consequence of the process being a localization. Of course, this does not mean that the time evolution of $\langle \mathbf{Q}|\rho|\mathbf{Q}\rangle$ is the same as the one given by the pure Schrödinger dynamics: some changes are expected in the time dependence of both position and momentum spreads, as a consequence of the presence of the localization process. An explicit evaluation of these effects is necessary in order to check that no unacceptable behaviour arises.

The equation for the statistical operator, in operator form, is written

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H,\rho(t)] + \gamma \int d^3x \left[ F(\mathbf{Q}-\mathbf{x})\rho(t)F(\mathbf{Q}-\mathbf{x}) - \frac{1}{2}\left\{F^2(\mathbf{Q}-\mathbf{x}),\rho(t)\right\}\right] \tag{8.34}$$

where we have retained also the Schrödinger term. We consider the case of a free macroscopic body, so that, in our notation, $H = P^2/(2M)$, $M$ being the total mass. For a dynamical variable $S$, we define the mean value

$$\langle\!\langle S \rangle\!\rangle \quad = \quad \mathrm{Tr}\left[S\rho\right]. \tag{8.35}$$

The time derivative of $\langle\!\langle S \rangle\!\rangle$, according to equation (8.34), is given by

$$\begin{aligned} \frac{d}{dt}\langle\!\langle S \rangle\!\rangle \quad &= \quad -\frac{i}{\hbar}\mathrm{Tr}\left([S,H]\rho\right) \; + \\ &\gamma \int d^3x \; \mathrm{Tr}\left[\left( F(\mathbf{Q}-\mathbf{x})SF(\mathbf{Q}-\mathbf{x}) - \frac{1}{2}\left\{S,F^2(\mathbf{Q}-\mathbf{x})\right\}\right)\rho\right]. \end{aligned} \tag{8.36}$$

¿From equation (8.36), through tedious but elementary calculations, one gets expressions for the average values of position, momentum and their combinations which are analogous to the QMSL equations (6.15)–(6.19). In particular we have:

$$\{Q_i\} \quad = \quad \{Q_i\}_{\mathrm{Sch}} \; + \; \gamma\delta_i \frac{\hbar^2}{6M^2}t^3, \tag{8.37}$$

$$\{P_i\} \quad = \quad \{P_i\}_{\mathrm{Sch}} \; + \; \gamma\delta_i \frac{\hbar^2}{2}t, \tag{8.38}$$

where

$$\delta_i \quad = \quad \int d^3y \left[\frac{\partial F(\mathbf{y})}{\partial y_i}\right]^2. \tag{8.39}$$

To evaluate the quantities $\delta_i$ the sharp scanning approximation is no longer sufficient, because here the derivative of the function $F$ is required. We then use the macroscopic density approximation (8.26). For definiteness and simplicity, we make reference to a homogeneous macroscopic parallelepiped of density $D_0$ having edges of lengths $L_i$ parallel to the coordinate axes. Then, as shown in [66], one has with high accuracy

$$\delta_i \quad = \quad \sqrt{\frac{\alpha}{\pi}}\, D_0^2\, S_i, \tag{8.40}$$

where $S_i = L_1L_2L_3/L_i$ is the transverse section of the macroscopic parallelepiped.

If the choice (8.32) and (8.33) is used for $\alpha$ and $\gamma$ together with $D_0 \approx 10^{24}$ cm$^{-3}$, one gets from equation (8.40) for the momentum diffusion coefficient,

$$\frac{1}{2}\,\gamma\,\delta_i\,\hbar^2 \quad \approx \quad 10^{-32}(\mathrm{g\ cm\ sec^{-1}})^2\,\mathrm{sec}^{-1}\,S_i\,\mathrm{cm}^{-2}. \tag{8.41}$$

For an ordinary macroscopic body, this value appears too small to give detectable effects. For a very small macroscopic particle, due to the $1/M^2$ factor in the extra term of equation (8.37), a non–negligible stochasticity could appear. For $L_j \approx 10^{-4}$ cm (this is the smallest order of magnitude for which the approximations leading to equation (8.40) remain valid), a time of the order of $10^2$ sec is required to make the extra term of the order of $10^{-10}$ cm$^2$. We do not know whether this kind of effect could be used to provide a significant experimental bound on the product $\gamma\sqrt{\alpha}$ contained in the momentum diffusion coefficient. We note, however, that the value (8.40) could overestimate $\delta_i$, because of the assumption of a rectangular profile for the object under consideration.

## 8.5 Reduction mechanism: a simple model

After this long and detailed analysis of the reduction mechanism within CSL, we propose here a second simpler way of looking at the localization process [68]; for simplicity's sake we restrict ourselves to a simplified version of CSL obtained by disregarding the hamiltonian term and discretizing the space.

We divide the space into cells of volume $(\alpha/2\pi)^{-3/2}$ and we denote by $N_i^{(k)}$ the number operator counting the particles of type $k$ in the $i$–th cell. As follows from the discussion of the preceding subsections in the considered case, the dynamical evolution drives the statevector into a manifold such that the number of particles present in any cell is definite. The simplified equation for the statistical operator (8.12) reads:

$$\frac{d}{dt}\,\rho(t) \;=\; \gamma\left(\frac{\alpha}{4\pi}\right)^{3/2}\sum_k\left[\sum_i N_i^{(k)}\,\rho(t)\,N_i^{(k)} \;-\; \frac{1}{2}\sum_i\{N_i^{(k)\,2},\rho(t)\}\right]. \qquad (8.42)$$

In accordance with relation (8.16), we will often use the QMSL rate parameter $\lambda$ in place of the expression $\gamma(\alpha/4\pi)^{3/2}$. If we denote by $|n_1^{(k)}, n_2^{(k)}, \ldots n_i^{(k)}, \ldots\rangle$ the state with the indicated occupation numbers for the various types of particles and for the various cells, the solution of equation (8.42) reads, in the considered basis:

$$\begin{aligned}\langle n_1^{(k)}, n_2^{(k)}, \ldots|\rho(t)|m_1^{(k)}m_2^{(k)},\ldots\rangle \;&=\\ &= e^{-\frac{\lambda}{2}\sum_{k,i}(n_i^{(k)}-m_i^{(k)})^2 t}\\ &\qquad\qquad \langle n_1^{(k)}, n_2^{(k)}, \ldots|\rho(0)|m_1^{(k)}m_2^{(k)},\ldots\rangle. \qquad (8.43)\end{aligned}$$

Equation (8.43) is an indirect proof[35] that linear superpositions of states containing different number of particles in the various cells are dynamically reduced to one of the superposed states with an exponential time rate depending on the expression

$$\frac{\lambda}{2}\sum_k\sum_i (n_i^{(k)}-m_i^{(k)})^2.$$

The amplification process in going from the micro to the macroscopic case and the preferred role assigned to position make it clear how such models overcome the difficulties of quantum measurement theory. In fact in measurement processes one usually assumes that different eigenstates of the measured micro-observable trigger (through the system–apparatus interaction) different displacements of a macroscopic pointer from its "ready" position. The unique dynamical principle of QMSL or CSL leads then, in extremely short times, to the dynamical suppression, with the appropriate probability, of all but one of the terms in the superposition, i.e., to the emergence of an outcome.

[35] The direct proof, as repeatedly stated, comes from the study of the localization mechanism at the wavefunction level.

## 8.6 Relating reductions to the mass density

In this subsection we consider a CSL type dynamics in which, in place of the operators $N(\mathbf{x})$ previously considered, we introduce the mass density operators [68]:

$$M(\mathbf{x}) \quad = \quad \sum_k m_k N_k(\mathbf{x}), \tag{8.44}$$

where $m_k$ and $N_k$ are the mass and the average mass density operator for a particle of type $k$, respectively. The Stratonovich stochastic evolution equation for the statevector is:

$$\frac{d}{dt}|\psi(t)\rangle \;=\; \left[-\frac{i}{\hbar}\,H \;+\; \int d^3x\, M(\mathbf{x})V(\mathbf{x},t) \;-\; \frac{\gamma}{m_0^2}\int d^3x\, M^2(\mathbf{x})\right]|\psi(t)\rangle, \tag{8.45}$$

where $m_0$ is a reference mass and $\gamma$ is the parameter appearing in standard CSL of section 8.1. We identify the mass $m_0$ with the nucleon mass; in this way the reduction rates for macroscopic objects are practically equal to those of the standard CSL model. With this choice the decoherence is governed by the mass of the nucleons in ordinary matter, the contribution due to electrons being negligible.

As usual the corresponding equation for the statistical operator is easily obtained:

$$\frac{d}{dt}\,\rho(t) \;=\; -\frac{i}{\hbar}\;[H,\rho(t)] + \frac{\gamma}{m_0^2}\int d^3x\, M(\mathbf{x})\rho(t)M(\mathbf{x}) - \frac{\gamma}{2m_0^2}\int d^3x\,\{M^2(\mathbf{x}),\rho(t)\}. \tag{8.46}$$

One of main motivations to replace the number density operators $N^{(k)}(\mathbf{x})$ in the CSL dynamics with the mass density operators $M(\mathbf{x})$ derives from the desire to relate reductions to gravity as suggested by various authors [102, 103, 104, 105, 99, 106] (a model with analogous characteristics will be presented in the next section). Another important feature of the above choice has been pointed out by P. Pearle and E. Squires [107]: while the reduction rates for macro–objects are practically the same as those of the standard CSL model, the probabilities of excitation or dissociation of microscopic bound systems turn out to be depressed by large factors [107, 108], thus leading to a smaller disagreement with the predictions of quantum mechanics for such systems. In particular, a simple computation within the quark model for nucleons (disregarding all relativistic effects which however could turn out to be very important at this level) gives a dissociation rate for the proton well below the experimental bounds, while the standard CSL model would lead to the violation of such bounds for the proton life-time. The advantages of taking the above position have also been discussed by A. Rimini [109].

## 8.7 A reduction model involving gravity

In 1989 L. Diòsi [99] proposed a modification of QMSL, different from CSL, with the explicit aim of eliminating the new constants of nature $\alpha$ and $\gamma$ and of relating the process to gravity. Such model has been called *quantum mechanics with universal density localization* (QMUDL) by the author.

Diòsi considers an extended macroscopic object, such as e.g. a homogeneous sphere, and introduces a mass density operator $f(\mathbf{x})$ at point $\mathbf{x}$. For the case of a homogeneous sphere of mass $M$ and radius $R$:

$$f(\mathbf{x}) \quad = \quad \frac{M}{V}\,\theta(R-|\mathbf{q}-\mathbf{x}|), \tag{8.47}$$

where $V$ is the volume of the sphere and $\mathbf{q}$ is the center of mass position operator. The stochastic dynamical equation is assumed to be:

$$d\,|\psi(t)\rangle \quad = \quad \left[-\frac{i}{\hbar}\,H\,dt \;-\; \frac{G}{2\hbar}\int\int d^3x_1\, d^3x_2 \frac{1}{x_{12}}[f(\mathbf{x}_1)-f_\psi(\mathbf{x}_1)]\times$$

$$\times\,[f(\mathbf{x}_2)-f_\psi(\mathbf{x}_2)]dt+\int d^3x[f(\mathbf{x})-f_\psi(\mathbf{x})]d\xi(\mathbf{x})\Big]\,|\psi(t)\rangle, \quad (8.48)$$

where $f_\psi(\mathbf{x})=\langle\psi|f(\mathbf{x})|\psi\rangle$; $\xi(\mathbf{x})$ is a real Wiener process satisfying:

$$\langle\!\langle d\xi(\mathbf{x})\rangle\!\rangle \;=\; 0 \qquad\qquad \langle\!\langle d\xi(\mathbf{x}_1)d\xi(\mathbf{x}_2)\rangle\!\rangle \;=\; \frac{G}{\hbar}\,\frac{1}{\mathbf{x}_{12}}\,dt. \quad (8.49)$$

In the above equation $\mathbf{x}_{12}=|\mathbf{x}_1-\mathbf{x}_2|$ and $G$ is Newton's gravitational constant.

Equation (8.48) can be put in the form (7.18), as shown in [110], the corresponding operators $A_i$ being functions of the center of mass position operator $\mathbf{q}$: the non Hamiltonian terms then induce the reduction onto states in which the position of the center of mass is more and more definite.

It is easy to prove by Itô stochastic calculus that equation (8.48) implies, for the statistical operator, the dynamical equation

$$\frac{d}{dt}\,\rho(t) \quad=\quad -\frac{i}{\hbar}\,[H,\rho(t)]\;-\;\frac{G}{2\hbar}\int\int d^3x_1\,d^3x_2\,\frac{1}{x_{12}}\,[f(\mathbf{x}_1),[f(\mathbf{x}_2),\rho(t)]]. \quad (8.50)$$

¿From equation (8.50), disregarding the Hamiltonian term, in the considered case of a homogeneous sphere, one gets

$$\langle\mathbf{q}'|\rho(t)|\mathbf{q}''\rangle \quad=\quad e^{\,\Gamma(\Delta)\,t}\,\langle\mathbf{q}'|\rho(0)|\mathbf{q}''\rangle, \quad (8.51)$$

where

$$\Gamma(\Delta)\;=\;-\frac{1}{\hbar}\,[U(0)-U(\Delta)], \qquad\qquad \Delta\;=\;|\mathbf{q}'-\mathbf{q}''|. \quad (8.52)$$

In equation (8.52) $U(\Delta)$ is the gravitational interaction energy of two homogeneous spheres of the considered mass and radius, with the centers at distance $\Delta$. For instance, for $R=1$ cm, $M=1$ g and $\Delta=10^{-5}$ cm, $\Gamma(\Delta)\simeq 10^9$ sec$^{-1}$, so that a linear superposition of two such states evolves into one of them in about $10^{-9}$ sec. This is comparable to the reduction rate implied by QMSL in such a case.

This model has many appealing features, e.g. the reduction mechanism is related to the gravitational potential without resorting to any physical constant, besides Newton's constant $G$. However, QMUDL, when taken literally, meets some serious difficulties: in fact, it has been proven [110] that the localization mechanism of QMUDL induces, in the case of macroscopic systems (with a number of constituents of the order of Avogadro's number), a total energy increase of about $10^3$ erg sec$^{-1}$, which clearly is unacceptable.

Ghirardi, Grassi and Rimini [110] have shown that it is possible to remove such a difficulty by making the following choice for the operators $f(\mathbf{x})$ (which replaces (8.47)):

$$f(\mathbf{x}) \quad=\quad m\,N(\mathbf{x}), \quad (8.53)$$

where $N(\mathbf{x})$ is given by equation (8.1) and $m$ is the mass of the particles created by $a^\dagger(\mathbf{y},s)$. Obviously if the system contains different types of particles, in (8.53) a sum over all types is understood. They chose for the parameter $\alpha$ again the value given by CSL.

It is not difficult to show that the model possesses all the appealing features of CSL, in particular it induces a fast suppression of the linear superpositions of states containing a macroscopic number of particles which are differently located, it does not alter in any appreciable way the dynamics of microsystems, and, in the case of a body with the internal coordinate sharply localized with respect to $1/\sqrt{\alpha}$, it allows the decoupling of the internal and center of mass motions, the internal motion being governed with high accuracy by the standard quantum dynamics. The price which has been paid is, with respect to Diósi's proposal which aimed to get rid of any new

constant, the introduction of a new physical constant, namely the localization width $1/\sqrt{\alpha}$. In our opinion, however, this is not a serious drawback. Actually, as we have stressed many times, in order to have a precise theory one needs to identify the borderline between quantum and classical, to get rid of the shifty character of the standard theory and of any proposed interpretation of it. The new parameter plays such a role in the just discussed model.

# 9 Dynamical reduction models with general gaussian noises

The dynamical reduction models analyzed in the previous sections have proven to yield (at the non relativistic level) a perfectly consistent solution to the macro–objectification problem of Quantum Mechanics. They are based on a stochastic modification of Schrödinger equation: besides the standard Hamiltonian, new terms are added, which contain Gaussian white noises.

It is an interesting question to analyze whether the most important features of CSL (and consequently of QMSL), in particular the localization mechanism, depend in any essential way on the white noise character of the stochastic processes considered [111, 112, 113, 114]. This is the subject of the present section.

There is a second, more important, reason to consider dynamical reduction models governed by more general noises. As we shall see in the fourth part of the report, relativistic CSL meets serious difficulties, since the reduction process yields an infinite increase of the energy (per unit time and unit volume). This divergent increase is due to the local coupling between quantum fields and the *white noise* stochastic filed: it is still an open problem to understand whether a non–white stochastic field can remove these divergences, and lead to a fully consistent relativistic model of dynamical reductions.

The content of the present section is the following. In subsection 9.1 we derive a modified Schrödinger equation, in which the new stochastic terms contain a general Gaussian noise. In subsection 9.2 we analyze two important cases in which the modified Schrödinger equation can be studied in detail. In subsection 9.3 we analyze the reduction properties of such an equation, while in subsection 9.4 we study the time evolution of the average value of physical quantities. In this way we prove that the considered model shares all the essential features of standard CSL.

The section ends with an explicit application of the above results to a specific model of dynamical reductions in *space*, like we did in section 8 for white noise models.

## 9.1 The modified Schrödinger equation

In this subsection we begin the analysis of dynamical reduction models in which the reduction mechanism is controlled by general Gaussian noises. The first task is to derive a modified Schrödinger equation generalizing equation (7.42), and preserving the average value of the square norm of vectors, so that the cooking prescription can be applied to it.

Let us then consider the following equation:

$$\frac{d|\psi(t)\rangle}{dt} \quad = \quad \left[-\frac{i}{\hbar}H_0 \; + \; \sum_i A_i w_i(t)\right]|\psi(t)\rangle, \tag{9.1}$$

where, as before, $H_0$ is the Hamiltonian of the system, $\{A_i\}$ is a set of commuting self–adjoint operators, and $w_i(t)$ are $c$–number gaussian stochastic processes whose

first two moments are[36]:

$$\langle\!\langle w_i(t) \rangle\!\rangle = 0, \qquad \langle\!\langle w_i(t_1) w_j(t_2) \rangle\!\rangle = \gamma\, D_{ij}(t_1, t_2). \tag{9.2}$$

We already know that the evolution described by equation (9.1) is not unitary and it does not preserve the norm of the statevector; we then follow the same prescription outlined in section 7. We consider as *physical* vectors the normalized ones:

$$|\phi(t)\rangle \quad = \quad \frac{|\psi(t)\rangle}{\||\psi(t)\rangle\|}, \tag{9.3}$$

and we assume that any particular realization of the stochastic processes $w_i(t)$ has a probability of occurrence $P_{\rm Cook}[w(t)]$ equal to:

$$P_{\rm Cook}[w(t)] \quad = \quad P_{\rm Raw}[w(t)]\, \||\psi(t)\rangle\|^2, \tag{9.4}$$

where $P_{\rm Raw}[w(t)]$ is now the gaussian probability distribution defined by (9.2). The above assumptions guarantee that the reduction probabilities reproduce the standard quantum mechanical probabilities.

Of course, we have to check that equation (9.4) correctly defines a probability distribution, i.e. that it sums to 1. Following the discussion of section 7, we know that this is equivalent to requiring that the time derivative of $\langle\!\langle \langle \psi(t) | \psi(t) \rangle \rangle\!\rangle$ is zero. Let us evaluate it:

$$\begin{aligned}
\frac{d}{dt} \langle\!\langle \langle \psi(t) | \psi(t) \rangle \rangle\!\rangle \quad &= \quad \left\langle\!\!\left\langle \left[ \frac{d\langle \psi(t)|}{dt} \right] |\psi(t)\rangle \right\rangle\!\!\right\rangle \; + \; \left\langle\!\!\left\langle \langle \psi(t)| \left[ \frac{d|\psi(t)\rangle}{dt} \right] \right\rangle\!\!\right\rangle \quad = \\
&= \quad \left\langle\!\!\left\langle \langle \psi(t)| \left[ + \frac{i}{\hbar} H_0 \; + \; \sum_i A_i w_i(t) \right] |\psi(t)\rangle \right\rangle\!\!\right\rangle \; + \\
&\qquad \left\langle\!\!\left\langle \langle \psi(t)| \left[ - \frac{i}{\hbar} H_0 \; + \; \sum_i A_i w_i(t) \right] |\psi(t)\rangle \right\rangle\!\!\right\rangle .
\end{aligned}$$

The two terms involving the Hamiltonian $H_0$ cancel out (in fact they describe the unitary part of the evolution); the noises $w_i(t)$, being $c$–numbers, can be taken out of the scalar product, so that:

$$\frac{d}{dt} \langle\!\langle \langle \psi(t) | \psi(t) \rangle \rangle\!\rangle \quad = \quad 2 \sum_i \langle\!\langle \langle \psi(t) | A_i | \psi(t) \rangle w_i(t) \rangle\!\rangle. \tag{9.5}$$

The right hand side of (9.5) can be rewritten with the help of the *Furutsu–Novikov formula* [115, 114]:

$$\langle\!\langle F[w(t)] w_i(t) \rangle\!\rangle \quad = \quad \gamma \sum_j \int_0^{+\infty} D_{ij}(t, s) \left\langle\!\!\left\langle \frac{\delta F[w(t)]}{\delta w_j(s)} \right\rangle\!\!\right\rangle ds \tag{9.6}$$

(for simplicity, throughout this subsection we take $t_0 = 0$ as the initial time). $F[w(t)]$ is any functional of the stochastic fields $w_i(t)$; in the present case case, $F[w(t)] = \langle \psi(t) | A_i | \psi(t) \rangle$.

The formal solution of equation (9.1) is:

$$|\psi(t)\rangle = T\, e^{\displaystyle - \frac{i}{\hbar} H_0 t \; + \sum_i A_i \int_0^t w_i(s)\, ds} |\psi(0)\rangle. \tag{9.7}$$

[36] There is no loss of generality in considering gaussian processes with zero mean. In fact, if $\langle\!\langle w_i(t) \rangle\!\rangle = m_i(t) \neq 0$, we can always define new processes $z_i(t) = w_i(t) - m_i(t)$, which have zero mean, and rewrite the modified Schrödinger equation (9.1) in terms of the processes $z_i(t)$.

Note that, since $|\psi(t)\rangle$ depends on the stochastic processes $w_i(s)$ only within the time–interval $[0,t]$, the functional derivative of $|\psi(t)\rangle$ with respect to $w_j(s)$ is zero if $s \notin [0,t]$. We then have:

$$\begin{aligned} \frac{d}{dt}\langle\!\langle \langle\psi(t)|\psi(t)\rangle \rangle\!\rangle &= 2\gamma \sum_{i,j} \int_0^t D_{ij}(t,s) \left\langle\!\!\left\langle \left[\frac{\delta\langle\psi(t)|}{\delta w_j(s)}\right] A_i |\psi(t)\rangle \right\rangle\!\!\right\rangle ds \\ &+ 2\gamma \sum_{i,j} \int_0^t D_{ij}(t,s) \left\langle\!\!\left\langle \langle\psi(t)| A_i \left[\frac{\delta|\psi(t)\rangle}{\delta w_j(s)}\right] \right\rangle\!\!\right\rangle ds \neq 0. \end{aligned} \tag{9.8}$$

Since the time derivative of the average value of the square norm of the statevector is not zero, we have to add an extra term to equation (9.1), as expected and as it happens also in the case of white noise. Relation (9.8) tells us which kind of term must be added. The conclusion follows: with reference to our procedure, the request that $P_{\text{Cook}}[w(t)]$ correctly defines a probability distribution, i.e. that the average value of the square norm of the statevector $|\psi(t)\rangle$ is conserved, leads to the stochastic Schrödinger equation:

$$\frac{d|\psi(t)\rangle}{dt} = \left[ -\frac{i}{\hbar} H_0 + \sum_i A_i w_i(t) - 2\gamma \sum_{i,j} A_i \int_0^t ds\, D_{ij}(t,s) \frac{\delta}{\delta w_j(s)} \right] |\psi(t)\rangle. \tag{9.9}$$

This is the main result of this subsection. Note that an equation like (9.9) has been derived also in [113, 116] by following a different line of thought.

Some comments are appropriate:

- Equation (9.9) no longer describes a Markovian evolution for the statevector unless the correlation functions $D_{ij}(t,s)$ are Dirac–$\delta$'s in the time variable — i.e. the stochastic processes $w_i(t)$ are white in time. As a consequence, the corresponding equation for the statistical operator is not of the quantum–dynamical–semigroup type[37], contrary to what happen for the case of CSL (see equation (7.51)).

- In general, the explicit form of the functional derivatives of $|\psi(t)\rangle$ with respect to the noise $w_i(t)$ cannot be evaluated exactly, except for few special cases, two of which will be considered in the next subsection. Therefore, in the general case it is difficult to analyze the time evolution of the statevector and the statistical properties of the ensemble of states generated by the stochastic processes. In particular, one cannot write a closed equation for the evolution of the statistical operator.

## 9.2 Two special cases

In order to understand the kind of difficulties one encounters when working with non–white stochastic processes, and in particular the reasons for which the functional derivative of the statevector $|\psi(t)\rangle$ in general cannot be computed exactly, let us reconsider equation (9.7), writing explicitly its perturbative expansion:

$$\begin{aligned} T\, e^{-\frac{i}{\hbar} H_0 t + \sum_i A_i \int_0^t w_i(s)\, ds} &= \\ &= \sum_{n=0}^{\infty} \left[-\frac{i}{\hbar}\right]^n \frac{1}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n\, T\left\{H(t_1)\ldots H(t_n)\right\}, \end{aligned} \tag{9.10}$$

[37] See section 6.6.

where we have defined the operator:

$$H(t) \quad = \quad H_0 \; + \; i\hbar \sum_i A_i \, w_i(t). \tag{9.11}$$

The functional derivative of $|\psi(t)\rangle$ with respect to $w_j(s)$ can be obtained deriving term by term the series[38] (9.10). The derivative of the term $n = 0$ is zero; the derivative of the term $n = 1$ is:

$$\frac{\delta}{\delta w_j(s)} \left[ -\frac{i}{\hbar} \int_0^t dt_1 \, H(t_1) \right] \; = \; -\frac{i}{\hbar} \int_0^t dt_1 \, \left[ i\hbar \, \delta(s - t_1) \, A_j \right] \; = \; A_j. \tag{9.12}$$

The next ($n = 2$) term is:

$$\left[ -\frac{i}{\hbar} \right]^2 \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \, T \left\{ H(t_1) \, H(t_2) \right\}. \tag{9.13}$$

The functional derivative of the time–ordered product $T\{H(t_1)\, H(t_2)\} \; = \; \theta(t_1 - t_2) H(t_1) H(t_2) + \theta(t_2 - t_1) H(t_2) H(t_1)$ is:

$$\begin{aligned} &\frac{\delta}{\delta w_j(s)} T \left\{ H(t_1) \, H(t_2) \right\} \; = \\ &= \quad i\hbar \, \theta(t_1 - t_2) \left[ \delta(t_1 - s) \, A_j \, H(t_2) \; + \; \delta(t_2 - s) \, H(t_1) \, A_j \right] \\ &+ \quad i\hbar \, \theta(t_2 - t_1) \left[ \delta(t_2 - s) \, A_j \, H(t_1) \; + \; \delta(t_1 - s) \, H(t_2) \, A_j \right]. \end{aligned} \tag{9.14}$$

We note that the first and third terms at the right hand side of (9.14) differ only for the exchange of the dummy variables $t_1 \leftrightarrow t_2$; the same is true for the second and the fourth term. The derivative of the $n = 2$ term (i.e. of Eq. (9.13)) is then:

$$A_j \left[ -\frac{i}{\hbar} \int_0^s dt_1 \, H(t_1) \right] \; + \; \left[ -\frac{i}{\hbar} \int_s^t dt_1 \, H(t_1) \right] A_j. \tag{9.15}$$

Equation (9.15) does not have a simple form, contrary to (9.12), and derivatives of higher terms are more and more complicated, due to the fact that the operators $A_j$ in general do not commute with the Hamiltonian $H_0$. In fact, would they commute, equation (9.15) would simplify to:

$$A_j \left[ -\frac{i}{\hbar} \int_0^t dt_1 \, H(t_1) \right], \tag{9.16}$$

i.e. the derivative of the second term would give $A_j$ times the first term. Moreover, if $[A_j, H_0] = 0$, the functional derivative of the term $n + 1$ gives $A_j$ times the $n$–th term so that :

$$\frac{\delta}{\delta w_j(s)} \, |\psi(t)\rangle \quad = \quad A_j \, |\psi(t)\rangle, \tag{9.17}$$

as we are going to prove. In fact, the hypothesis that the operators $A_i$ commute with the Hamiltonian $H_0$ is equivalent to the (more elegant) requirement that the operators $H(t)$ defined in (9.11) commute at different times. In this case, the time–ordered product in the exponential series (9.10) can be omitted, and the functional derivative of the $n$–th term is:

$$\frac{\delta}{\delta \, w_j(s)} \left[ -\frac{i}{\hbar} \right]^n \frac{1}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n \, \left\{ H(t_1) \ldots H(t_n) \right\} \; =$$

[38] We assume that the initial state $|\psi(0)\rangle$ does not depend on the stochastic processes $w_i(t)$.

$$= \left[-\frac{i}{\hbar}\right]^n \frac{1}{n!}\sum_{i=1}^{n}\int_0^t dt_1\cdots\int_0^t dt_n\left\{H(t_1)\ldots\frac{\delta\,H(t_i)}{\delta\;w_j(s)}\ldots H(t_n)\right\} =$$

$$= \left[-\frac{i}{\hbar}\right]^n \frac{1}{(n-1)!}\int_0^t dt_1\cdots\int_0^t dt_n\left\{\frac{\delta\,H(t_1)}{\delta\;w_j(s)}\ldots H(t_n)\right\} =$$

$$= A_j\left[-\frac{i}{\hbar}\right]^{n-1}\frac{1}{(n-1)!}\int_0^t dt_1\cdots\int_0^t dt_{n-1}\;\{H(t_1)\ldots H(t_{n-1})\}\,. \qquad (9.18)$$

This completes the proof. Note also that, when $s = t$, an extra factor $1/2$ appears in (9.17), because in this case the Dirac delta function arising from the functional derivative of $H(t)$ is centered in one of the two extreme points of the interval of integration.

Recently, L. Hughston [117], S. Adler and P. Horwitz [118, 119] have proposed a white–noise model of dynamical reductions in which the operators $A_i$ are taken to be functions of the Hamiltonian $H_0$; this implies that the stochastic terms of equation (7.42) drive the statevector into the *energy* eigenmanifolds of the physical system. Making such a choice in the non–white equation (9.9), the operators $H(t)$ at different times commute among themselves, the functional derivatives of the statevector $|\psi(t)\rangle$ can be computed, and equation (9.9) becomes:

$$\frac{d|\psi(t)\rangle}{dt} = \left[-\frac{i}{\hbar}H_0 + \sum_i A_i w_i(t) - 2\gamma\sum_{i,j}A_iA_j\int_0^t D_{ij}(t,s)\,ds\right]|\psi(t)\rangle, \qquad (9.19)$$

with $A_i = A_i(H_0)$. Equation (9.19) is exact and, correspondingly, one can easily derive a closed equation for the time evolution of the statistical operator. All the statistical properties concerning the physical system can be evaluated exactly.

We conclude the subsection showing that the functional derivatives of $|\psi(t)\rangle$ can be explicitly evaluated also in the case of general white noise stochastic processes, without having to require that $H_0$ commutes with $A_i$. Moreover, we will prove that in this case equation (9.9) reduces to (7.42), as expected.

Under the assumption of white–noise stochastic processes ($D_{ij}(t_1,t_2) = \delta_{ij}\,\delta(t_1 - t_2)$), the Furutsu–Novikov relation

$$\langle\!\langle F[w(t)]w_i(t)\rangle\!\rangle = \gamma\left\langle\!\!\left\langle\frac{\delta F[w(t)]}{\delta w_i(t)}\right\rangle\!\!\right\rangle \qquad (9.20)$$

leads to the following expression for the time derivative of the average value of the square norm of the statevector $|\psi(t)\rangle$ satisfying equation (9.1):

$$\frac{d}{dt}\langle\!\langle\langle\psi(t)|\psi(t)\rangle\rangle\!\rangle = 2\gamma\sum_i\left\langle\!\!\left\langle\left[\frac{\delta\langle\psi(t)|}{\delta w_i(t)}\right]A_i|\psi(t)\rangle\right\rangle\!\!\right\rangle + 2\gamma\sum_i\left\langle\!\!\left\langle\langle\psi(t)|A_i\left[\frac{\delta|\psi(t)\rangle}{\delta w_i(t)}\right]\right\rangle\!\!\right\rangle. \qquad (9.21)$$

We now have to evaluate the functional derivatives of the statevector, taking into account that the noises $w_i$ (appearing in the derivatives) are taken at time $t$.

The derivative of the term $n = 1$ is equal to $(1/2)A_j$ (see equation (9.12)), the factor $(1/2)$ deriving from the Dirac delta function $\delta(t-t_1)$ which is integrated between 0 and $t$. For the derivative of the $n = 2$ term, let us look at expression (9.15). If we take $s = t$, the second term goes to zero, while the first one gives[39]:

$$\frac{1}{2}\,A_j\left[-\frac{i}{\hbar}\int_0^t dt_1\,H(t_1)\right]. \qquad (9.22)$$

[39]The factor (1/2) appears for the same reason as before.

In general, the functional derivative of any term of the exponential series (9.10) gives $(1/2)A_j$ times the previous term, so that, in general:

$$\frac{\delta}{\delta w_j(t)}\,|\psi(t)\rangle \quad = \quad \frac{1}{2}\,A_j\,|\psi(t)\rangle. \tag{9.23}$$

This means that the square–norm–preserving Schrödinger equation is:

$$\frac{d|\psi(t)\rangle}{dt} = \left[-\frac{i}{\hbar}H_0 + \sum_i A_i w_i(t) - \gamma \sum_i A_i\,A_j\right]|\psi(t)\rangle, \tag{9.24}$$

which coincides with the original CSL equation (7.42). An alternative and quicker way to derive the white–noise limit is to replace $D_{ij}(t,s)$ with $\delta_{ij}\delta(t-s)$ in equation (9.9) and to show that (9.23) is a consistent solution.

## 9.3 The reduction mechanism

Here, we will analyze under which conditions the new terms in the modified Schrödinger equation (9.9) induce, for large times, the reduction of the statevector to one of the common eigenstates of the commuting operators $A_i$.

For this purpose, let us disregard the Hamiltonian $H_0$; under this assumption the operators $H(t)$ commute at different times and (as discussed in the previous subsection) the functional derivatives of the statevector $|\psi(t)\rangle$ give the operators $A_i$ times $|\psi(t)\rangle$. Equation (9.9) becomes then[40]:

$$\frac{d|\psi(t)\rangle}{dt} = \left[\sum_i A_i\,w_i(t) - 2\gamma\sum_{i,j} A_iA_j\int_{t_0}^{t} D_{ij}(t,s)\,ds\right]|\psi(t)\rangle. \tag{9.25}$$

The equation for the statistical operator can now be easily derived; using the definition (7.49), we get:

$$\frac{d\rho(t)}{dt} \quad = \quad -\gamma\sum_{i,j}[A_i,[A_j,\rho(t)]]\int_{t_0}^{t} D_{ij}(t,s)\,ds, \tag{9.26}$$

which is a consistent generalization of the CSL equation (7.51) when the Hamiltonian $H_0$ is omitted: in fact, if the stochastic processes $w_i(t)$ are independent and white ($D_{ij}(t_1,t_2) = \delta_{ij}\,\delta(t_1-t_2)$), then (9.26) reduces exactly to (7.51).

In order to test the reduction properties, we will show first of all how the reduction mechanism works for the statistical operator. As in section 7, let us suppose that the common eigenmanifolds of the operators $A_i$, which we assume to have a purely discrete spectrum, are one–dimensional; let $|\alpha\rangle$ be the vector spanning the $\alpha$–eigenmanifold. The equation for the matrix elements $\langle\alpha|\rho(t)|\beta\rangle$ is:

$$\frac{d\langle\alpha|\rho(t)|\beta\rangle}{dt} \;=\; -\,\gamma\sum_{i,j}(a_{i\alpha}-a_{i\beta})(a_{j\alpha}-a_{j\beta})\int_{t_0}^{t} D_{ij}(t,s)\,ds\,\langle\alpha|\rho(t)|\beta\rangle. \tag{9.27}$$

Making use of the symmetry property of the correlation functions:

$$D_{ij}(t_1,t_2) \quad = \quad D_{ji}(t_2,t_1), \tag{9.28}$$

we can write the solution of equation (9.27) in the following form:

$$\langle\alpha|\rho(t)|\beta\rangle = e^{-\frac{\gamma}{2}\sum_{i,j}(a_{i\alpha}-a_{i\beta})(a_{j\alpha}-a_{j\beta})\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\,D_{ij}(t_1,t_2)}\,\langle\alpha|\rho(t_0)|\beta\rangle. \tag{9.29}$$

[40] Here and in what follows, we consider a generic initial time $t_0$.

¿From equation (9.29), we sees that if $|\alpha\rangle = |\beta\rangle$, the exponent is zero: as in CSL, the diagonal elements of the density matrix do not change in time. If, on other other hand $|\alpha\rangle \neq |\beta\rangle$, the evolution of the matrix element depends on the time behavior of the correlation functions $D_{ij}(t_1, t_2)$.

If we want the off–diagonal elements to be damped at large times, two conditions must be satisfied. The first one is that the **exponent** in (9.29) must be **negative**: this is always true, since the correlation function of a Gaussian process is positive definite[41].

The second condition is that the **double integral** of the correlation function must **diverge** for large times:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, D_{ij}(t_1, t_2) \;\longrightarrow\; +\infty \qquad \text{for } t \to +\infty, \tag{9.30}$$

so that the off–diagonal elements of the density matrix go to zero. This condition is not *a priori* satisfied by a generic Gaussian stochastic field. At any rate, physically reasonable stochastic fields always satisfy it: here we present just a couple of meaningful examples.

Suppose the stochastic fields $w_i(t)$ are equal and independent, with a (normalized) Gaussian correlation function:

$$D_{ij}(t_1, t_2) \quad = \quad \delta_{ij} \, \frac{1}{\sqrt{2\pi}\tau} \, e^{-\dfrac{(t_1 - t_2)^2}{2\tau^2}}. \tag{9.31}$$

Let us also take $t_0 = -\infty$. Equation (9.27) then becomes:

$$\frac{d\langle\alpha|\rho(t)|\beta\rangle}{dt} \;=\; -\frac{\gamma}{2} \sum_i (a_{i\alpha} - a_{i\beta})^2 \, \langle\alpha|\rho(t)|\beta\rangle, \tag{9.32}$$

which is independent from the correlation time $\tau$, and moreover it corresponds exactly to the CSL evolution. Note that if we take the limit $\tau \to 0$, the gaussian process becomes a white noise process with a Dirac–$\delta$ correlation function and we recover, again, the CSL theory.

As a second example, suppose the correlation function is:

$$D_{ij}(t_1, t_2) \quad = \quad \delta_{ij} \, \frac{1}{2\tau} \, e^{-\dfrac{|t_1 - t_2|}{\tau}}. \tag{9.33}$$

Equation (9.27) becomes:

$$\frac{d\langle\alpha|\rho(t)|\beta\rangle}{dt} \;=\; -\frac{\gamma}{2} \left[ 1 - e^{-\dfrac{(t - t_0)}{\tau}} \right] \sum_i (a_{i\alpha} - a_{i\beta})^2 \, \langle\alpha|\rho(t)|\beta\rangle. \tag{9.34}$$

As before, the off–diagonal elements are exponentially damped and, in the limit $t \to +\infty$ we recover the behaviour of CSL. Note that the effect of a non–white correlation function is that of decreasing the reduction rate of the localization mechanism.

We now analyze how the reduction mechanism works at the wavefunction level. As in section 7.4, we consider a simplified dynamics in which only one operator $A$ appears in equation (9.25). This operator is coupled to a single stochastic process $w(t)$, whose correlation function is $D(t_1, t_2)$. Finally, we assume that at the initial time $t_0$ the statevector is:

$$|\psi(t_0)\rangle \quad = \quad P_\alpha |\psi(t_0)\rangle \;+\; P_\beta |\psi(t_0)\rangle, \tag{9.35}$$

[41] We assume that the correlation function is non degenerate.

where $P_\alpha$ and $P_\beta$ are projection operators onto the eigenmanifolds of $A$ corresponding to two different eigenvalues $\alpha$ and $\beta$, respectively. The solution of equation (9.25) is:

$$|\psi(t)\rangle \;=\; e^{\alpha x(t) - \alpha^2 \gamma f(t)} P_\alpha |\psi(t_0)\rangle \;+\; e^{\beta x(t) - \beta^2 \gamma f(t)} P_\alpha |\psi(t_0)\rangle, \tag{9.36}$$

where

$$x(t) \;=\; \int_{t_0}^{t} w(s)\, ds, \qquad\qquad f(t) \;=\; \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2\, D(s_1, s_2). \tag{9.37}$$

Note that $\gamma f(t) = \langle\!\langle x^2(t) \rangle\!\rangle$, i.e. such a quantity is the variance of the stochastic process $x(t)$.

Since the "raw" probability distribution of the process $x(t)$ is:

$$P_{\rm Raw}[x(t)] \quad = \quad \frac{1}{\sqrt{2\pi\gamma f(t)}}\, e^{-\dfrac{1}{2\gamma f(t)}\, x^2(t)}, \tag{9.38}$$

taking into account the cooking prescription (9.4) we obtain:

$$\begin{aligned} P_{\rm Cook}[x(t)] \quad &= \quad \|P_\alpha|\psi(t_0)\rangle\|^2\, \frac{1}{\sqrt{2\pi\gamma f(t)}}\, e^{-\dfrac{1}{2\gamma f(t)}\, [x(t) - 2\alpha\gamma f(t)]^2} \\ &+ \quad \|P_\beta|\psi(t_0)\rangle\|^2\, \frac{1}{\sqrt{2\pi\gamma f(t)}}\, e^{-\dfrac{1}{2\gamma f(t)}\, [x(t) - 2\beta\gamma f(t)]^2}. \end{aligned} \tag{9.39}$$

Equation (9.39) implies that, if $f(t) \to +\infty$ when $t \to +\infty$, the stochastic process $x(t)$ will take either a value close to $2\alpha\gamma f(t)$ — within an interval of width $\sqrt{\gamma f(t)}$ — or a value close to $2\beta\gamma f(t)$, within the same interval[42]. Of course, the requirement that $f(t) \to +\infty$ as time increases is exactly the same as requirement (9.30) which guarantees the damping of the off–diagonal elements of the density matrix.

Suppose now that the actual realization of the stochastic process $x(t)$ occurs around $2\alpha\gamma f(t)$; the corresponding probability is $\|P_\alpha|\psi(0)\rangle\|^2$. We then have:

$$\frac{\|P_\beta|\psi(t)\rangle\|^2}{\|P_\alpha|\psi(t)\rangle\|^2} \;\simeq\; e^{-2\gamma(\alpha-\beta)^2 f(t)}\, \frac{\|P_\beta|\psi(0)\rangle\|^2}{\|P_\alpha|\psi(0)\rangle\|^2} \;\to\; 0 \qquad \text{as } t \to \infty, \tag{9.40}$$

which means that the statevector $|\psi(t)\rangle$ is driven into the eigenmanifold of the operator $A$ corresponding to the eigenvalue $\alpha$. By the same reasoning, it is immediate to see that, with a probability equal to $\|P_\beta|\psi(0)\rangle\|^2$, the statevector is driven into the eigenmanifold associated to the eigenvalue $\beta$. We have thus proved that the statevector $|\psi(t)\rangle$ undergoes a random spontaneous reduction to one of the two eigenmanifolds of the operator $A$, with a probability which coincides with the one assigned by standard Quantum Mechanics to the outcomes of an experiment aimed to measure the observable $A$.

## 9.4 The average value of observables

When one disregards the Hamiltonian term $H_0$, it is not difficult to see how the stochastic terms affect the average value of physical quantities. Its time derivative

[42] As noted in section 7.4, even though the interval $\sqrt{\gamma f(t)}$ tends to infinity as time increases, the ratio $\sqrt{\gamma f(t)}/2(\alpha-\beta)\gamma f(t)$ goes to zero.

can be calculated following almost the same steps which, in the previous subsection, have led to equation (9.26) for the statistical operator; the final equation is:

$$\frac{d\langle O\rangle}{dt} \quad = \quad -\gamma \sum_{i,j} \langle\!\langle \langle\psi(t)| \left[A_i, [A_j, O]\right] |\psi(t)\rangle \rangle\!\rangle \int_{t_0}^{t} D_{ij}(t,s)\, ds, \tag{9.41}$$

to be compared with the corresponding CSL–white noise equation:

$$\frac{d\langle O\rangle}{dt} \quad = \quad -\frac{\gamma}{2} \sum_{i} \langle\!\langle \langle\psi(t)| \left[A_i, [A_i, O]\right] |\psi(t)\rangle \rangle\!\rangle. \tag{9.42}$$

The analysis of the previous subsection should have made clear how (9.41) differs from (9.42), so we will not repeat it here.

## 9.5 Connection with CSL

We now apply the formalism introduced in the previous subsections to derive an equation with the property of localizing macroscopic systems in *space*, as it happens for CSL. In other words, we specify the choice of the "preferred basis" $\{A_i\}$ in such a way to have a physically meaningful theory for our purposes.

The most natural choice for the operators $A_i$ is the number density operator for a system of identical particles:

$$A_i \quad \longrightarrow \quad \mathcal{N}(\mathbf{x}) \;=\; \sum_s a^\dagger(\mathbf{x}, s)\, a(\mathbf{x}, s). \tag{9.43}$$

Correspondingly, the noises $w_i(t)$ are replaced by a stochastic field $w(\mathbf{x}, t)$, whose correlation function is $D(\mathbf{x}, t_1; \mathbf{y}, t_2)$.

In section 13, the transformation and invariance properties of dynamical reduction models will be discussed in detail. In particular, it will be proved that, in order for the physics of the model to be invariant under Galilean transformations (we speak of *stochastic Galilean invariance*), the correlation function $D(\mathbf{x}, t_1; \mathbf{y}, t_2)$ itself must be invariant under the considered group of transformations, i.e.

$$D(\mathbf{x}, t_1; \mathbf{y}, t_2) \quad = \quad D(|\mathbf{x} - \mathbf{y}|, t_1 - t_2); \tag{9.44}$$

the easiest way to construct a function like (9.44) is to take the product of two functions of the space and time variables, respectively:

$$D(\mathbf{x}, t_1; \mathbf{y}, t_2) \quad = \quad g(|\mathbf{x} - \mathbf{y}|)\, h(t_1 - t_2). \tag{9.45}$$

As regards $g(|\mathbf{x} - \mathbf{y}|)$, a reasonable choice is a gaussian function, like in CSL:

$$g(|\mathbf{x} - \mathbf{y}|) \quad = \quad \gamma \left(\frac{\alpha}{4\pi}\right)^{\frac{3}{2}} e^{-\frac{\alpha}{4}(\mathbf{x} - \mathbf{y})^2}, \tag{9.46}$$

with $1/\sqrt{\alpha} \simeq 10^{-5}$ cm.

It is natural to choose a gaussian function also for $h(t_1 - t_2)$:

$$h(t_1 - t_2) \quad = \quad \left(\frac{\beta}{4\pi}\right)^{\frac{1}{2}} e^{-\frac{\beta}{4}(t_1 - t_2)^2}. \tag{9.47}$$

In making the above choice, we have introduced a new parameter ($\beta$); this can be considered as a drawback of the model. However, we note that it always is possible to define $\beta$ in terms of $\alpha$, $\gamma$ and fundamental constants of nature, so that, essentially,

no new arbitrary parameter is introduced into the model. As an example, we can choose $\beta = c^2\alpha \simeq 10^{30}$ sec$^{-2}$, where $c$ is the speed of light. This choice is particularly appropriate from the point of view of a possible relativistic generalization of the theory, which we will discuss in the fourth part of this report. Moreover, such a choice corresponds to an extremely small correlation time, so that for ordinary systems (moving slower than the speed of light) the behaviour of the model is similar to the one arising from the white–noise CSL.

The modified equation (9.9) for the statevector evolution becomes now:

$$\begin{aligned}\frac{d|\psi(t)\rangle}{dt} &= \left[-\frac{i}{\hbar}H_0 + \int d^3x\,\mathcal{N}(\mathbf{x})w(\mathbf{x},t) - \right. \\ &\left. - 2\gamma\int d^3x d^3y\,\mathcal{N}(\mathbf{x})g(|\mathbf{x}-\mathbf{y}|)\int_{t_0}^{t} ds\,h(t-s)\frac{\delta}{\delta w(\mathbf{y},s)}\right]|\psi(t)\rangle.\end{aligned} \tag{9.48}$$

If we ignore the free Hamiltonian $H_0$, i.e. if we confine our considerations to the reduction mechanism[43], equation (9.48) becomes:

$$\frac{d|\psi(t)\rangle}{dt} = \left[\int d^3x\,\mathcal{N}(\mathbf{x})w(\mathbf{x},t) - \gamma(t)\int d^3x d^3y\,\mathcal{N}(\mathbf{x})g(|\mathbf{x}-\mathbf{y}|)\mathcal{N}(\mathbf{y})\right]|\psi(t)\rangle \tag{9.49}$$

with:

$$\gamma(t) = 2\gamma\int_{t_0}^{t} ds\,h(t-s). \tag{9.50}$$

The corresponding equation for the statistical operator is:

$$\frac{d}{dt}\,\rho(t) = -\frac{\gamma(t)}{2}\int d^3x\,d^3y\,[\mathcal{N}(\mathbf{x}),[\mathcal{N}(\mathbf{y}),\rho(t)]]\,g(|\mathbf{x}-\mathbf{y}|). \tag{9.51}$$

Equation (9.48) can be rewritten in a form closer to equation (8.6), a fact which will be useful for the subsequent discussion. Let us define a new Gaussian stochastic process $\overline{w}(\mathbf{x},t)$, which is connected to $w(\mathbf{x},t)$ by the relation:

$$w(\mathbf{x},t) = \left(\frac{\alpha}{2\pi}\right)^{\frac{3}{2}}\int d^3x\,e^{-\frac{\alpha}{2}(\mathbf{x}-\mathbf{y})^2}\overline{w}(\mathbf{y},t). \tag{9.52}$$

The process $\overline{w}(\mathbf{x},t)$ has zero mean and correlation function

$$\langle\!\langle \overline{w}(\mathbf{x},t_1)\,\overline{w}(\mathbf{y},t_2)\rangle\!\rangle = \gamma\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,h(t_1-t_2). \tag{9.53}$$

Using the following relation:

$$\begin{aligned}\frac{\delta}{\delta\overline{w}(\mathbf{x},s)}|\psi(t)\rangle &= \int d^3y\,\frac{\delta w(\mathbf{y},s)}{\delta\overline{w}(\mathbf{x},s)}\frac{\delta}{\delta w(\mathbf{y},s)}|\psi(t)\rangle = \\ &= \left(\frac{\alpha}{2\pi}\right)^{\frac{3}{2}}\int d^3y\,e^{-\frac{\alpha}{2}(\mathbf{x}-\mathbf{y})^2}\frac{\delta}{\delta w(\mathbf{y},s)}|\psi(t)\rangle,\end{aligned} \tag{9.54}$$

it can be easily seen that (9.48) is equivalent to the equation:

$$\begin{aligned}\frac{d|\psi(t)\rangle}{dt} &= \left[-\frac{i}{\hbar}H_0 + \int d^3x\,N(\mathbf{x})w(\mathbf{x},t) - \right. \\ &\left. - 2\gamma\int d^3x\,N(\mathbf{x})\int_{t_0}^{t} ds\,h(t-s)\frac{\delta}{\delta\overline{w}(\mathbf{x},s)}\right]|\psi(t)\rangle,\end{aligned} \tag{9.55}$$

with $N(\mathbf{x})$ defined by (8.1).

[43] For the physically interesting cases, e.g. for the dynamical evolution of macrosystems, such an assumption is justified by the fact that the effect of the reduction is much faster than the typical times in which the Hamiltonian can induce appreciable dynamical changes of the statevector.

### 9.5.1 Dynamics for macroscopic rigid bodies

As for CSL, it is not difficult to discuss the physical implications of equation (9.48) — or equation (9.55) — for the case of an almost rigid macroscopic body, i.e. a body such that the wavefunctions of its constituents can be considered very well localized with respect to the localization length $1/\sqrt{\alpha}$. Under the same assumptions of section 8.2, we obtain that if $|\Psi\rangle$ and $|\chi\rangle$ satisfy the equations

$$\begin{aligned}\frac{d|\Psi(t)\rangle}{dt} &= \left[-\frac{i}{\hbar}H_{\mathbf{Q}} + \int d^3x\,\mathcal{N}(\mathbf{x})\overline{w}(\mathbf{x},t) - \right. \\ &\left. - 2\gamma\int d^3x\,\mathcal{N}(\mathbf{x})\int_{t_0}^{t} ds\,h(t-s)\frac{\delta}{\delta\overline{w}(\mathbf{x},s)}\right]|\Psi(t)\rangle, \end{aligned} \tag{9.56}$$

$$\frac{d|\chi(t)\rangle}{dt} = \left[-\frac{i}{\hbar}H_r\right]|\chi(t)\rangle, \tag{9.57}$$

then $|\psi(t)\rangle = |\Psi(t)\rangle\,|\chi(t)\rangle$ satisfies equation (9.55) or, equivalently, equation (9.48).

Equations (9.56) and (9.57) imply that the center of mass and internal motion decouple, and that the stochastic terms affect only the center of mass and not the internal structure, as it happens for CSL.

Following the same arguments of section 8.3, it can also be proven that the localization rate of the center of mass wavefunction grows linearly with the number of particles of the rigid body. In fact, by disregarding the Hamiltonian $H_0$, one derives an evolution equation for the matrix elements $\langle\mathbf{Q}'|\rho_{\mathbf{Q}}(t)|\mathbf{Q}''\rangle$ of the statistical operator of the centre of mass, which is similar to (8.23), with $\gamma(t)$ replacing $\gamma$:

$$\frac{\partial\langle\mathbf{Q}'|\rho_{\mathbf{Q}}(t)|\mathbf{Q}''\rangle}{\partial t} = -\tilde{\Gamma}(\mathbf{Q}',\mathbf{Q}'',t)\,\langle\mathbf{Q}'|\rho_{\mathbf{Q}}(t)|\mathbf{Q}''\rangle \tag{9.58}$$

with

$$\begin{aligned}\tilde{\Gamma}(\mathbf{Q}',\mathbf{Q}'',t) &= \gamma(t)\int d^3x\,\left[\frac{1}{2}F^2(\mathbf{Q}'-\mathbf{x}) + \frac{1}{2}F^2(\mathbf{Q}''-\mathbf{x}) - \right. \\ &\left. F(\mathbf{Q}'-\mathbf{x})F(\mathbf{Q}''-\mathbf{x})\right]. \end{aligned} \tag{9.59}$$

This proves that also in the present model the reduction rate of the center of mass of the system grows linearly with the number of its constituents. Moreover, taking a large value for $\beta$, as it has been suggested previously, $\gamma(t) \to \gamma$ in very short times, so that the reducing dynamics is practically the same as the one of CSL.

# Part III
# The Interpretation of GRW and CSL Models

## 10 The mass density function

In the second part of this report we have made plausibe that the Dynamical Reduction Models allow to overcome the macro-objectification problem. However, in accordance with the appropriate and strict requests by J.S. Bell (see below) about the fact that a theory must first of all make perfectly clear what it is actually about, it is necessary to supplement the formal apparatus with a precise interpretation which specifies how the

mathematical entities entering into play are related to the physical aspects of natural processes we experience. Accordingly, in this section we tackle the subtle problem of working out a consistent and unambiguous interpretation of the theoretical models under study, in the non relativistic case. We will show how, by taking advantage of their specific features, one can give a description of the world in terms of the mean values $\mathcal{M}(\mathbf{r},t)$, at different places and at different times, of appropriately defined mass density operators. The presentation is organized as follows.

We start with a historical account of J.S. Bell's contribution to the elaboration of a sensible interpretation of QMSL (subsection 10.1). Next, we introduce the mass–density function $\mathcal{M}(\mathbf{r},t)$ and we show that, within standard Quantum Mechanics, i.e. in the absence of a mechanism restricting the possible states of the Hilbert space of "our universe", one unavoidably meets situations which cannot be consistently described in terms of $\mathcal{M}(\mathbf{r},t)$ (subsection 10.2).

Fortunately, since the universal dynamics of the reduction models does not permit the persistence for [32] *more than a split second* of the just mentioned unacceptable states, it allows to identify the function $\mathcal{M}(\mathbf{r},t)$ as the basic element for the description of the world (subsection 10.3). In terms of it one can define an appropriate "topology" (subsection 10.4) which is the natural candidate for establishing a satisfactory psycho–physical correspondence. We conclude the section with a general discussion of how a theory should describe the physical world, and how this is accomplished by dynamical reduction models (subsection 10.5).

## 10.1 The position of J. Bell about dynamical reduction models

J. S. Bell has always been in the forefront of the struggle for clarifying the conceptual status of quantum theory. He has repeatedly stressed the points he considered as essential to have "an exact theory" i.e., in his words [87], one which *neither needs nor is embarrassed by an observer*. We may appropriately recall some of his more passionate statements [120]:

> A good word is 'beable' from the verb 'to be', 'to exist'. In your theory you should identify some things as being really there, as distinct from the many mathematical concepts you can easily devise — like the projection of the side of a triangle to infinity and so on. We must decide that some things are really there and that you are going to take them seriously. These are the beables, and if you are going eventually to have 'observers', for example, they must be made out of beables. ... Another good word is 'kinematics'. Many accounts of quantum mechanics start by telling you how to calculate probabilities; and I consider them to be dynamics. The kinematics should list the possibilities that you are envisaging, then afterwords you can attach probabilities to the different possibilities. I won't accept as a list of possibilities the 'possible results of experiments', because that is to try again to begin with these vague concepts. I would want the kinematics of your theory tell me what it is you are talking about before you tell me what about it.

And it is just in the spirit of the above sentences that he has analyzed the GRW theory, a theory that [120]:

> looks like a rather neat resolution of the problem of quantum mechanics. It is very close to what one does in practice, but instead of having this funny jump at an arbitrarily defined act of 'measurement', it has it as something which happens all the time and more often in systems which are big — big in a way which is controlled by the parameters of the theory ... .

Since his first writing on this theory [32], Bell has proposed an interesting interpretation for it in terms of beables:

> There is nothing in this theory but the wavefunction. It is in the wavefunction that we must find an image of the physical world, and in particular of the arrangement of things in ordinary 3–dimensional space. But the wavefunction as a whole lives in a much bigger space, of $3N-$dimensions. It makes no sense to ask for the amplitude or phase or whatever of the wavefunction at a point in ordinary space. It has neither amplitude nor phase nor anything else until a multitude of points in ordinary 3–space are specified. However, the GRW jumps (which are part of the wavefunction, not something else) are well localized in ordinary space. Indeed each is centered on a particular spacetime point $(\mathbf{x}, t)$. So we can propose these events as the basis of the 'local beables' of the theory. These are the mathematical counterparts in the theory to real events at definite places and times in the real world (as distinct from the many purely mathematical constructions that occur in the working out of physical theories, as distinct from things which may be real but not localized, and as distinct from the 'observables' of other formulations of quantum mechanics, for which we have no use here). A piece of matter then is a galaxy of such events. As a schematic psychophysical parallelism we can suppose that our personal experience is more or less directly of events in particular pieces of matter, our brains, which events are in turn correlated with events in our bodies as a whole, and they in turn with events in the outer world.

The suggestion about the possibility of establishing an appropriate psycho–physical parallelism has been subsequently proved [121] to be perfectly appropriate (see also the discussion of section 12): a perception process requires the displacement of a certain number of particles within our brain. The definite perception corresponds therefore unambiguously to a mini–galaxy of localizations in the axons or the cerebral cortex, such mini–galaxies referring to different 'brain regions' according to the precise situation which triggers the perception[44]. This is the precise sense in which the GRW theory allows to 'close the circle', i.e., it yields a picture of natural processes which agrees with quantum predictions at the micro–level but also with our definite perceptions and conceptualizations at the macroscopic one. We do not see what more, on an ontological basis, can be required from a theory of natural processes.

Starting from these remarks we can now pass to discuss what we have denoted as the mass density interpretation and raise the question: why can't one take the mass density function seriously as the "local beable" of the theory and, in particular, replace the 'galaxy' of definite localizations occurring in definite space regions with the existence of precise regions in which the mass density is "accessible" (the exact meaning of this expression will be discussed later) and the 'mini–galaxies' of localizations in our bodies and our brains with the locations of the macroscopic number of ions which are involved in the transmission of a nervous signal and in triggering the conscious perception?

Before coming to deepen this point we have to mention that subsequently J. Bell himself has slightly changed his mind. In reference [2] he wrote:

> The GRW type theories have nothing in their kinematics but the wavefunction. It gives the density (in a multidimensional configuration space!)

[44] Actually, even triggering processes involving superpositions of states of microsystems which are such to induce precise perceptions (due to the extreme sensitivity, e.g. of the visual process) lead to definite perceptions and not to a confused state of mind, just because they imply different modalities of the localization processes in the brain (see reference [121]).

> of stuff. To account for the narrowness of that stuff in macroscopic dimensions, the linear Schrödinger equation has to be modified, in the GRW picture by a mathematically prescribed spontaneous collapse mechanism.

One of us (G.C.G.) has exchanged with him various letters devoted to deepening this point. In a letter of October 3, 1989, J.S. Bell wrote:

> As regards $\Psi$ and the density of stuff, I think it is important that this density is in the $3N-$dimensional configuration space. So I have not thought of relating it to ordinary matter or charge density in 3−space. Even for one particle I think one would have problems with the latter. So I am inclined to the view you mention 'as it is sufficient for an objective interpretation' ... And it has to be stressed that the 'stuff' is in $3N-$space — or whatever corresponds in field theory.

This concludes our analysis of the stimulating suggestions of J.S. Bell about the interpretation of dynamical reduction theories.

As already anticipated, in the next subsection we will analyze a different and more precise proposal, based on the consideration of an appropriately averaged mass density distribution in ordinary space, which has been put forward in ref. [68]. Such a proposal, among other interesting features, will make perfectly clear the links, at the macroscopic level, between the formal elements which characterize the states of macroscopic systems according to the theory, the properties which can be considered as objectively possessed by them and the practical way of testing such properties. In fact one could remark that J.S. Bell has been not wholly explicit about the problem of establishing precise relations between the specific formal features characterizing macro–systems (*the narrowness of the stuff*) and the experiments aimed to ascertain the associated properties, and has simply suggested (*as it is sufficient for an objective interpretation*) that it could be easily settled within the GRW theory.

## 10.2 Mass density function

In order to prepare the grounds for the analysis we are going to perform it is useful to recall that the universal dynamics of dynamical reduction models strives to make some precise properties as objectively possessed by individual physical systems. The very fact that no analogous mechanism is at work within the standard theory forbids to adopt, within such a theory, the natural interpretation we are going to propose as the basic ontology for CSL. As we have discussed in great details, the Dynamical Reduction Theories make almost definite the positions of massive particles. Accordingly, the natural quantity which will end up having an objective (i.e. independent of any measurement process) value is the locally averaged (over the characteristic volume of the theory, i.e., $10^{-15} cm^3$) mass distribution of the universe.

In this subsection we characterize in a mathematically precise way the $c$-number function representing the just mentioned average mass density in ordinary 3-dimensional space, the quantity to which we will attach an absolutely prominent role for the interpretation of the theory. We will also make clear that, as already mentioned, within standard quantum mechanics such a quantity exhibits problematic features (which parallel the ones connected with the macro-objectification problem) implying that an ontology based on the mass density function cannot be consistently adopted within such a theory. The clarification of this important point will pave the road for the proof that, on the contrary, such an ontology leads, within the dynamical reduction formalism, to a clear, precise and fully consistent worldview which fits perfectly our experience with the reality around us.

To begin with, let us then consider a physical system $S$ which will constitute "our universe" and let us denote by $\mathcal{H}(S)$ the associated Hilbert space. Let $|\psi(t)\rangle$ be the

normalized statevector describing our individual system at time $t$; in terms of it we define an average mass density $c$–number function $\mathcal{M}(\mathbf{r},t)$ in ordinary space as

$$\mathcal{M}(\mathbf{r},t) \quad = \quad \langle\psi(t)|M(\mathbf{r})|\psi(t)\rangle, \tag{10.1}$$

where $M(\mathbf{r},t)$ is the mass density operator defined in section 8.6. Equation (10.1) establishes, for a given t, a mapping of $\mathcal{H}(S)$ into the space of positive and bounded functions of $\mathbf{r}$.

Obviously this map is many to one; in particular, to better focus on this point as well as for making clear the difficulties of the standard theory with such a function, it turns out to be useful to compare two statevectors $|\psi^{\oplus}\rangle$ and $|\psi^{\otimes}\rangle$ defined as follows. Let us consider a very large number $N$ of particles and two space regions $A$ and $B$ with spherical shape and radius $R$. The state $|\psi^{\oplus}\rangle$ is the linear superposition, with equal amplitudes, of two states $|\psi^A_N\rangle$ and $|\psi^B_N\rangle$ in which the $N$ particles are well localized with respect to the characteristic length ($10^{-5}$ cm) of the model and uniformly distributed in regions $A$ and $B$, respectively, in such a way that the density turns out to be of the order of 1 gr/cm$^3$. On the other hand, $|\psi^{\otimes}\rangle$ is the tensor product of two states $|\phi^A_{N/2}\rangle$ and $|\phi^B_{N/2}\rangle$ corresponding to $N/2$ particles being distributed in region $A$ and $N/2$ in region $B$, respectively:

$$|\psi^{\oplus}\rangle \;=\; \frac{1}{\sqrt{2}}\left[|\psi^A_N\rangle + |\psi^B_N\rangle\right] \qquad\qquad |\psi^{\otimes}\rangle \;=\; |\phi^A_{N/2}\rangle \otimes |\phi^B_{N/2}\rangle \tag{10.2}$$

It is trivial to see that the two considered states give rise to the same function $\mathcal{M}(\mathbf{r})$ and it is clear that if one attempts to attach some meaning to it one has to be very careful in keeping in mind from which state $\mathcal{M}(\mathbf{r})$ originates.

In particular, it is quite obvious that in the case of $|\psi^{\oplus}\rangle$, $\mathcal{M}(\mathbf{r})$ cannot be considered as describing an "actual" mass density distribution. To see this, let us suppose that one can use standard quantum mechanics to describe the gravitational interaction between massive bodies and let us consider the following gedanken experiment: a test mass is sent through the middle point of the line joining the centers of regions $A$ and $B$ with its momentum orthogonal to it (see figures 2a and 2b).

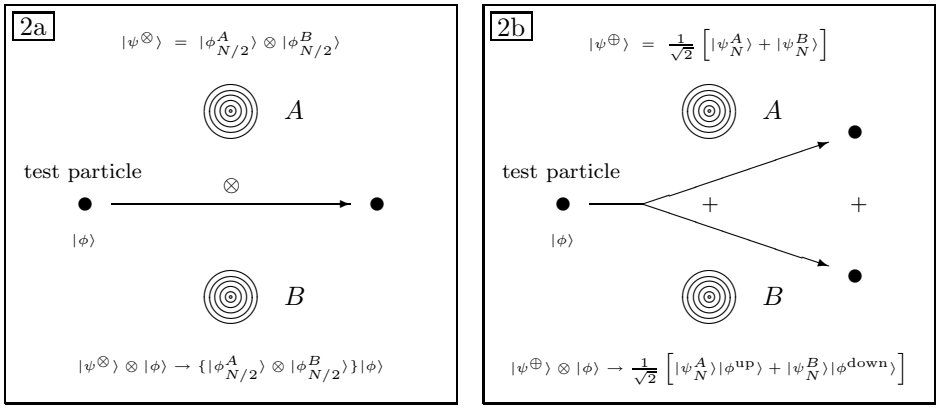



Figure 2: Accessible and non–accessible mass density distribution $\mathcal{M}(\mathbf{r})$. In case 7a, corresponding to the factorized state $|\psi^{\otimes}\rangle$, the mass density in regions $A$ and $B$ is accessible and the test particle, interacting with $|\psi^{\otimes}\rangle$, behaves in such a way as to give rise to the appropriate density along its natural trajectory. In case 7b, corresponding to the superposition $|\psi^{\oplus}\rangle$, the densities in $A$ and $B$ are non–accessible and the same holds for the density distribution generated by the interaction of the test particle with $|\psi^{\oplus}\rangle$.

In the case of the state $|\psi^{\otimes}\rangle$ for the system of the $N$ particles standard quantum mechanics predicts that the test particle will not be deflected. On the other hand, if

the same test is performed when the state is $|\psi_N^A\rangle$ ($|\psi_N^B\rangle$), quantum mechanics predicts an upward (downward) deviation of the test particle. Due to the linear nature of the theory this implies that if one would be able to prepare the state $|\psi^{\oplus}\rangle$ the final state would be

$$|\psi\rangle \quad = \quad \frac{1}{\sqrt{2}}\left[|\psi_N^A\rangle \otimes |\psi^{\rm UP}\rangle \; + \; |\psi_N^B\rangle \otimes |\psi^{\rm DOWN}\rangle\right] \tag{10.3}$$

with obvious meaning of the symbols. If one includes the test particle into the "universe" and considers the mass density operator in regions corresponding to the wavepackets $|\psi^{\rm UP}\rangle$ and $|\psi^{\rm DOWN}\rangle$, one discovers once more that nowhere in the universe one can "detect" or "perceive" a density corresponding to the density of the test particle. In a sense, if one would insist in giving a meaning to the density function he would be led to conclude that the particle has been split by the interaction into two pieces of half its density. This analysis shows that great attention should be paid in assuming that the function $\mathcal{M}(\mathbf{r})$ describes the actual state of affairs.

Before going on we consider also another quantity which will be useful in what follows. It is the mass density variance at $\mathbf{r}$ at time $t$ defined by the following map from $\mathcal{H}(S)$ into $\mathbf{R}^3$:

$$\mathcal{V}(\mathbf{r},t) \quad = \quad \langle\psi(t)|[M(\mathbf{r}) - \langle\psi(t)|M(\mathbf{r})|\psi(t)\rangle]^2|\psi(t)\rangle \tag{10.4}$$

$|\psi(t)\rangle$ being a normalized statevector.

With these premises we have all the elements which are necessary to discuss the problems one meets when dealing with $\mathcal{M}(\mathbf{r})$ and the way to overcome them. We will do this in the next subsection. Before doing that, we consider it appropriate to simply mention the obvious fact that the states giving rise to puzzling, non objective, density functions are those corresponding to superpositions of differently located macroscopic bodies, i.e. the infamous states which are at the centre of the long debated problems about the meaning of quantum mechanics at the macro–level.

For future purposes it is useful to introduce a mathematical criterion which permits to clarify the different status of the mass densities in the two above considered cases (corresponding to the states $|\psi^{\oplus}\rangle$ and $|\psi^{\otimes}\rangle$, respectively). This is more easily expressed by resorting to a discretization of space in analogy with what has been done in section 8.5. Obviously, in place of the space functions $\mathcal{M}(\mathbf{r},t)$ and $\mathcal{V}(\mathbf{r},t)$ we will consider the mean value $\mathcal{M}_i(t)$ and the variance $\mathcal{V}_i(t)$ of the mass operator in the $i$–th cell. For any cell $i$ we define the ratio:

$$\mathcal{R}_i^2 \quad = \quad \mathcal{V}_i/\mathcal{M}_i^2 \tag{10.5}$$

We then state that the mass $\mathcal{M}_i$ is "accessible" if $\mathcal{R}_i$ turns out to be much smaller than one, that is:

$$\mathcal{R}_i \quad \ll \quad 1 \tag{10.6}$$

This criterion is clearly reminiscent of the probabilistic interpretation of the statevector in standard quantum mechanics. Actually, within such a theory equation (10.6) corresponds to the fact that the spread of the mass operator $M_i$ is much smaller than its mean value. Even though in this paper we take a completely different attitude with respect to the mean value $\mathcal{M}_i$, it turns out to be useful to adopt the above criterion also within the new context. In fact, as we will discuss in what follows, when one has a space region such that for all cells contained in it (10.6) holds, it behaves as if it would have the "classical" mass corresponding to $\mathcal{M}_i$. This remark should clarify the reason for having characterized as "accessible" the mass (or equivalently the mass density) when the above conditions are satisfied.

With reference to the previous example we stress that in the case of $|\psi^{\otimes}\rangle$ all cells within regions $A$ and $B$ are such that criterion (10.6) is very well satisfied. In the

case of $|\psi^{\oplus}\rangle$ one has for the same cells:

$$\mathcal{M}_i \simeq \frac{n}{2} M_0, \qquad \mathcal{V}_i \simeq \frac{n^2}{4} M_0^2 \tag{10.7}$$

where $n$ is the number of particles per cell. It follows that

$$\mathcal{R}_i \simeq 1. \tag{10.8}$$

## 10.3 The mass density function within dynamical reduction models

In the previous subsection we have presented a meaningful example of the difficulties one meets when one keeps the standard quantum dynamics and tries to base a description of the world and an acceptable psycho–physical correspondence on the mass density function $\mathcal{M}(\mathbf{r})$. The unacceptable features find their origin in the fact that, when the macrostate is $|\psi^{\oplus}\rangle$, while the density function takes the value of about $1/2$ gr/cm$^3$ within regions $A$ and $B$, if one performs a measurement of the density in the considered regions, or if a measurement like process (such as the passage of the test particle in between $A$ and $B$) occurs, things proceed in such a way that is incompatible with the above density value. Actually one could state that no outcome emerges in the measurement. To understand fully the meaning of this statement one could identify, e.g., the final position of the test particle with a pointer reading; the pointer would then not point to the middle position (corresponding to equal densities in $A$ and $B$) but would be split into "two pointers of half density" pointing upward and downward, respectively (compare with figure 7b).

If one takes an analogous attitude with reference to dynamical reduction theories one does not meet the same difficulties because they imply that linear superpositions of states corresponding to far–apart macroscopic systems are dynamically suppressed in extremely short times and measurements have outcomes. Therefore, we can guess that, within the context of the dynamical reduction program, the description of the world in terms of the mass density function $\mathcal{M}(\mathbf{r})$ is a good description precisely because it becomes dynamically accessible at the macroscopic level; moreover it is such as to allow one to base on it a sensible psycho–physical correspondence.

Obviously, if one naively looks at the dynamical reduction models by sticking to some sort of "classical" ontology, one can be tempted to claim that some fuzzy situations can occur also in this context, when the mass density, as it may very well happen for a microsystem, is not "accessible", i.e. when (in the simplified discretized version) criterion (10.6) is not satisfied. However, as we are going to show, this does not give rise to any difficulty whatsoever for the program we are furthering.

In order to show this we will examine, along the above lines, the status of the mass density function $\mathcal{M}(\mathbf{r})$ for the various possible states which are not forbidden by the reducing dynamics. We will discuss the cases of microsystems and macro–systems, and, with reference to the latter, we will identify two physically relevant classes of states which can occur. As we have done previously we will deal with a discretized space.

### 10.3.1 Microscopic systems

For the sake of simplicity, let us consider a single nucleon. As it is well known, the reducing dynamics does not forbid the persistence, for extremely long times, of linear superpositions of far–away states of the particle, typically states like:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|0,0,\ldots,1_i,\ldots,0_j,\ldots\rangle + |0,0,\ldots,0_i,\ldots,1_j,\ldots\rangle\right] \tag{10.9}$$

where $i$ and $j$ are two distinct and far apart cells. Such microscopic states which are not eigenvectors of the operators $M_i$ will be called "microscopically non definite", the term "non definite" making reference to the characteristic preferred basis of the model. As is evident from (10.9) the mean values of $M_i$ and $M_j$ are $(1/2)m_0$ and criterion (10.6) is not satisfied at the space regions of both cells. As it is (inconsistently) assumed within standard quantum mechanics and as it is (rigorously) implied by CSL, a measurement of the mass in one of these two cells would give the *definite outcome* 0 or $m_0$ with equal probability (corresponding to the fact that wavepackets of microsystems diffuse, but however the reaction of a detector devised to reveal them remains spotty) and not $(1/2)m_0$, the value taken by the density function within the considered cells. Accordingly, in the considered case the mass in the cells is not accessible. This discrepancy, this *non classical* character of $\mathcal{M}_i$ and $\mathcal{M}_j$, cannot however be considered a difficulty for the theory with the proposed interpretation, in particular, it does not forbid to take seriously, i.e. to attach an objective status to the mass density function; it simply amounts to the recognition that we cannot legitimately apply our *classical* pictures to the microworld. On the contrary we must allow [122] *microsystems to enjoy the cloudiness of waves*. The crucial point is that within the theory we are discussing, in spite of the mass density having the value $(1/2)m_0$ in both regions, any attempt to detect its value in one of them by an amplification process, implies, as a rigorous consequence of the dynamical equation governing all physical processes, that the outcome will be either 0 or $m_0$, in perfect agreement with our experience.

### 10.3.2 Macroscopic systems

The theory allows the persistence of two general classes of states for macroscopic systems, i.e. those describing almost rigid bodies with sharply defined (with respect to the characteristic length of the model) center of mass position, and those corresponding to a macroscopic number of microsystems in microscopically non definite states. Due to the fact that the center of mass of the wavefunction has, in general, non compact support, the first class obviously includes also states which, being brought in by the reducing dynamics, have "tails". The so called "tail problem" will be discussed in the following section.

States of the first class have been extensively analyzed in sections 6, 7 and 8; we have seen that superpositions of different macroscopic states are reduced — in a very short time — to one of their terms; correspondingly, the classical properties of macroscopic systems are restored. Thus, for example, a state like $|\psi^{\oplus}\rangle$ is spontaneously transformed, by the reducing dynamics, either into the state $|\psi^N_A\rangle$ or $|\psi^N_B\rangle$, i.e. into states which have an accessible mass–density distribution.

Concerning states of the second class, it is of extreme relevance to make clear that they have a conceptual status which is very different[45] from the one of the superpositions of macroscopically distinguishable states like $|\psi^{\oplus}\rangle$; moreover, they represent rather peculiar situations which mainly have an "academic" character, since states of this kind certainly do not appear often in practice. However, it is worthwhile to discuss them in some detail.

Let us consider a system of $N$ nucleons and a discretization of space in small cubes of linear dimensions $10^{-8}$ cm. We consider again two macroscopic regions $A$ and $B$, and we label by the indices $k_A$ and $k_B$ pairs of cubes within $A$ and $B$ respectively. For $k_A \neq \tilde{k}_A$ the two cubes are disjoint and the union of all cubes $k_A$ ($k_B$) covers the region $A$ ($B$). The index $k$ runs from 1 to $N$, a very large number; typically if $A$ and

[45] This important difference has already been appropriately stressed by A. Leggett [123], who, even though in a different context, has introduced the mathematically precise concept of disconnectivity to distinguish states of this type from states like $|\psi^{\oplus}\rangle$.

$B$ have volumes of the order of 1 dm$^3$, $N$ will be of the order of $10^{27}$.

Let us denote by $|\psi_{k_A}\rangle$ and $|\psi_{k_B}\rangle$ the states of a particle whose coordinate representation are well localized within $k_A$ and $k_B$, respectively. As an example we could choose

$$\langle \mathbf{r}|\psi_{k_A}\rangle \quad = \quad \chi(k_A) \tag{10.10}$$

$\chi(k_A)$ being the characteristic function of the cube $k_A$. We now consider the following microscopically non definite state for the $k$–th particle:

$$|\psi^k\rangle \quad = \quad \frac{1}{\sqrt{2}}\,[|\psi_{k_A}\rangle \,+\, |\psi_{k_B}\rangle] \tag{10.11}$$

and the factorized state of the $N$ particles

$$|\psi\rangle \quad = \quad |\psi^1\rangle \otimes \ldots \otimes |\psi^k\rangle \otimes \ldots \otimes |\psi^N\rangle. \tag{10.12}$$

In spite of the fact that the state $|\psi\rangle$ is a direct product of microscopically non definite states it is nevertheless "almost" an eigenstate of the operators $M_i$ (remember that the linear dimensions of the cell to which the index $i$ refers are of the order of $10^{-5}$ cm so that one such cell contains about $10^9$ cubes of the kind of $k_A$ ($k_B$)). In fact, denoting by $n \simeq 10^9$ the number of $k_A$ ($k_B$) small cubes contained in the $i$–th cell, one can easily see that $|\psi\rangle$ gives rise to "accessible" mass $\mathcal{M}_i$ in regions $A$ and $B$ respectively[46]:

$$\langle M_{i(A,B)}\rangle \;\simeq\; \frac{1}{2}\,n\,m_0, \qquad\qquad \langle M^2_{i(A,B)}\rangle \;\simeq\; \frac{1}{4}\,(n^2+n)\,m_0^2 \tag{10.13}$$

hence

$$\mathcal{V}^2_{i(A,B)} \;\simeq\; \frac{1}{4}\,n\,m_0^2, \qquad\qquad \mathcal{R}_{i(A,B)} \;\simeq\; \frac{1}{\sqrt{n}} \;\ll\; 1 \tag{10.14}$$

To clarify the physical implications of the state $|\psi\rangle$, from the point of view which interests us here, we can imagine performing once more the gedanken experiment with a test particle we have already considered in the previous subsection, assuming, for simplicity, that the interactions between the test particle and the considered $N$ particles do not change the state[47] of the latter. By substituting equation (10.11) into equation (10.12), we see that $|\psi\rangle$ is a superposition of $2^N$ states in which each particle is well localized. In such a superposition all states have an equal amplitude $1/\sqrt{2^N}$ and almost all states correspond to about $N/2$ particles being in regions $A$ and $B$ respectively. Therefore, in the language of dynamical reduction models, the probability of occurrence of a realization of the stochastic potential leading to the "actualization" of an almost completely undeflected trajectory for the test particle is extremely close to one[48]. This shows that the mass density function $\mathcal{M}(\mathbf{r})$ corresponding to the state behaves in a "classical way", so that no trouble arises in this case.

It has to be noted that, obviously, the mass $\mathcal{M}_i$ corresponding to the state (10.12) coincides with the one corresponding to the state $|\psi^{\otimes}\rangle$ of the previous subsection, in spite of the fact that both states are dynamically allowed and are quite different as

[46] In making the computations we have identified the operators $M_i$ with the sum of the projectors (multiplied by the nucleon mass $m_0$) of the various particles in the $i$–th cell.

[47] At any rate, possible changes in such a state would be symmetrical with respect to the middle plane, so that the subequent considerations would still hold true.

[48] It could be useful to remark that if one would analyze the same experiment in terms of the linear quantum dynamics, the test particle would end up in the linear superposition of an extremely large number of states. However, since such states correspond to trajectories which are very near and almost undeflected, the evaluation of the mass density associated to the final statevector would show that in the "middle" region there would practically be the total mass of the test particle. Therefore, this represents a case in which even without any reduction process the mass density referring to the test particle would correspond to a precise outcome of the measurement.

physical states. However, as we have shown, the masses $\mathcal{M}_i$ in the two cases behave practically in the same way and are fully unproblematic, contrary to what happens in the case of $|\psi^{\oplus}\rangle$.

Concluding, we have made plausible that in the context of the dynamical reduction program one can consistently describe the world, at a given time, in terms of the mass density function $\mathcal{M}(\mathbf{r})$ and that, due to the fact that such a function becomes accessible at the macrolevel, such a description matches our experience with the reality around us. Obviously, since with the elapsing of time the state of the world changes, a complete description requires the consideration of the motion picture of the density, i.e. of $\mathcal{M}(\mathbf{r}, t)$ defined in equation (10.1). We will discuss in greater detail this crucial point in the next section.

## 10.4 Defining an appropriate topology for the CSL model

Let us consider a system $S$ of finite mass which will constitute our "universe" and its associated Hilbert space $\mathcal{H}(S)$. We denote by $U(S)$ the unit sphere in $\mathcal{H}(S)$ and we consider the nonlinear map[49] $\mathcal{M}$ associating to the element $|\varphi\rangle$ of $U(S)$ the element $\mathbf{m} = \{\mathcal{M}_i(|\varphi\rangle)\}$ of $l_2$, $\mathcal{M}_i(|\varphi\rangle)$ being the quantity $\langle\varphi|M_i|\varphi\rangle$.

On $U(S)$ we define a topology by introducing a mapping $\Delta : U(S) \otimes U(S) \rightarrow \mathbf{R}^+$ according to:

$$\Delta(|\varphi\rangle, |\psi\rangle) \;=\; d(\mathbf{m}, \mathbf{n}) \;=\; \sqrt{\sum_i (m_i - n_i)^2} \tag{10.15}$$

where $\mathbf{m} = \{\mathcal{M}_i(|\varphi\rangle)\}$, $\mathbf{n} = \{\mathcal{M}_i(|\psi\rangle)\}$. Such a mapping is not a distance since, as it emerges clearly from the analysis of the previous subsection, it may happen that $\Delta(|\varphi\rangle, |\psi\rangle) = 0$ even though $|\varphi\rangle \neq |\psi\rangle$. However $\Delta$ meets all other properties of a distance:

$$\Delta(|\varphi\rangle, |\psi\rangle) \;=\; \Delta(|\psi\rangle, |\varphi\rangle) \;\geq\; 0 \tag{10.16}$$

and

$$\Delta(|\varphi\rangle, |\psi\rangle) \;\leq\; \Delta(|\varphi\rangle, |\chi\rangle) \;+\; \Delta(|\chi\rangle, |\psi\rangle) \tag{10.17}$$

as one easily proves by taking into account the fact that $d$ is a distance in $l_2$.

¿From now on we will limit our considerations to the proper subset $A(S)$ of $U(S)$ of those states which are allowed by the CSL dynamics. In the previous subsection we have already identified, even though in a rough way, the set $A(S)$. One could obviously be very precise about such a set by adopting e.g. the following criterion: let $|\varphi\rangle \in U(S)$, and let us consider the ensemble $A(S)(|\varphi\rangle)$ of states which have a non negligible (this obviously requires the definition of a threshold) probability of being brought in by the reducing dynamics after a time interval of the order of $10^{-2}$ sec, which is the characteristic perception time of a human being[50], for the given initial condition $|\varphi\rangle$. The union of all subsets $A(S)(|\varphi\rangle)$ for $|\varphi\rangle$ running over $U(S)$ is then $A(S)$. For our purposes, however, it is not necessary to go through the cumbersome management of a very precise definition of the set $A(S)$; the consideration of the cases we have discussed in the previous subsection is sufficient to lead to the interesting conclusions.

For any element $|\varphi\rangle$ of $A(S)$ we consider the set of states of $A(S)$ for which $\Delta(|\varphi\rangle, |\psi\rangle) \leq \epsilon$. Here the quantity $\epsilon$ has the dimensions of a mass and is chosen of the order of $10^{18} m_0$, with $m_0$ the nucleon mass. From the properties of the map $\Delta$ it follows that:

[49] To be rigorous, one should consider the map $\mathcal{M}$ from the unit sphere of $\mathcal{H}(S)$ into the space $L^2$ of the square integrable functions of $\mathbf{r}$. However, we can deal, without any loss of generality, with the discretized version of the model.

[50] See the discussion of section 12.

1. $\{\Delta(|\varphi\rangle, |\psi\rangle) \leq \epsilon \text{ and } \Delta(|\varphi\rangle, |\chi\rangle) \leq \epsilon\}$ implies $\Delta(|\chi\rangle, |\psi\rangle) \leq 2\epsilon$.

2. $\{\Delta(|\varphi\rangle, |\psi\rangle) \gg \epsilon \text{ and } \Delta(|\varphi\rangle, |\chi\rangle) \leq \epsilon\}$ implies $\Delta(|\chi\rangle, |\psi\rangle) \gg \epsilon$.

We have introduced the parameter $\epsilon$ in such a way that it turns out to be sensible to consider similar to each other states whose "distance" $\Delta$ is smaller than (or of the order of) $\epsilon$. More specifically, when

$$\Delta(|\varphi\rangle, |\psi\rangle) \quad \leq \quad \epsilon \tag{10.18}$$

we will say that $|\varphi\rangle$ and $|\psi\rangle$ are "physically equivalent". More about this choice in what follows.

To understand the meaning of this choice it is useful to compare it with the natural topology of $\mathcal{H}(S)$. We begin by pointing out the inappropriateness of the Hilbert space topology to describe the concept of similarity or difference of two macroscopic states. In fact suppose our system $S$ is an almost rigid body and let us consider the following three states: $|\varphi^A\rangle$, $|\varphi^B\rangle$ and $|\tilde{\varphi}^A\rangle$. The state $|\varphi^A\rangle$ corresponds to a definite internal state of $S$ and to its center of mass being well localized around $A$, the state $|\varphi^B\rangle$ is simply the translated of $|\varphi^A\rangle$ so that it is well localized in a distant region $B$, the state $|\tilde{\varphi}^A\rangle$ differs from $|\varphi^A\rangle$ simply by the fact that one or a microscopic number of its "constituents" are in states which are orthogonal to the corresponding ones in $|\varphi^A\rangle$.

It is obvious that, on any reasonable assumption about similarity or difference of the states of the universe, $|\tilde{\varphi}^A\rangle$ must be considered very similar (identical) to $|\varphi^A\rangle$ while $|\varphi^B\rangle$ must be considered very different from $|\varphi^A\rangle$. On the other hand, according to the Hilbert space topology

$$\| |\varphi^A\rangle - |\tilde{\varphi}^A\rangle \| \;=\; \| |\varphi^A\rangle - |\varphi^B\rangle \| \;=\; \sqrt{2} \tag{10.19}$$

This shows with striking evidence that the Hilbert space topology is totally inadequate for the description of the macroscopic world. As a consequence such topology is also quite inadequate to base on it any reasonable psycho-physical correspondence.

We now discuss the "distorted" (with respect to the Hilbert space one) topology associated to the "distance" $\Delta$. First of all we stress that the two states $|\varphi^A\rangle$ and $|\tilde{\varphi}^A\rangle$ which are maximally distant in the Hilbert space topology, turn out to be equivalent, i.e. to satisfy condition (10.18) in the new topology. This represents an example showing how such a topology takes more appropriately into account the fact that, under any sensible assumption, the "universes" associated to the considered states are very similar.

Obviously, one problem arises. Criterion (10.18) leads us to consider as equivalent states which are quite different from a physical point of view, even at the macroscopic level. To clarify this statement we take into account two states $|\varphi\rangle$ and $|\psi\rangle$ corresponding to an almost rigid body located, at $t = 0$, in the same position but with macroscopically different momenta, let us say $P = 0$ and $P$, respectively. Even though the two states are physically quite different, their distance at $t = 0$ is equal to zero. However, if one waits up to the time in which the state $|\psi\rangle$ has moved away from $|\varphi\rangle$, the "distance" $\Delta(|\varphi(t)\rangle, |\psi(t)\rangle)$ becomes large and the two states are no longer equivalent. We will discuss the now outlined problem in great details in the next section.

Before concluding this part it is important to analyze the case of two states $|\psi\rangle$ and $|\psi_T\rangle$ such that $|\psi\rangle$ corresponds to an almost rigid body with a center of mass wavefunction which is almost perfectly localized while $|\psi_T\rangle$ corresponds to the same body with a "tail" in a distant region. As we have already discussed, the CSL dynamics allows the existence of this latter type of states; however it tends to depress more and more the tail in such a way as to make the mass in the distant region extremely close to zero (much less than one nucleon mass) in very short times. As a

consequence, according to the topology that we propose the two states $|\psi\rangle$ and $|\psi_T\rangle$ turn out to be identical. This is quite natural. In fact, in the same way in which taking away a single particle from a macroscopic system would be accepted as being totally irrelevant from a macroscopic point of view, when one chooses, as we do, to describe reality in terms of mass density, one must consider as equivalent situations in which their difference derives entirely from the location of a small fraction of the mass of a nucleon in the whole universe. We remark that $|\psi\rangle$ and $|\psi_T\rangle$ are extremely close to each other also in the standard Hilbert space topology.

## 10.5 Deepening the proposed interpretation

We consider it appropriate to devote this subsection to discuss in great generality the problem of giving an acceptable description of the world within a given theory. Usually one tries to do so by resorting to the notion of observable. As repeatedly remarked, such an approach meets, within standard quantum mechanics, serious difficulties since the formal structure of the theory allows only probabilistic statements about measurement outcomes conditional under the measurement being performed. In brief, the theory deals with *what we find* not with *what is*. This is why J.S. Bell has suggested [124] replacing the notion of observable with the one of "beable", from the verb to be, to exist. Obviously, the identification of the beables, of what is real, requires the identification of appropriate formal ingredients of the theory we are dealing with.

### 10.5.1 The case of the Pilot–Wave theory

To clarify our point, it turns out to be useful to analyze the de Broglie–Bohm Pilot–Wave theory. It describes the world in terms of the wavefunction and of the actual positions of the particles of our "universe", each of which follows a definite trajectory. Therefore, in such a theory it is quite natural to consider as the beables the positions (which are the local elements accounting for reality at a given instant) and the wavefunction (which is nonlocal and determines uniquely the evolution of the positions). It is important to remember that, within the theory under discussion, all other "observables" (in particular, e.g. the spin variables) turn out, in general, to be contextual. This simply means that *the truth value* of a statement about the outcome of the measurement of one such observable (which in turn is simply a statement about the future positions of some particles) may in general depend (even nonlocally) on the *overall* context. This obviously implies that the attribution of a value to the considered observable cannot be thought as corresponding, in general, to an "objective property" of the system.

Before coming to discuss the problem of the beables within CSL we would like to call attention to the fact that [87] within the Pilot–Wave theory, one can construct, from the microscopic variables $\mathbf{r}$, macroscopic variables $\mathbf{R}$ including pointer positions, images on photographic plates etc. Obviously this requires some fuzziness, but such a limitation is not relevant for a consistent account of reality. Thus, in this theory we are led to suppose that it is from the $\mathbf{r}$, rather than from the wavefunction, that the observables we use to describe reality are constructed. The positions are also the natural candidates to be used in defining a psycho–physical parallelism, if we want to go so far. An appropriate way to express the now discussed features of the theory derives from denoting, as J.S. Bell proposed, as "exposed variables" the positions of particles and as a "hidden variable" the wavefunction $\psi$.

### 10.5.2 The case of CSL

Let us now perform a corresponding analysis for the dynamical reduction models. Since, as should be clear from the discussion given in the previous subsection, the most relevant feature of the modified dynamics is that of suppressing linear superpositions corresponding to different mass distributions, one is actually led to identify as the local beables of the theory the mass density function $\mathcal{M}(\mathbf{r}, t)$ at a given time. Obviously, also within CSL just as for the Pilot–Wave the wavefunction plays a fundamental role for the evolution so that it too acquires the status of a nonlocal beable.

It has to be remarked that in the interpretation we are proposing, even though the wavefunction is considered as one of the beables of the theory, the "exposed variables" are the values of the mass density function at different points. It is then natural to relate to them, as we have done in the previous section, the concept of similarity or difference between universes.

In doing so one is led to consider equivalent, at a fixed time $t$, two "universes" which are almost identical in the exposed beables (i.e. they satisfy the condition (10.18)). Obviously the fact that the above condition holds at $t$ does by no means imply that the two universes will remain equivalent as time elapses. It has to be stressed that the just mentioned feature is not specific of the model and the interpretation we are proposing, but is quite general and occurs whenever one tries to make precise the idea of "similarity" of physical situations. In fact within all theories we know, and independently of the variables we choose to use to define nearness, situations can occur for which *nearby states* at a given time can evolve in extremely short times in *distant states*.

To focus on this important fact we can consider even classical mechanics with the assumption that both positions and momenta are the beables of the theory[51]. As it is obvious, even if such an attitude is taken there are at least two reasons for which nearby points in phase space can rapidly evolve into distant ones. First of all one must take into account that many systems exhibit dynamical instability so that the distance between "trajectories" grows exponentially with time. Secondly, even for a "dynamically standard" situation one can consider cases in which just the present conditions can give rise to completely different evolutions depending on some extremely small difference in the whole universe. Suppose in fact you consider two universes $A$, $\tilde{A}$ differing only in the *direction* of propagation of a single particle (such universes have to be considered as very close in any sensible objective interpretation). If the trajectory of the considered particle in $\tilde{A}$ is such that in a very small time it triggers e.g. the discharge of a Geiger counter, which in turn gives rise to some relevant macroscopic effect, while in $A$ it does not, the evolved universes soon become quite different. An analogous argument obviously holds for standard quantum mechanics, the Pilot–Wave theory and, as previously remarked, for CSL too.

It is appropriate to stress that, in a sense, the above considerations favor taking a position about reality which can be described in the following terms. One considers the sensible "beables" for its theory at a fixed time and one distinguishes similar or different universes on the basis of such a snapshot. Obviously, one must then also pay attention to the way in which the beables evolve, i.e. to compare snapshots at different times[52].

[51] Obviously, within classical mechanics any function of these variables can be considered as a beable, but since all information about the system can be derived from the positions and the momenta, consideration of such variables is sufficient.

[52] From this point of view, one could state that also the classical world would be most appropriately described in terms of positions at fixed time.

### 10.5.3 The role of mass density

The previous analysis has shown that the proposed interpretation (mass density = exposed beable) can be consistently taken within CSL. Obviously it gives an absolutely prominent role to the mass in accordance with the fact that mass is the handle by which the reduction mechanism induces macro–objectification.

Other features of natural phenomena, such as the effects related to the charge are, in a sense, less fundamental since to become objective they need mass as a support. To clarify this point we remark that one could consider, e.g., a condenser with two plates of about 1 $cm^2$, at a distance of 1 cm. The plates are supposed to be perfectly rigid and in perfectly defined positions[53]. Let us also consider the following *gedanken* situation: the condenser can be prepared in the superposition of two states, $|C_0\rangle$ and $|C_c\rangle$, the first corresponding to its plates being neutral, the second to its plates having being charged by displacing $10^{12}$ *electrons* from one plate to the other. We remark that for the two states the decoupling rate (recall that electrons, which are very light, are quite ineffective in suppressing superpositions) is about $10^{-8}$ $sec^{-1}$, i.e. that the superposition can persist for more than ten years. The electric field within the plates is zero or about $10^8$ V/m in the two states, respectively. Suppose now we consider a small sphere of radius $10^{-5}$ cm and density $10^{-2}$ $gr/cm^3$ carrying a charge corresponding to $10^4$ electrons. We send such a test particle through the plates of the condenser. What happens? The final state is the entangled state

$$|\psi(t)\rangle \;=\; \frac{1}{\sqrt{2}}\,[|C_0\rangle\,|\text{undeflected}\rangle \;+\; |C_c\rangle\,|\text{deflected}\rangle] \tag{10.20}$$

the location of the particle in the state $|\text{undeflected}\rangle$ and $|\text{deflected}\rangle$ differing by macroscopic amounts. According to the CSL model of section 8.6, one can easily evaluate the rate of suppression of the superposition. As already remarked the contribution of the electrons on the plates is totally negligible so that the decoherence is governed mainly by the mass of the particle. Then, with the above choices for the radius and the density of the test particle, the superposition will persist for more than one minute. In spite of the fact that macroscopically relevant forces enter into play no reduction takes place for such a time interval. On the contrary, if we put the same charge on a particle of normal density and of radius $10^{-3}$ cm, we see that the macroscopic force acting on it when the condenser is in the state $|C_c\rangle$ leads to a displacement of the order of its radius in about $10^{-5}$ sec and that within the same time the reducing effect of the dynamics suppresses one of the two terms of the superposition.

This example is quite enlightening since it shows that superpositions of charge distributions generating different forces which are relevant at the macroscopic level, are not suppressed unless they induce displacements of masses. It goes without saying that any attempt to relate reduction to charge is doomed to fail since it will not suppress superpositions of macroscopically different but electrically neutral mass distributions.

We hope to have made clear, with this perhaps tedious analysis, the real significance of treating the mass function as the "exposed beables" allowing one to describe reality.

[53] This assumption must be made because we are just discussing the role of the charge with respect to the one of the mass within the model. If one would allow deformations and/or displacements of the plates, once more the ensuing reduction would be due to the mass and not directly to the charge density difference in states $|C_0\rangle$ and $|C_c\rangle$.

# 11 The "tail problem" in dynamical reduction models

Dynamical reduction models have been repeatedly criticized because the reducing dynamics does not lead, in the case of macroscopic systems, to perfectly localized wavefunctions, i.e. to wavefunctions having a compact support corresponding to a small (with respect to the characteristic length $1/\sqrt{\alpha}$) volume of space. On the contrary, wavefunctions describing macroscopic systems always have (very small, as we shall see) "tails" spreading out to infinity.

Due to this feature, many authors [125, 45, 126, 127, 128] have suggested that dynamical reduction models do not guarantee the emergence of a objective (and classical) world at the macroscopic level, a world in which macroscopic systems occupy a precise position in space.

In the present section we will show why the "tail problem" is not a problem within QMSL and CSL. After a brief introduction (subsection 11.1), in subsection 11.2 we list the criticisms which have been put forward by the above quoted authors. In subsection 11.3 we give a quantitative estimate of the order of magnitude of the tails, in the case of a macroscopic system, showing that they represent an extremely small portion of the wavefunction.

In subsections 11.4 and 11.5 we reply to the criticisms, by means of the mass interpretation introduced in the previous section: the tails of the wavefunction do not forbid the (classical) description of macroscopic systems. However, this does not mean that the tails do not have any physical relevance, as we will prove in subsection 11.6. We conclude the section with a classical analog of the tail problem (subsection 11.7).

## 11.1 Historical remarks

J. Bell vividly described Schrödinger's trials to give a consistent interpretation of the wavefunction [2]:

> In the beginning Schrödinger tried to interpret his wavefunction as giving somehow the density of stuff of which the world is made. He tried to think of an electron as represented by a wavepacket ... a wavefunction appreciably different from zero only over a small region in space. The extension of that region he thought of as the actual size of the electron ... his electron was a bit fuzzy. At first he thought that small wavepackets, evolving according to the Schrödinger equation, would remain small. But that was wrong. Wavepackets diffuse, and with the passage of time become indefinitely extended, according to the Schrödinger equation. But however far the wavefunction has extended, the reaction of a detector to an electron remains spotty. So Schrödinger's "realistic" interpretation of his wavefunction did not survive.
>
> Then came the Born interpretation. The wavefunction gives not the density of stuff, but gives rather (on squaring its modulus) the density of probability. Probability of what, exactly? Not of the electron being there, but of the electron being found there, if its position is "measured".
>
> Why this aversion to "being" and insistence on "finding"? The founding fathers were unable to form a clear picture of things on the remote atomic scale. They became very aware of the intervening apparatus, and of the need for a "classical" base from which to intervene on the quantum system. And so the shifty split.

Some remarks having a direct connection with the central problem of the present section emerge quite naturally. Schrödinger too, just as von Neumann, was certainly aware of the fact that the electron wavefunction cannot have compact support, according to his equation. In spite of that he was quite keen to interpret a well localized wavefunction as describing a "bit fuzzy" electron, or better the "stuff" of which the electron is made (its mass and charge density). The compelling reasons for abandoning such a position did not come, as we all know, from the fact that even extremely well localized wavefunctions have tails, but, as appropriately stressed by Bell, from the fact that well concentrated wavepackets become appreciably different from zero over macroscopic regions in extremely short times. Thus, Schrödinger's realistic position had to be abandoned, to be replaced by the probabilistic Born interpretation.

There is no doubt that Schrödinger too was perfectly aware of the fact that the integral over the whole space of the squared modulus of the wavefunction does not change with time, as a very consequence of Schrödinger's equation. Consequently he certainly had perfectly clear that by adopting the "density of stuff" interpretation he had to accept also that, in spite of the fact that his electron was there (in the small region in space in which its wavefunction was concentrated), a negligible part of its mass (or charge) would not be confined to that region. The extremely relevant difference of the model theories we are analyzing here with the case of Schrödinger, however, derives from the fact that superpositions of functions of macro–systems, appreciably different from zero over macroscopic distances, are dynamically forbidden within QMSL, contrary to Schrödinger's case.

## 11.2 Criticisms about the "tail problem"

As the reader has certainly grasped, the localization of the wavefunction does not lead to an infinitely precise localization of the pointer. Actually, after a localization the wavefunction (like all conceivable wavefunctions, both within standard Quantum Mechanics and CSL) unavoidably exhibits tails extending over the whole space. In fact, since $|\psi\rangle$ has a noncompact support in the position representation, multiplying it times a Gaussian leaves it different from zero everywhere[54] (recall the example at the end of section 6.1). This fact is at the basis of the uneasiness of various people who naively transfer the ontology of standard quantum mechanics to the new theory. The first criticism of this kind has been formulated by A. Shimony [125] who has put forward many desiderata for a modified quantum dynamics, one of them being that:

> If a stochastic dynamical theory is used to account for the outcome of a measurement, it should not permit excessive indefiniteness of the outcome, where "excessive" is defined by consideration of sensory discrimination. This desideratum tolerates outcomes in which the apparatus variable does not have a sharp value, but it does not tolerate "tails" which are so broad that different parts of the range of the variable can be discriminated by the senses, even if very low probability amplitude is assigned to the tail.

It goes without saying that the perspective chosen by Shimony with respect to the dynamical reduction models is entirely based on *the standard probabilistic interpretation* of quantum mechanics (*even if very low probability amplitude is assigned to the tail*) about the possible 'outcomes' of a *measurement* process.

A quite similar criticism has been raised by Albert and Loewer [45]:

> Our worry is that GRW collapses almost never produce definite outcomes even when outcomes are recorded in distinct positions of macro-

[54] We stress that it would have been totally useless to make the localization function of compact support, since the kinetic energy part of the hamiltonian would immediately make the wavefunction different from zero everywhere.

> scopically many particles. The reason is that a GRW jump does not literally produce a collapse into an eigenstate of position. A GRW collapse yields one of the states with tails in which almost all the amplitude is concentrated in the region around one of the two components but there is nonzero, though very small, amplitude associated with other regions. ... This means that the post collapse state is not an eigenstate of position and so does not actually assign a definite position to the pointer.

Once more we stress the adherence of the authors to the standard formalism: the only variables which *have values* are those of which the statevector is an *eigenstate*. One should also note that the request that the statevector be an eigenstate of position is very peculiar: the localization mechanism should map a state of the Hilbert space onto a non–normalizable state and, at any rate, such a state would immediately spread everywhere, loosing what is considered its fundamental feature of assigning *a definite position to the pointer*. Subsequently, the same authors [45] have suggested that the GRW proposal could be saved provided one would be keen to release the eigenvector–eigenvalue link. More about this in what follows.

The tail problem leads in a straightforward way to the enumeration anomaly which is the subject of the papers of Lewis [126] and Clifton and Monton [127, 128] (who use the term 'conjunction introduction' instead of 'enumeration'). The idea is quite simple. Lewis considers a macroscopic marble and a very large box, he denotes the normalized eigenstates of the marble being inside and outside the box as $|\text{in}\rangle$ and $|\text{out}\rangle$, respectively, and he remarks that starting from a state:

$$\frac{1}{\sqrt{2}}\left(|\text{in}\rangle \ + \ |\text{out}\rangle\right) \tag{11.1}$$

the GRW dynamics will transform it, almost immediately, into a state like

$$a|\text{in}\rangle \ + \ b|\text{out}\rangle \tag{11.2}$$

or into a state like

$$b|\text{in}\rangle \ + \ a|\text{out}\rangle \tag{11.3}$$

where $1 > |a|^2 \gg |b|^2 > 0$ ($|a|^2 + |b|^2 = 1$). Lewis recalls that the GRW theory requires us to interpret each one of these states as one in which the marble is inside (in case (11.2)) or outside (in case (11.3)) the box[55]. Then he considers a system of $n$ non interacting marbles, each in a state like (11.2):

$$|\Psi\rangle_{\text{all}} \ = \ (a|\text{in}\rangle_1 + b|\text{out}\rangle_1) \otimes (a|\text{in}\rangle_2 + b|\text{out}\rangle_2) \otimes ... \otimes (a|\text{in}\rangle_n + b|\text{out}\rangle_n)\,. \tag{11.4}$$

The counting anomaly is then easily derived: for a state like (11.4) the GRW theory allows us to claim that: 'particle 1 is in the box', 'particle 2 is in the box', ..., 'particle $n$ is in the box' but for the same state the probability that all marbles *be found* in the box is $|a|^{2n}$, which, for $n$ sufficiently large, can be made arbitrarily small.

Clifton and Monton [127] agree, in principle, with Lewis, and they also prove that the suggestion of releasing the eigenvalue–eigenvector link put forward by Albert and Loewer [45] does not allow one to overcome the difficulty. In fact, what Albert and Loewer propose is to weaken the eigenvalue–eigenvector link for position according to a rule they call PosR:

> 'Particle $x$ is in region R' if and only if the proportion of the total squared amplitude of $x$'s wavefunction which is associated with points in region R is greater or equal to $1 - p$,

[55] We point out that, with reference to the above states, it would have been much more appropriate to assert that they represent a marble which is located in the precise region where its wavefunction is sharply peaked. Obviously that region is inside the box for state (11.2) and outside for state (11.3).

with an appropriately chosen (and small) $p$. Clifton and Monton suggest, first of all, to generalize PosR for a multi–particle system resorting to what they call the **fuzzy link** criterion:

> 'Particle $x$ lies in region $R_x$ and $y$ lies in $R_y$ and $z$ lies in $R_z$ ...' if and only if the proportion of the total squared amplitude of $\psi(t, r_1, ..., r_N)$ that is associated with points in $R_x \times R_y \times R_z$ ... is greater than or equal to $1 - p$.

It should be clear that, while according to the fuzzy link criterion for a state like (11.4) the propositions $A_i \equiv$ 'particle $i$ is in the box' are true for any $i$, the conjuction $A_1 \wedge A_2 \wedge ... \wedge A_n$ is false. In other words, the proposal entails a failure of *conjunction introduction*. Having remarked this, the above authors feel the necessity of pointing out that the tail problem, even though present, can never become manifest within the GRW theory. This is proved by *operationalizing the procedure of counting marbles*, i.e. by considering apparata aimed *to detect* whether particle 1 is in the box, particle 2 is in the box and so on, and another apparatus aimed *to detect* how many particles there are in the box and comparing their outcomes. Then, by some assumptions (which seem to us useless and inappropriate) one shows that a situation resembling the one of the von Neumann chain emerges: in spite of the different possible final outcomes there is always consistency between the individual and global detections.

To be more precise we recall the reasoning of reference [127]. It goes as follows:

> An ideal measurement of whether marble 1 is in the box would correlate orthogonal states of a macroscopic measuring apparatus to the $|\text{in}\rangle$ and $|\text{out}\rangle$ states of the marble.

Obviously one has to resort to $n$ such apparatuses. The evolution leads to the state:

$$(a|\text{in}\rangle_1|\text{`in'}\rangle_{M1} + b|\text{out}\rangle_1|\text{`out'}\rangle_{M1}) \otimes ... \otimes (a|\text{in}\rangle_n|\text{`in'}\rangle_{Mn} + b|\text{out}\rangle_n|\text{`out'}\rangle_{Mn}) \quad (11.5)$$

where the states $|\text{`in'}\rangle_{Mk}$ ( $|\text{`out'}\rangle_{Mk}$) are eigenstates of the observable: 'the $k$–th apparatus has recorded that the marble is inside (outside) the box'. At this stage the authors need a further apparatus $M$ (again working ideally) to see how many marbles are in the box. A new step in the chain is necessary and one ends up with an entangled state of the kind:

$$|\psi\rangle_{\text{count}} = \sum_{k=0}^{n} a^{n-k} b^k \, |\phi(n-k, \text{in}; k, \text{out})\rangle \, |\text{`}O = n-k\text{'}\rangle \quad (11.6)$$

where the last factor refers to the eigenstate of $M$ corresponding to the eigenvalue '$n-k$ particles are inside the box', and the states $|\phi(n-k, \text{in}; k, \text{out})\rangle$ are (in general) linear superpositions of states in which there are $n-k$ factors of the type $|\text{in}\rangle_j|\text{`in'}\rangle_{Mj}$ and $k$ factors $|\text{out}\rangle_s|\text{`out'}\rangle_{Ms}$. And here comes the conclusion:

> The state $|\psi\rangle_{\text{count}}$ is highly unstable given the GRW dynamics, ... since its various terms differ as to the location of the pointer on $M$'s dial that registers the value of O.

Moreover, even if the state collapses to a term $|\phi(n-k, \text{in}; k, \text{out})\rangle|\text{`}O = n-k\text{'}\rangle$, since the terms of the first factor differ as to the location of at least one marble and since the marbles and the apparata $M_r$ are macroscopic, then one would end up, e.g., with the state:

$$(|\text{out}\rangle_1|\text{`out'}\rangle_{M1}|\text{in}\rangle_2|\text{`in'}\rangle_{M2} \ldots |\text{in}\rangle_n|\text{in}\rangle_{Mn})|\text{`}O = n-1\text{'}\rangle \quad (11.7)$$

in which the records of the various individual apparata $M_s$ agree with the one of $O$. Thus, the counting anomaly cannot become manifest.

In what follows we will put forward precise motivations showing that this operationalization process is useless. But we want to call immediately the attention of the reader to an important fact. Suppose we consider, for simplicity, only two terms of one of the states $|\phi(n-k,\mathrm{in};k,\mathrm{out})\rangle$, e.g.:

$$|\mathrm{in}\rangle_1|\text{`in'}\rangle_{M1}|\mathrm{out}\rangle_2|\text{`out'}\rangle_{M2}\ldots+|\mathrm{out}\rangle_1|\text{`out'}\rangle_{M1}|\mathrm{in}\rangle_2|\text{`in'}\rangle_{M2} \tag{11.8}$$

and recall that these authors denote as $|\mathrm{in}\rangle_1$ the part of the GRW Gaussian–like wavefunction lying inside the box and as $|\mathrm{out}\rangle_1$ the tail outside the box. Suppose particle 1 suffers a localization within the box. In configuration space one sees immediately that multiplying the wavefunction corresponding to (11.8) times a Gaussian centered within the box near the point at which the wavefunction is peaked and normalizing the resulting statevector makes the second of the two above terms of much smaller norm than the first, but in no way suppresses it. Obviously the same argument applies for any localization of the pointers of the individual apparata $M_i$ and $M$. So the state will never take precisely to form (11.7). The authors have implicitly *assumed* that at some level the *standard quantum mechanical reduction* (and not the spontaneous localization characterizing the GRW theory) takes place[56].

## 11.3 A quantitative analysis

Let us consider a macroscopic object which is in an "almost" eigenstate of the mass operators $M_i$ (we consider once more the discretized version of CSL) but which however have tails. Let $|\psi\rangle$ be the normalized state

$$|\psi\rangle \quad = \quad \alpha|\psi_N^A\rangle \;+\; \beta|\psi_N^B\rangle \tag{11.9}$$

where $|\psi_N^A\rangle$ and $|\psi_N^B\rangle$ are the states appearing in (10.3) and $|\beta|^2$ is extremely close to zero. In region $A$ we have

$$\mathcal{M}_{i(A)} \;\simeq\; |\alpha|^2 nm_0, \quad \mathcal{V}_{i(A)} \;\simeq\; |\alpha|^2|\beta|^2 n^2 m_0^2, \quad \text{and} \quad \mathcal{R}_{i(A)} \;\simeq\; |\beta|^2 \;\ll\; 1 \tag{11.10}$$

so that the masses $\mathcal{M}_{i(A)}$ are accessible and practically equal to those corresponding to the state $|\psi_N^A\rangle$. In region $B$ we have

$$\mathcal{M}_{i(B)} \;\simeq\; |\beta|^2 nm_0, \quad \mathcal{V}_{i(B)} \;\simeq\; |\alpha|^2|\beta|^2 n^2 m_0^2, \quad \text{and} \quad \mathcal{R}_{i(A)} \;\simeq\; |\beta|^{-2} \;\gg\; 1 \tag{11.11}$$

hence the masses $\mathcal{M}_{i(B)}$ are not accessible.

Our aim is to make a quantitative estimate of $\mathcal{R}_{i(A)}$ and of the total mass in region $B$. To this purpose (as it is evident from equations (11.10) and (11.11)) one has to explicitly evaluate the order of magnitude of the parameter $|\beta|^2$ implied by the reducing dynamics. In order to do this, to cover also the case of non homogeneous bodies, we consider again two far apart regions $A$ and $B$, each containing $K$ cells and a system of nucleons which at time $t=0$ is in a (normalized) state of the type (the overall phase factor being irrelevant)

$$\begin{aligned} |\psi\rangle \quad = \quad & \alpha(0)|n_{1(A)},\ldots,n_{K(A)},\ldots,0,\ldots,0\rangle \;+ \\ & \beta(0)e^{i\gamma(0)}|0,\ldots,0,\ldots,n_{1(B)},\ldots,n_{K(B)}\rangle \end{aligned} \tag{11.12}$$

[56] In [128], Clifton and Monton have tried to de–emphasize their previous assertions by stating: *In our exposition we were simply dividing the collapse process into different stages for ease of exposition. Certainly we were aware that the marbles themselves will be almost continually subject to GRW collapses.* Our answer is quite simple: first, why not confine the analysis to the marbles? What is the purpose of introducing the apparata? Secondly, we invite the reader to introduce in the game the appropriate description of the whole process by assuming that, in turn, the registration of the outcome is given by the location of a pointer whose wavefunction unavoidably has tails. He will easily realize that there is no advantage in operationalising the process of counting marbles: he will find himself back to square one.

where $\alpha(0)$ and $\beta(0)$ are comparable positive numbers and $n_{i(A,B)}$ represents the occupation number in the $i$–th cell in regions $A$ and $B$ respectively[57]. We then study the ensemble of systems brought in by the reducing dynamics after a time interval of the order of, e.g., $10^{-2}$ sec (the reason for this choice will become clear in what follows).

According to the CSL model of section 8.6, after such a time interval the normalized state corresponding to a definite realization of the stochastic potential would be of the type

$$\begin{aligned} |\psi_B(t)\rangle \quad &= \quad \alpha_B(t)|n_{1(A)},\ldots,n_{K(A)},\ldots,0,\ldots,0\rangle \; + \\ &\qquad \beta_B(t)e^{i\gamma(0)}|0,\ldots,0,\ldots,n_{1(B)},\ldots,n_{K(B)}\rangle \end{aligned} \tag{11.13}$$

with $\alpha_B(t)$ and $\beta_B(t)$ as positive numbers. The ensemble of systems corresponding to all possible realizations of the stochastic potential would be described by the statistical operator

$$\rho(t) \quad = \quad \int dB_1 \ldots dB_{2K}\, P_{\rm cook}[B(t)]\, |\psi_B(t)\rangle\langle\psi_B(t)| \tag{11.14}$$

satisfying[58]

$$\begin{aligned} &\langle n_{1(A)},\ldots,n_{K(A)},\ldots,0,\ldots,0|\rho(t)|0,\ldots,0,\ldots,n_{1(B)},\ldots,n_{K(B)}\rangle \; = \\ &\qquad e^{-\lambda\, t \sum_i^K n_i^2} \\ &\qquad\qquad \langle n_{1(A)},\ldots,n_{K(A)},\ldots,0,\ldots,0|\rho(0)|0,\ldots,0,\ldots,n_{1(B)},\ldots,n_{K(B)}\rangle \end{aligned} \tag{11.15}$$

with $\lambda t \simeq 10^{-18}$. From (11.15) we see that the matrix elements of $\rho(t)$ between the considered states are exponentially damped by a factor which is proportional to $\sum_i^K n_i^2$.

In the following we consider only situations in which $\sum_i^K n_i^2$ turns out to be much greater than $10^{18}$, so that in the considered time interval of $10^{-2}$ sec the linear superposition (11.12) is actually suppressed, i.e. either $\alpha_B(t)$ or $\beta_B(t)$ of equation (11.13) become very small. The states at time $t$ are then typical states with "tails", i.e. states whose existence is considered as a drawback of the theory by the authors of references [125, 45, 126, 127, 128]. Equation (11.15) implies (taking into account equations (11.13) and (11.14)) that

$$\int dB_1 \ldots dB_{2K}\, P_{\rm cook}[B(t)]\, \alpha_B(t)\, \beta_B(t) \quad = \quad \alpha(0)\, \beta(0)\, e^{-\lambda\, t \sum_i^K n_i^2} \tag{11.16}$$

¿From (11.16), since $\alpha_B(t)$ and $\beta_B(t)$ are positive, one can easily deduce that the probability of occurrence of realizations of the stochastic potential which would lead to a value for the product $\alpha_B(t)\beta_B(t)$ much greater than $e^{-\lambda\, t \sum_i^K n_i^2}$ must be extremely small. Therefore, one can state that in practically all cases

$$\alpha_B(t)\beta_B(t) \quad \simeq \quad e^{-\lambda\, t \sum_i^K n_i^2} \tag{11.17}$$

[57] We disregard the cells which are not contained in regions $A$ and $B$ since they are irrelevant for the following discussion.

[58] Even though we are using the CSL model relating decoherence to the mass, the formulas of this subsection coincide with the analogous ones of standard CSL. This is due to the fact that we deal only with nucleons and that we have chosen the coupling to the noise to be governed by the ratio $\gamma/m_0^2$, taking the standard CSL value.

If we assume that $\alpha_B(t) \simeq 1$, so that we consider an individual case for which the reduction leads to the state corresponding to the nucleons being in region $A$, $|\beta_B(t)|^2$ must be of the order of $e^{-\lambda t \sum_i^K n_i^2}$. On the basis of this fact we can then estimate the value of $|\beta|^2$, e.g., for a homogeneous marble of normal density (so that $n_i = n \simeq 10^9$ is the number of particles per cell) and of size 1 dm$^3$ (so that $K \simeq 10^{18}$ is the number of cells in regions $A$ and $B$) getting a figure of the order of $e^{-10^{18}}$. Correspondingly, we have

$$\mathcal{R}_{i(A)} \quad \simeq \quad e^{-10^{18}} \tag{11.18}$$

while for the total mass in region $B$ we get the value

$$\mathcal{M}_B \quad \simeq \quad e^{-10^{18}} 10^{27} m_0 \tag{11.19}$$

Equation (11.18) shows that the mass in region $A$ is "accessible" to an extremely high degree of accuracy and equation (11.19) shows that the total mass in region $B$ is incredibly much smaller than the mass of a nucleon. If we consider a situation in which $K$ or $n$ are greater than those of the example we have discussed now, we find values for $\mathcal{R}_{i(A)}$ and $\mathcal{M}_B$ which are even smaller[59] than those of equations (11.18) and (11.19). This fact by itself (see also the analysis of the following subsection) shows that the states with "tails" allowed by CSL cannot give rise to difficulties for the proposed interpretation of the theory. For example, if we would perform the usual gedanken experiment with the test particle it would be deflected just as if in region A there would be the "classical" mass $Knm_0$.

## 11.4 Mass density interpretation and marbles

Before replying to the criticisms previously mentioned [125, 45, 126, 127, 128], and in the light of the analysis of the previous section, we call the reader's attention to the following properties of the mass density interpretation within dynamical reduction models which are relevant for the discussion about the "tail problem":

1. In the case of a marble the mass density is accessible just where the marble is located. Any test devised to 'reveal' the mass density distribution will agree with the statements which make reference to the accessible mass density. One can easily evaluate the contribution of the tails[60] to any possible gravitational test and conclude that there would be no physically testable difference whatsoever between the case of a marble whose wavefunction is a Gaussian of width $10^{-11}$ cm, and one for which its wavefunction has compact support.

2. This should make clear why we claim that there is no need to operationalize the process of counting marbles. We already know that the regime condition induced by the GRW dynamics corresponds precisely to the statement that the particles are where they are, i.e., in those regions in which there is an accessible mass density, and that any test will confirm such statements.

3. It is important to stress a point which seems to have been underestimated in [127] and [128], i.e. the crucial role played by the fact that the GRW theory leads to precise regions in which the mass is accessible. For this reason it does not seem a good choice to have schematized the 'tail problem' by resorting to the states $|\text{in}\rangle$ and $|\text{out}\rangle$ (even though they are those which matter for the counting anomaly). If the dynamics would allow a marble in the box to be

[59] Note that this holds also for objects like a galaxy or a neutron star.

[60] In the previous subsection it has been proved that the integral of the mass density extended to all space exception made for the region in which the Gaussian is centered amounts to an incredibly small fraction of the mass of a nucleon.

> in the superposition of two states corresponding to two Gaussians, or even (at a certain instant) to two wavefunctions of comparable norm with disjoint and far apart (let us say 1 meter) compact supports, the associated mass density function would not be accessible anywhere in ordinary space, and any talk about the location of the marble would be devoid of any meaning. We can speak of the position of a marble just because there is a region coinciding (practically) with its location in which there is an accessible mass density. If there is no such region, to be allowed to say something about where the marble is one should resort to 'a measurement' and to some recipe leading to an accessible mass density of the pointer of the measuring device. But fortunately, within the GRW theory, when one reaches the macroscopic level, there is no need to invoke observables or measurement processes. This argument shows also how inappropriate it is to make reference, as is repeatedly done in references [127] and [128], to microscopic systems replacing the marbles: for such systems there is no mechanism forbidding the spread of the wavefunction and consequently no talk about their location has any meaning[61].

As already mentioned, Clifton and Monton have pointed out that the mass density interpretation requires a divorce of mass talk from position talk. We believe that what they want to stress is that since the integral over the whole space of the operator $M(r)$ gives the the total mass operator of all particles in the universe, and since the GRW theory does not contemplate creation and/or annihilation processes, the mass associated to the region in which it is accessible (i.e. in which, according to our views, the marble is located) cannot coincide with the total mass of the marble. This is true, but, as we have already remarked this divorce amounts to an extremely small fraction of the mass of a nucleon and fits perfectly the ontology of a theory which allows microsystems [122]:

> to enjoy the cloudiness of waves, while allowing tables and chairs, and ourselves, and black marks on photographs, to be rather definitely in one place rather than another, and to be described in classical terms.

We do not want to be misunderstood. We are not playing once more with the possibility of locating the split between micro and macro, reversible and irreversible, quantum and classical, where we consider it appropriate for our purposes. To clarify this point we can consider a state of (the particles of) the marble in which, for some physical interaction, a portion of $10^{-15}$ cm$^3$ of the marble has been separated from the marble itself. We stress that, within the GRW theory with the proposed interpretation this situation is 'objectively' different from that of an unsplit marble and can be perfectly described in classical terms. In particular, the superposition of a state of an 'unbroken marble' and the one of the marble which 'has lost a piece' will not last for the perception times, and, if reduction takes place to the second state, the mass density will turn out to be accessible just where the marble and where its fragment are located. But if, instead of a fragment, the marble has been deprived of some elementary particles, then, while the mass is still accessible where the marble is, it is no more so for the region over which the wavefunctions of the lost particles extends, and this precisely for the reason that their wavefunction spread almost instantaneously. We think that one has to keep clearly in mind this situation

[61] Obviously, also the mass density of an elementary particle can be accessible, if its wavefunction is extremely well peaked in a region of, let us say, less tan $10^{-6}$ cm. But as we all know, the hamiltonian evolution will make it unacessible almost immediately due (not to the tails) but to the extremely rapid increase of its spread without any localization balancing the hamiltonian spread. In our opinion this is the appropriate way to read the statement of J.S. Bell: *even for one particle I think one would have problems* with the mass density and makes clear while, at any rate, one cannot apply it to the macroscopic case.

to understand why the remarks of Clifton and Monton about the location of the marble and the mass density are not cogent (more about this later).

## 11.5 Proof of the internal consistency of the mass density interpretation: reply to criticisms

¿From the previous analysis it should be clear why we consider not cogent the criticisms raised by Shimony, Albert and Loewer and Lewis, and why we consider superfluous and inappropriate the suggestion by Clifton and Monton to operazionalize the counting process (see also [129, 130, 131]). Let us be more specific.

1. The tail problem is not a problem, it simply requires us to abandon the idea that the presence of the tail implies that there is a certain probability that a measurement gives the outcome: 'the marble has been found in the region of the tail'. In the GRW theory there are no measurements. If we are interested in testing the 'properties' of a microsystem then we have to invent a device correlating its different microstates to different regions in which the mass density of the pointer becomes accessible. From the knowledge of such an accessible mass density we can infer 'the outcome of the measurement'. If the object we are interested in is already macroscopic then there is nothing to measure about its position: the object is where the associated mass density is accessible. Measuring where the object is becomes then, in a sense, tautological and has only a practical interest if, e.g., we have no access to the object. In such a case the so called 'measurement procedure' consists simply in establishing a correlation between the region where the mass density of the measured object is accessible and the region in which the mass density of the pointer is accessible, period.

2. According to the remarks under 1), in the state (11.4) considered by Lewis the mass density is accessible only inside the box, since one can easily check that the only points in space where the mass is accessible lie inside it, while outside the box there are no points where this occurs. This, in turn, implies that in such a state any test aimed to ascertain where the marble is will almost always give as a result 'the marble is inside the box'. We stress that we have added the specification 'almost', not because the regions in which the mass density is accessible are (due to the tails) to some extent imprecise in a state like (11.4), but to take into consideration that any physical test requires a certain time and that, when the number of marbles increases beyond any limit, the peculiar rapid variations of the centers of the Gaussians which define the accessibility regions could lead us to detect a particle outside the box, just because it can make a sudden enormous jump during the test process. But this is a story which has nothing to do with the enumeration principle and with the problem of the tails: it originates entirely from the dynamical structure of the theory. At any rate, as we have already argued, even when the number of marbles which one takes into account becomes enormous, at any given time one will always be dealing with precisely $n$ Gaussians peaked around $n$ precise positions. In brief: there is no probability of finding marbles in different places than those where their mass density is accessible and so there is no counting anomaly. The statement that the probability of *finding* all the particles within the box is $|a|^{2n}$ derives entirely from adopting the standard **probabilistic** interpretation and the standard position about measurements.

3. For the above reasons, there is also no need to operationalize the counting process: such a procedure, if developed rigorously, will simply lead to more

and more macro–systems in states which are characterized by an accessible mass density in the regions in which the theory allows us to say they actually are, plus more and more tails spread out over the whole universe. These tails require a divorce of position talk from mass talk, but contrary to what the remarks of reference [128] seem to suggest, this divorce is absolutely negligible and experimentally undetectable. In fact even though one could naively argue that when the marbles become more and more numerous the mass density in the tails becomes more and more relevant (and might amount to the mass of even 1000 marbles) one cannot forget that increasing the number of marbles increases correspondingly their gravitational effects (masking more and more the gravitational contribution of the tails) and that the mass density of the tails is never accessible. Stated in rather brute terms: the tails cannot in any way conspire to produce an accessible mass density which can be identified with 'a marble'. Thus, asserting that in a state like (11.4) one could legitimately claim that $n-$ 1000 marbles are in the box is completely nonsensical[62]. The analogy with a classical situation we will present in the last subsection will allow us to deepen this point.

4. Clifton and Monton claim that the same arguments which "serve to motivate the mass density criterion, also serve to motivate speaking of a particle (or particles) as being located in a region whenever its wavefunction assigns high probability to its being detected in that region; that is they also motivate the fuzzy link. Unfortunately, [Bassi and Ghirardi] never say why the fuzzy link is 'inappropriate' and not a 'valid' way to understand reduction theories".

   It is easy to find in this very sentence the reasons for the inappropriateness of the Clifton and Monton analysis: the authors use the terms *probability* and *being detected* which shows that they are still bound to the orthodox interpretation. But the whole sentence reveals that they have not grasped the real meaning of the idea of accessibility: if a superposition of two states of a marble with comparable weights and both with compact support entirely within the box but separated by an appreciable distance would be possible, then the mass would not be accessible. It is just because of a lack of understanding of this point that these authors feel comfortable in replacing a marble with a particle, as the above sentence shows. But this is totally inappropriate and this is why the fuzzy link is inappropriate. In brief: the main reasons to reject the fuzzy–link are that it puts exactly on the same ground micro and macro–systems and it does not take at all into account the most relevant feature of the GRW theory,

[62] With reference to this point we would like to point out that the same problem of the relations between position and mass occurs even in Classical Mechanics and, in general, in field theories. In fact the equivalence between mass and energy implies that the mass of a classical object is not given only by the masses of its constituents, but also by the energy of the fields which keep the constituents together. And such fields, of course, are spread out in space. Thus, if one wants to be very pedantic, he could say that not all of the mass–energy of the object is where we see the object to be, since a very small portion of it is spread in outer space. From this point of view the situation is quite similar to the one of the GRW theory: if one considers an incredibly large number $n$ of macroscopic classical objects inside a box and calculates the total mass inside it, he could very well find as a result $m\,(n-1000)$. Do we have to conclude that not only the GRW theory but also Classical Mechanics and Field Theory violate the enumeration principle? We think that this is not the case; we believe that it is Lewis' and Clifton and Monton's points of view which are too strict and inadequate to the interpretation of such theories. A field (quantum or classical), in general is never well localized in space and thus concepts like 'being located' are not well suited to describe its properties. This is why when we want to speak of a field as located in a certain region of space, we have to accept a certain amount of fuzziness. Thus, for a field, being confined to a certain region means that almost all the field is confined in such region, and if particles and matter have to be described in terms of fields (and this is the trend in modern physics), we have to accept some fuzziness, otherwise nothing would be located anywhere in space.

i.e., that what the dynamical evolution makes accessible is precisely and solely the mass density at the appropriate macro level.

## 11.6 The stochastic nature of the evolution

So far we have discussed the description of the world allowed by the CSL theory in terms of the values taken by the mass density function $\mathcal{M}(\mathbf{r})$ which have been recognized to constitute the exposed beables of the theory. According to equation (10.1) it is the wavefunction associated to the system which determines $\mathcal{M}(\mathbf{r})$. It is useful to analyze the evolution of the beables. Let us consider, for convenience, the linear formulation of CSL: as we have discussed in sections 7 and 8, the dynamical evolution equation for the wave function is fundamentally stochastic, being governed by the stochastic processes $w(\mathbf{r}, t)$. The "cooked" probability of occurrence of such processes, based on the analog of equation (7.40), depends on the wavefunction which describes the system and this fact is of crucial importance for getting the "right" (i.e. the quantum) probabilities of measurement outcomes. Therefore, in the CSL theory, the wavefunction has both a descriptive (since it determines $\mathcal{M}(\mathbf{r})$) and a probabilistic (since it enters in the prescription for the cooking of the probability of occurrence of the stochastic processes) role.

Also the "tails" of the wavefunction have a precise role. In fact, suppose our "universe" is described at $t = 0$ by a normalized state

$$|\psi(0)\rangle \quad = \quad \alpha(0)\,|a\rangle \;+\; \beta(0)\,|b\rangle \tag{11.20}$$

with $|\beta(0)|^2$ being extremely small. The "reality" of the universe at $t = 0$ is "determined" by the state $|a\rangle$, as we have explicitly shown in the previous subsection. However, one cannot ignore the (extremely small) probability $|\beta(0)|^2$ that a realization of the stochastic potential occurs which, after a sufficiently long time, leads to a normalized state

$$|\psi(t)\rangle \quad = \quad \alpha(t)\,|\tilde{a}\rangle \;+\; \beta(t)\,|\tilde{b}\rangle \tag{11.21}$$

with $|\alpha(t)|^2$ being extremely small and with $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ two of the most probable states at time $t$ for the initial conditions $|a\rangle$ and $|b\rangle$, respectively. Then, the "reality" at time $t$ is that associated to the state $|\tilde{b}\rangle$ which has its origin in the negligible component $|b\rangle$ at time $t = 0$. Thus, some "memory" of a situation which at time zero did not correspond to the "reality" of the world remains at time $t$. Obviously, if such an extremely improbable case would occur one would be tempted (wrongly) to retrodict that "reality" at $t = 0$ was the one associated to $|b\rangle$ and not the one associated to $|a\rangle$. However, we stress that such peculiar events, which we could denote as the "reversal of the status of the universe", have absolutely negligible probabilities. As made plausible by the estimate for the values of $\beta(t)$ given in the previous section, the "risk to be wrong" in retrodicting from the present to the past "status of the world" is comparable with the probability of being wrong when, having observed now a table standing on the floor, and knowing that it has been kept isolated, we claim that it was standing there even one hour ago, in spite of the fact that thermodynamically a very peculiar situation corresponding to its "levitation" at that time could in principle have occurred.

## 11.7 A classical analogue of the tail problem

The criticisms which have been put forward concerning the tail problem and which we have discussed in the previous subsections, claim that the GRW theory, just due to the tail problem, is fundamentally unsatisfactory from an ontological point of view. To answer such criticisms we consider it appropriate to consider a quite simple classical situation which shares many of the aspects of the dynamical reduction models.

Suppose we have a collection of macroscopic objects of a precise shape and with precise mass density: to be specific let us consider an assembly of identical tables. Let us suppose that they are at rest and that the dynamics allows processes in which each of the tables has an extremely small probability (let us say of the order of $e^{-[10^{34}]}$) of emitting just one nucleon or one electron in a certain time. The specification *just one* is intended to be strict: a table can emit one such particle but if the emission takes place, no more particles can be emitted. To go on, we *postulate* (this is the analog of adopting the accessibility criterion within the GRW theory) that a table which has lost just one nucleon or one electron can still be called a table[63]. We now consider a universe made by an unphysically large number $n$ (of the order of some $10^{24}e^{[10^{34}]}$) of tables. After a while, a certain number of nucleons and protons (proportional to $n$) will be emitted. Thus the physical situation will be: we have $n-k$ tables in their original status (with $k << n$), $k$ tables which have lost a particle, and $k$ particles propagating in space. We think that everybody would agree with the statement that there are still $n$ tables in our universe. And we also believe that nobody would suggest that the emitted particles could, in principle, be used to 'build up a new table'[64]. At any rate, if one would take seriously such a possibility one would be lead to conclude that he started with $n$ tables and he ended up with $n+1$ tables without changing the total mass. How is this possible? Is there something terribly wrong in the classical model we are envisaging? Do we have to declare that the ontology of the model leads to inconsistencies? Or does not this simply mean that the links between objects, their masses and their locations is not so strict as Shimony, Albert, Loewer, Clifton and Monton believe and that, in particular, in the unrealistic and purely speculative case in which the number of tables which enter into play is made unphysically large one cannot avoid facing quite peculiar but logically prefectly consistent situations?

# 12 The psycho–physical parallelism within CSL

The most characteristic and appealing feature of CSL and of its interpretation consists in the fact that it allows one to give a satisfactory account of reality, to take a realistic view about the world, to talk about it as if it is really there even when it is not observed. However, one cannot avoid raising the problem of including also conscious observers in the picture, for [87] *what is interesting if not experienced?* Thus one is led to consider the problem of the psycho–physical parallelism within dynamical reduction models: this is the subject of the present section.

The section begins with a challenge for dynamical reduction models, put forward by D. Albert and L. Vaidman [132, 133]: do they always guarantee definite outcomes to measurement processes (subsection 12.1)? In order to answer this question, in

[63] Obviously, the reader should have grasped why we allow the tables to loose just one elementary constituent and we forbid the emission processes to continue indefinitely. If one would have taken such an attitude one would be compelled to define precisely up to which point one is keen to consider the system to be still 'a table', i.e. to specify that a loss of, e.g., $10^5$ particles is acceptable but that after that limit it is no longer legitimate to call the system 'a table'. Since within GRW the accessibility derives from the fact that the center–of–mass of a macro–system gets localized with an extremely precise accuracy (of about $10^{-11}$ cm) and for ordinary objects like marbles or tables this defines quite precisely the 'mass in the tail' (which turns out to be an extremely small fraction of a nucleon mass) the above assumption is quite appropriate to develop the analogy we are interested in.

[64] Taking the risk of being pedantic we stress once more that both in the considered example as well as within the GRW theory with the mass density interpretation, our 'disregarding', in some sense, the emitted particles (the tails) for what concerns the consideration of the tables which are in our universe does not mean we wish to deny to them a real status and to ignore the physically relevant effects they can trigger. One can think, e.g., of having in the universe also many Geiger counters; there is no doubt that one of the emitted particles could trigger a counter inducing relevant dynamical changes. The same is true within the GRW theory: the tails can trigger further appreciable effects, as we have already stressed with particular emphasis.

subsection 12.2 we analyze what we expect a "measurement process" and a "measurement outcome" to be. This leads us (subsection 12.3) to consider how the process of perception unfolds in time and how the reduction mechanism works within the nervous system, thus proving that an observer always has definite perceptions about measurement outcomes.

We conclude the section (subsections 12.4 and 12.5) with some further comments about the relation between measurement outcomes and human perceptions.

## 12.1 Introduction

Some years ago, two quite interesting papers, one by D. Albert and L. Vaidman [132] and the other by D. Albert [133] (see also [62]), challenging QMSL, have appeared. In them some critical remarks have been put forward which, in the authors' intention, should prove that dynamical reduction models suffers from some serious limitations, in particular that they do not satisfy fundamental requirements that conventional wisdom imposes on a workable theory of collapse. The papers offer the opportunity of a clarification about the model and its aims and deserve various comments.

In reference [132] the authors start by listing the features that any theory pretending to describe the collapse should exhibit. Let us summarize them:

1. It ought to guarantee that *measurements always have outcomes* after they are over, i.e., after a *recording* of the outcomes exists in the measuring device.

2. It ought to imply that, for any given eigenvalue of the measured observable, the probability that the statevector ends up in the associated eigenmanifold coincides with the probability that standard quantum mechanics attributes to such an outcome.

3. It should not contradict any experimentally established quantum mechanical predictions about physical systems, in particular *the fact that isolated microscopic systems have never yet been observed to undergo collapses*.

As regards QMSL, Albert and Vaidman agree that, due to the extremely low probability of occurrence of a localization and the fact that the localization distance is large on the atomic scale, the theory satisfies requirement 3). They also seem to agree on the fact that, when consideration is given to a system containing many particles which is in a superposition of two states $|\psi_1\rangle$ and $|\psi_2\rangle$, such that in $|\psi_1\rangle$ a large number $N$ of particles have spatial positions which differ more than $1/\sqrt{\alpha}$ from those they have in $|\psi_2\rangle$, then such a state is transformed dynamically into either $|\psi_1\rangle$ or $|\psi_2\rangle$ in a time of the order of $10^{16}N^{-1}$ sec, which, for $N$ of the order of Avogadro's number, means in less than a microsecond. So, when consideration is given to cases in which the measurement outcomes are indicated by some sort of a macroscopic pointer taking macroscopically different spatial positions, QMSL does satisfy also 1) and 2) besides 3).

Then, why do Albert and Vaidman assert that the theory runs into difficulties with the first two requirements? The reason for this is that they consider it incorrect to assume that all measuring instruments work in the above indicated way. They illustrate this point with an example, which we briefly report here.

A spin–1/2 system is prepared in an eigenstate of $\sigma_x$ and then it is passed in a Stern–Gerlach arrangement with non–uniform magnetic field in the $z$ direction. The two spin states $|z+\rangle$ and $|z-\rangle$ are then correlated to upward and downward "trajectories," respectively. These trajectories hit a fluorescent screen at two different points $A$ and $B$ which are separated by a macroscopic distance. The screen is such that the particle impinging on it causes some of the electrons of the atoms around the point of impact to jump into excited orbitals. De–excitations of these electrons

give then rise to a luminous dot, which can be directly seen by an experimenter. The situation, before any observer enters into play, can then be described by the following statevector

$$\begin{aligned} |\psi(1,2)\rangle &= \frac{1}{\sqrt{2}}\left[|z+, x=A\rangle_{MP}|1_{ex},\ldots,n_{ex};(n+1)_{gr},\ldots,2n_{gr}\rangle\right. \\ &+ \left.|z-, x=B\rangle_{MP}|1_{gr},\ldots,n_{gr};(n+1)_{ex},\ldots,2n_{ex}\rangle\right], \end{aligned} \tag{12.1}$$

where $MP$ refers to the measured particle and the electrons from 1 to $n$ are near point $A$, the remaining ones are near $B$. The indices $ex$ and $gr$ refer to excited and ground states for the indicated electrons, which subsequently decay, emitting photons from the indicated points, respectively.

With reference to this example, the authors argue then as follows: since in the whole process few particles are involved and, in any case, the displacements of the particles which are involved are totally negligible with respect to $1/\sqrt{\alpha}$, the reduction mechanism of QMSL cannot be effective in suppressing one of the two states in equation (12.1). The consideration of the subsequent emission of photons does not change the situation since, on one hand, the spontaneous localizations of QMSL do not affect photons and, on the other hand, the photon wavefunctions originating from $A$ and $B$ immediately overlap almost completely. So, in spite of the fact that, corresponding to the impinging of the particle on the screen having occurred at $A$ or at $B$, there is a luminous record in different places which can be perceived by a naked human eye, QMSL does not imply that the suppression of one of the two states has occurred. Therefore QMSL does not satisfy the basic requirement 1). To put it in different words: while everybody would agree that the measurement has been completed after the emission of the photons from the fluorescent screen and before any observer actually looks at the light spot, QMSL is not able to attribute a definite outcome to the measurement.

The argument of the authors reduced to its essence consists in stressing that not all conceivable processes which everybody would agree to call measurement processes involve macroscopic displacements of a macroscopic number of particles, and that, under such conditions, dynamical reduction models cannot yield an actual dynamical reduction of the wavepacket. With reference to this pertinent remark and to the above example, we consider it important to distinguish, for conceptual clarity, two mechanisms which could make QMSL ineffective in inducing the reduction:

1. That the number of particles involved in the process (e.g., in the set–up considered, the electrons which have to be excited in order that the emitted photons be perceivable by the unaided eye of a human experimenter) be very small.

2. That, even if many particles are involved, the changes in their states which occur as a consequence of their interactions with the measured microsystem (in the example considered the transitions from the ground to an excited state) involve position changes which are much smaller than the localization distance $1/\sqrt{\alpha}$ of the spontaneous process. The above two alternatives will be discussed in the next subsection.

## 12.2 Measurements and outcomes

Let us begin by stating that we agree perfectly with the conclusions drawn in [132] about the specific situation discussed in it. Due to the extremely low rate of the spontaneous localizations and the large value (on the atomic scale) of the localization distance of QMSL, any superposition of states in which only few particles are in appreciably different spatial positions or in which many particles are only slightly displaced is not dynamically suppressed by the theory.

We then agree that in the considered example the linear superposition will persist for long times. Is this a feature which shows the failure of the model in accounting for measurement processes? We think that one should be careful in drawing such a conclusion.

As a starting point it is useful to recall the lucid position of J. Bell about such kind of problems. In [124] he stated:

> The usual approach, centered on the notion of "observable", divides the world somehow into parts: "system" and "apparatus". The "apparatus" interacts from time to time with the "system", "measuring" "observables"... There is nothing in the mathematics to tell what is "system" and what is "apparatus", nothing to tell, which natural processes have the special status of "measurements".

Furthermore, in [122], he stressed:

> Surely the big and the small should merge smoothly with one another. And surely in fundamental physical theory this merging should be described not just by vague words but by precise mathematics.

These quotations appropriately point out that, in what the author would have considered a serious theory, the very mathematics of the theory and not "vague words" should define what is a measurement. In this respect it seems to us that in [132] what has to be considered a measurement, what plays the role of an apparatus, is still defined by terms that are to some extent vague. For instance, in footnote 1, the authors, trying to make precise what is a recording by an instrument, require that it consists in a change of the measuring device which is *macroscopic, irreversible, and visible to the unaided eye of a human experimenter*.

It is just to try to be very precise about the meaning of such terms that we have chosen to make at the end of the previous subsection a clear distinction between the two crucial points which are at the basis of the arguments of Albert and Vaidman.

Let us consider point 1. If the number of particles which are involved in the process under consideration is really very small (as it is legitimate to assume due to the sensitivity of the human eye to light quanta — the threshold for visual perception being of about 6 photons), then the changes of the measuring device are surely not macroscopic, even though they can be irreversible and visible to the human unaided eye. Is it correct to pretend that also in such a case the process has to be considered as a genuine measurement which should have an outcome before anything else happens?

We can raise some doubts about the correctness of taking this attitude by considering the following gedanken experiment. Suppose we have an atom which can be prepared in an excited state $|\mathrm{Ex}\rangle$ of spin 1/2 and with the following characteristics: the excited state decays with a quite long lifetime, of the order of several seconds or longer, in a state $|\mathrm{SLC}\rangle$ which is the ancestor of a short lived sequential decay chain with various steps, all lifetimes in the sequence being extremely short (of the order of nanoseconds). We prepare the state $|\mathrm{Ex}\rangle$ in a spin state that is an eigenstate of $\sigma_x$, and we perform a Stern–Gerlach experiment devised to measure the $z$ component of the spin. The long lifetime of the initial state then allows us to correlate the two $\sigma_z$ components to two different positions of the atom, which we will denote by $A$ and $B$, respectively, just as in the example considered by Albert and Vaidman. After a time of the order of the lifetime for the first transition, the atom decays; and, going through the cascade process, it emits several photons, let us say a number larger than the perception threshold. There is no doubt that the situation is very similar to the one considered in [132]: the sequential decay is an irreversible process in the same sense in which the excitation and decay of few fluorescent electrons is. Moreover, the emission from two macroscopically distant regions of a number of photons which is

sufficient to be perceived by the naked eye of a human experimenter allows a direct detection of whether the atom has been found with spin up or with spin down. Could anybody pretend that a spin measurement has been performed on the atom before any observer would look at it? We think that this would be a wrong request and that actually nobody would feel embarrassed in considering the system to be in the linear superposition of the two final states. So, there is no reason to require that an acceptable theory of collapse should guarantee that this specific "measurement" has an outcome at this stage.

Let us now consider possibility 2. We think that it is perfectly plausible to imagine interaction mechanisms between a microscopic and a macroscopic system such that, on one hand, many constituents of the macroscopic one can change their quantum state in a way which depends on the state of the triggering system, and, on the other hand, that such changes do not involve displacements of the particles considered of amounts larger than $10^{-5}$ cm. The simplest example which one can think of is just the one in which many particles change their energy by amounts summing up to a macroscopic energy change without appreciably changing their position (on the relevant scale). However, the important question is: can this change be really macroscopic and have no other effect? In particular, would it not induce, at least indirectly, macroscopic changes in the positions of a macroscopic number of particles? It is not easy to exclude this, since one cannot ignore, e.g., the interactions of the system with its environment[65]. In such a case, at least for reasons of thermal equilibrium, changes of this sort must be recognized to occur. Obviously one could object that the body could very well be kept almost perfectly isolated. But how would then one check that it has changed its state? The final answer would be: by direct detection with an appropriate apparatus or by direct observation (if this is possible, since, e.g., the body radiates) by a conscious observer.

We can now raise our basic question: what would make it unacceptable to consider that such a body remains in a superposition of the two macroscopically different states under consideration before it is detected and the result is recorded?

The above statement can obviously be meaningful only if one makes precise the meaning of the expressions "detected" and "recorded". We completely agree on this, but we stress once more that it is just the theory itself which must give a precise meaning and make absolutely definite the significance of these expressions. QMSL does this: the body is detected and the result is recorded at the moment into which it evolves into, or, via its interactions with other systems, it induces the occurrence of a linear superposition of states containing a macroscopic number of particles which are differently located in space by an amount of the order of $10^{-5}$ cm or larger. At that moment it has to choose one of its possible ways! And it is the dynamics of the theory that guarantees that this will happen.

One may like or dislike this picture, but one has to recognize that it is consistent and represents a step towards a possible clarification of some of the puzzles of the quantum world. Obviously the above position is tenable only provided one can guarantee a fundamental fact: since our perceptions are definite, at least in connection with any act of perception, a situation must occur such that the reduction mechanism, whose taking place is precisely defined by the theory, becomes effective. The

[65] It has to be stressed that here our resorting to the environment is conceptually radically different from the procedure followed in approaches like the one considered by Joos and Zeh [30] to solve the measurement problems (see section 4.3). There the environment plays the role of a system whose correlations with the measured system and measuring apparatus cannot be detected, so that, to make physical predictions, one must take the partial trace on the environment variables. As already discussed, the reduction does not then occur at the individual level. Here the changes of the environment we are interested in are those in which a macroscopic number of particles are displaced. When they occur the universal dynamical mechanism of QMSL actually induces reductions at the individual level.

next subsection is devoted to making plausible that this is the case by a discussion of visual perception.

## 12.3 Reduction within the nervous system

As stressed in the previous subsection, the very possibility of considering QMSL as yielding a unified description of all physical phenomena rests on the fact that one can show that the physical processes occurring in sentient beings, leading to definite perceptions, involve a displacement of a sufficient number of particles over appropriate distances to allow the reduction to take place within the perception time.

We will describe now, in its essential aspects, the visual perception process, to make plausible that the above situation actually occurs. The reason to choose to discuss explicitly the case of the visual perception should be obvious. First, vision is directly involved in the example discussed in the previous subsection. Secondly, the visual channel is presently the most studied of all sensory channels and therefore the best understood. Finally, other sensory modalities show similar characteristics, both in the distal refined transduction mechanisms and in the excitation pattern reaching the central nervous system.

We will divide our description into the three main cascades of events that take place following the absorption of one photon in a photoreceptor cell of the retina:

1. multiplicative chain in the photoreceptor cell,
2. transmission of the electrical signals along the fibers of the optic nerve,
3. excitation of neurons in the cortical visual area.

All these events are necessary for seeing. We will make rough estimates of the number of particles moving in these processes.

An observation is relevant: In general, sensory cells have no threshold in responding to external stimuli. In the case of photoreceptors, the absorption of a photon by a pigment molecule (retinene) determines a isomeric transition (cis–trans). But the same transition can be also determined for example by thermal excitation. The detection of external stimuli is therefore based on statistical decision about the signal to noise ratio [134]. For this reason, the psycho–physical threshold is set at the average level of six absorbed photons.

We now analyze the multiplicative chain in the photoreceptor cell. The excited state $R$ of the retinene, inside the rhodopsin (a protein molecule with molecular weight $M_w = 39,000$) has a lifetime sufficiently long on the disks of the rod to activate about 100 transducin molecules $T$, present in the interdisk space (about of the same $M_w$). This is used to release the inhibition of the enzyme PDE (phosphodiesterase, $M_w = 180,000$), able to hydrolyze very rapidly about $1,000$ cyclic nucleotide c–GMP (guanosinmonophosphate). All together we obtain a multiplication in molecule number of about $100,000$ for each absorbed photon. The c–GMP molecules (cooperatively 3 of them) normally keep open the channels of the plasma membrane, enclosing the vertebrate rod outer segment. Their hydrolysis determines the closure of the channels and the consequent hyperpolarization of the cell and starts the electrical signal that will be transmitted to a chain of neurons of the retina [135]. The electric current change through the membrane is of the order of a pA: one can estimate that the ions (mostly Na) affected in their movement are about $10^8$.

But we are interested in following the chain of events further on. The hyperpolarization of the inner segment of the rod is followed by the release of a chemical transmitter, kept in vesicles near the membrane in such a way to diffuse rapidly (about 1 msec) in the synaptic gap towards other neurons present in the retina: the horizontal cells, the amacrine and bipolar cells. We will not try to estimate how much

these interactions will increase the number of particles involved and their movement, keeping in mind that those interactions develop not only in the forward direction but also laterally and in the backward direction. To be very conservative, we can indicate a factor 10 in the number of molecules. The signals transmitted, both electrical or chemical, in these stages are graded, modulated in linear and nonlinear fashion. The last step is acted on the final output cells of the retina, the ganglion cells. These cells, whose long axons (about one million) form the optic nerve, send the electric impulses (spikes) towards the CNS (central nervous system). The propagation is regenerative and saltatory, going along a series of Ranvier nodes, where the membrane of the nerve fiber is free of the insulating sheets of the Schwann cells.

The above figures about the displaced particles are not sufficient to guarantee by themselves that the reduction has already occurred. However, we stress that all estimates we have made are very conservative and correspond to having chosen for the numbers of particles their minimal values. The high complexity of the system and its connectivity would justify the introduction of various amplifying factors which would remarkably raise the above values. These considerations should have made plausible that the number of particles and their displacements have reached the level which is sufficient to make effective the reduction mechanism. This makes correct the conviction of everybody working and analyzing what is going on in the nervous system that, at this stage, the quantum aspects of the phenomenology have already come down to the level for which the classical description is adequate.

To be more specific and remove any possible doubt about the definiteness of conscious perception, we show now, examining in more details the signal propagation, that even if one limits his considerations to this last step of the visual perception process, the estimates of the number and of the displacements of the particles which are involved would lead to the same conclusion. For this purpose, we simply analyze what happens to a neuron involved in the transmission of such a signal. The neuron has a main cell body with a nucleus and a long tube, the axon, extending from the cell body and having at its end a variety of hair–like structures connecting it to other nerve cells. As already remarked, in the case of the optic nerve, the axon is wrapped in a series of small sheaths of myelin, an insulating substance, separated at intervals of the order of one millimeter by nodes referred to as Ranvier's nodes (see figure 3).

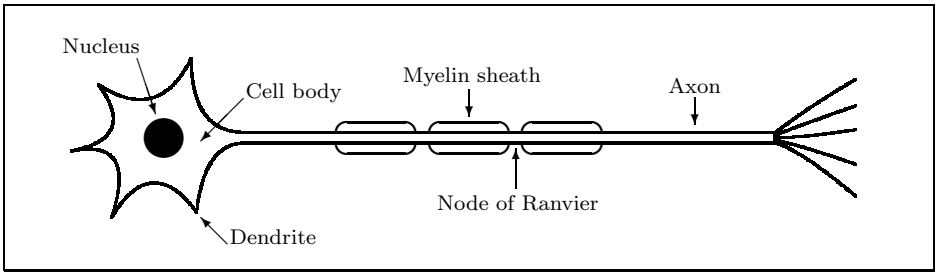



Figure 3: Drawing of a nerve cell.

To make the following discussion clearer, it is useful to mention that axon's diameter is of the order of $10^{-4}$ cm, the myelin sheath thickness is greater than $10^{-5}$ cm, and the membrane thickness at Ranvier's node is of the order of $10^{-6}$ cm. The transmission mechanism goes as follows: when an impulse is generated, at Ranvier's nodes, ions channels open in the membrane of the axon, through which $Na^+$ and $K^+$ ions flow. Thus, in the course of the depolarization of the membrane, circular currents, connecting two nearby Ranvier's nodes and closing through the external conducting medium, arise (see figure 4).

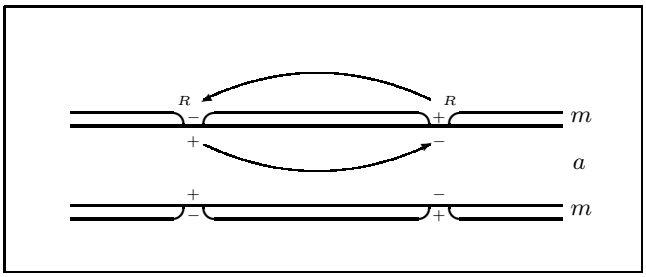



Figure 4: The scheme of impulse transmission. $R$ = Ranvier node. $m$ = myelin. $a$ = axoplasm.

Important facts to be taken into account are the following: during one impulse $\simeq 6{\cdot}10^6$ sodium ions pass a Ranvier's node; the time necessary to restore the resting potential in the considered region of the axon is of the order of $10^{-3}$ sec; finally the internal ion current flows near the axon membrane. With these premises, we can try to evaluate the efficiency of the reduction mechanism. To do this, in place of QMSL, we will make use of CSL: in such a theory, as we have seen, the decoupling rate for superpositions of states involving differently located particles is approximately given by:

$$e^{-\lambda\, t \sum_i (n_i - n'_i)^2}, \tag{12.2}$$

where $\lambda = 10^{-16}$ sec$^{-1}$, and $n_i$, $n'_i$ are the numbers of particles present in the cell (of volume $10^{-15}$ cm$^3$) labeled by the index $i$, in the two states $|\psi\rangle$ and $|\psi'\rangle$ whose superposition we are considering, respectively. Obviously $|\psi\rangle$ is associated with the occurrence of the impulse transmission, while $|\psi'\rangle$ corresponds to no transmission.

In our case, we consider all cells surrounding the internal membrane of the axon between two Ranvier's nodes. There are $10^5$ such cells. Taking into account that the atomic number of $Na^+$ is 11, so that one ion contains more than 30 particles and that we have disregarded the $K^+$ ions, we have to fit about $10^9$ particles in the $10^5$ cells. We then have that, when the signal is transmitted, for about $10^5$ cells $|n_i - n'_i|$ turns out to be $\sim 10^4$. These values, when substituted in equation (12.2), taking into account that the impulse lasts at least for $10^{-3}$ sec, give an exponent $10^{-6}$.

To complete the description, we have to consider two further steps, each involving a multiplicative factor. The optic fibers arrive at lateral geniculate bodies (LGB), where they branch out making contacts with many cells. The neurons in these bodies send their axons again to the visual striate cortex, an essential step for seeing. In both these stations a conservative estimate of the multiplicative effect [136] can be a minimum of $10^2$, therefore we can obtain at least a factor of $10^4$.

In our calculation we have completely disregarded further displacements of particles induced by the macroscopic current around the axon. In one of his books [137], R. Penrose has considered an analogous problem, to reach the Planck mass level, which in his approach would mark the setting up of the reduction. To get the desired result, he needs to assume that such further displacements imply a further amplification of a factor $10^8$, a figure that he considers reasonable. We obviously need a much smaller factor, e.g., one of the order of $10^4$ would be largely sufficient. So, we have made perfectly plausible that the number of displaced particles and the displacements which are involved imply the dynamical reduction mechanism of QMSL (or better of CSL) becomes fully effective in suppressing the superposition of the two states (nervous signal)–(no signal) before any act of conscious perception occurs. The fundamental requirement which has to be imposed on the model to account for our definite perceptions is therefore certainly satisfied.

## 12.4 Does reduction require observers?

It has to be firmly stressed that the fact that we have felt the necessity of performing an analysis of the visual perception mechanism is not intended as the acceptance, on our part, of the point of view that consciousness has a specific role in the reduction process. QMSL states that reduction will take place whenever the above indicated precise conditions for the reduction mechanism to be effective occur. So, if in place of a human being we put a spark chamber or a macroscopic pointer which is displaced, or a device producing ink spots on a computer output, reduction will take place. In the context of the previous subsection, the human nervous system is simply something which has the same function as one of these devices, if no such device interacts with the system before the human observer does.

With respect to Albert and Vaidman's example, our attitude should then be quite clear: the state is not reduced up to the moment in which the dynamical evolution leads to the occurrence of a superposition of states differing in the positions of a macroscopic number of particles. Whether this happens because one puts in front of the fluorescent screen a spark chamber or the nervous system of a human observer is totally irrelevant.

With reference to the argument we have just developed, we think it is appropriate to point out that some sentences of reference [133] could turn out to be misleading. For instance, at the beginning of section IV, the author, after having recalled the situation of [132], makes the following general statement: "If we want to stick with QMSL, then we would have to insist ... that no measurement is absolutely over, no measurement absolutely requires an outcome, until there is a sentient observer who is actually aware of that outcome". As we have proven, this is not true in general but only in very peculiar cases. Actually, according to QMSL, not only practically all measuring experiments of our laboratories but also all those measurement–like processes which, as J. Bell [2] has stressed, "we are obliged to admit ... are going on more or less all the time, more or less everywhere", have definite "outcomes" even in absence of any sentient being.

## 12.5 Some general comments about an alleged general impossibility proof of dynamical reduction models

We come now to examine further criticisms to QMSL, which have been put forward in sections IV and V of [133] (see also [62]). The whole argument can be briefly summarized as follows: the author contemplates the possibility of the existence of unusual sentient beings whose peculiar features consist in the fact that their conscious beliefs are assumed to be 100% correlated with the position of a single particle. To be more precise, the author considers a science–fiction character, John, who, due to a peculiar surgical implant in his brain, can perform a measurement of a spin component, e.g. of an electron, by making it cross a tunnel in his brain and interact with a microscopic particle $P$ in the implanted device. John is such that his "ready–to–perceive state" $|P_0\rangle$ corresponds to the particle $P$ being in a certain position, while the perceptions "spin up" and "spin down" are uniquely correlated to $P$ being up ($|P_+\rangle$) or down ($|P_-\rangle$) with respect to the initial position.

It is stipulated:

1. that no other change in John's body occurs as a consequence of the process (i.e., no other of John's atoms or electrons beside $P$ is involved in this process);

2. that, nevertheless, John is *consciously aware as vividly and as completely as he is ... or has ever been aware of anything in his life* of what the value of the spin component is.

Given these premises, provided one takes the attitude to believe what John says, the argument is quite straightforward: if one wants to account for the definite perceptions of John by a dynamical reduction mechanism, one must consider a process which suppresses the linear superpositions of position states of a single particle, and this would surely imply an easily detectable violation of the predictions of quantum mechanics for microsystems. So, no physically acceptable reduction mechanism whatsoever could do the desired job of making definite John's perceptions.

The first obvious remark is that the very consideration of such a being would upset any neurophysiologist. In fact, all we know about consciousness points undoubtedly towards the necessity of very complex systems like the human brain to support it. The features of such systems which are commonly considered essential for consciousness are, on the one hand, the fact that they contain a gigantic number of transmitting units (like neurons) and, on the other hand, that these units are wired up in an extremely complicated network. For these reasons, one could disregard objections which require as an essential ingredient the consideration of sentient beings whose different beliefs are correlated to states that are only microscopically different.

However, we can raise the question: even if one would accept that "in principle" the possibility of the existence of "microscopic" sentient beings (whatever precise meaning one could attribute to the word sentient) cannot be excluded, should one consider as cogent the criticism put forward in the two last sections of reference [133]? Answering the above question requires a detailed analysis. First of all, it turns out to be appropriate to recall the precise and relevant terms of the debate about the foundations of quantum mechanics and the reasons which make programs like the dynamical reduction one interesting. The situation can be summarized as follows. The conceptual structure of the quantum scheme does not allow, as it stands, the elaboration of an articulate, systematic and coherent picture of natural phenomena, i.e., what A. Shimony [1] calls a "philosophical world view". It is the desire to build such a coherent picture which has led to the consideration of various "alternative theories", which give for all practical purposes (FAPP) the same predictions as quantum mechanics plus the wavepacket reduction postulate. Each of them implies a certain world view, and each of them is based on different specific assumptions.

Let us start by considering two of the more appealing "alternative theories" that have been proposed:

1. The dynamical reduction models;

2. The de Broglie–Bohm pilot wave theory, or, more generally, the hidden–variable theories.

It is useful to remark that both these theories aim at giving an account of physical processes without having to make reference to observers. The other alternative which deserves to be discussed, and which Albert seems to prefer, is the many–minds theory. We will consider it below.

Due to the fact that present–day technology does not allow us to verify whether, at the macroscopic level and under appropriate circumstances, wavepacket reduction occurs, there is no experimental objective criterion to choose, e.g., between the two above alternatives 1) and 2). Such a choice must then be based on the identification of the assumptions which are essential for them as well as on the consideration of possible difficulties they meet or unpleasant features they exhibit. As concerns the assumptions, we remark that, as appropriately stressed by J. Bell [2], the characteristic element distinguishing program 1) from 2) consists in the fact that 1) assumes that the wavefunction gives a complete description of natural phenomena, while 2) accepts from the very beginning that additional variables are necessary. Concerning what one could consider the limitations of the corresponding programs, we mention that 1)

requires the acceptance of the fact that Schrödinger's equation is not exactly true and the introduction of "ad hoc" parameters; 2) requires us to accept contextuality (i.e., that a complete specification of a state assigns definite truth values to a proposition only relative to a specific context).

Albert [133], instead of following the above outlined procedure to choose among the "alternative schemes," seems to adopt as a crucial criterion for accepting or rejecting one of them the fact that it can accommodate beings like John. Various comments are appropriate: first, we do not view it as legitimate to adopt such a criterion to judge a theory, particularly in view of the fact that, as already remarked, the very idea that such beings exist seems to conflict with all we know to be needed to support consciousness. Secondly, we want to stress that the author's request amounts to plainly postulating that program 1) has to be rejected. In fact, the very hypothesis that John exists and is reliable leads to the denial of the fundamental assumption lying at the basis of the dynamical reduction program, i.e., that the wavefunction gives a complete description of natural phenomena, since the same overall statevector

$$\frac{1}{\sqrt{2}}\left[|z_+\rangle\,|P_+\rangle\ +\ |z_-\rangle\,|P_-\rangle\right] \tag{12.3}$$

would be associated in some cases with a situation in which John believes that "the spin is up" and in others to one in which John believes that "the spin is down[66]". Finally, it is worth noticing that even scheme 2) cannot accommodate, without encountering serious conceptual difficulties concerning the "status" that can be attributed to their perceptions, beings such as John. The reason for this lies just in the fact that the particle $P$ in John's device is a microscopic system. As a consequence, the linear superpositions of states corresponding to different positions of $P$ are not, even FAPP, equivalent to a statistical mixture of states corresponding to $P$ having a definite position. Then, the evolution leading the particle from its "ready state" to the final state is, not only in principle but practically, reversible (so that one can "undo" what has been done). Moreover, observables which do not commute with the position of $P$ must be admitted to be actually measurable, contrary to what would happen if $P$ would be a macroscopic system.

To illustrate the obvious fact that the process leading to John 's definite perceptions about the spin value can be reversed, let us assume, for simplicity, that the displacement of $P$ induced by its interaction with the spin of the electron occurs as a consequence of a $\sigma_z p_z$ coupling ($p_z$ being the $z$ component of the linear momentum of $P$) lasting for the time interval $\delta t$ taken by the electron to go through John's implanted device. Then it follows that, if $P$ is in the up (down) position, feeding into the device another electron with spin down (up), will cause $P$ to return to its ready state position. We can then consider the following experiment. We have two electrons in the singlet state, which travel towards John's device with a certain common velocity and which reach the device at times differing by an amount $\Delta t$ larger than $\delta t$. Then when the first electron crosses John's device, we have the statevector evolution:

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left[|z_{2-}\rangle\,|z_{1+}\rangle\ -\ |z_{2+}\rangle\,|z_{1-}\rangle\right]|P_0\rangle\ \longrightarrow\\ &\longrightarrow\ \ \frac{1}{\sqrt{2}}\left[|z_{2-}\rangle\,|z_{1+}\rangle\,|P_+\rangle\ -\ |z_{2+}\rangle\,|z_{1-}\rangle\,|P_-\rangle\right],\end{aligned} \tag{12.4}$$

where $|P_0\rangle$, $|P_+\rangle$, $|P_-\rangle$ represent the ready, up, and down states of $P$, respectively, and where we have disregarded the spatial variables of the electrons. When the state

[66] We do not want to be misunderstood: we are not assuming that it is possible to give a purely physical explanation of mental events, but we are simply imposing the obvious condition that different beliefs must be related to statevectors which somehow differ.

on the right hand side of equation (12.4) obtains, due to the fact that within the de Broglie–Bohm theory all particles at all times have definite positions, $P$ is either up or down and, as a consequence, John can "legimately" be said to have a definite belief about the $z$ component of the spin of the first electron. However, it has to be remarked that in the de Broglie–Bohm theory the spin being a contextual variable is not an observable quantity. As a consequence, in the case considered, the "outcome" of John 's measurement cannot be related to any objective property of the spin of the electron before the measurement (note that this would hold also in the case in which $P$ would be a macroscopic measuring device). Subsequently the second electron crosses John 's device, bringing $P$ back to its ready position $P_0$. As a consequence, due to the assumption that no other change but the displacement of $P$ has occurred in the first process, the state after the passage of the second electron turns out to be again exactly the initial state, i.e., the one on the left hand side of equation (12.4). It is important to stress that in such a state the spin variables are not correlated to the position of any particle and that the process just described (i.e., the interaction of the second electron with $P$) does not involve in any way the first electron on which the "measurement" of the spin had been previously performed.

Let us now suppose that another observer performs a measurement of the same spin component of the first electron. It is obvious that also this second measurement turns out to be contextual, and therefore its outcome is not related to the outcome of the "measurement" performed by John. This puts into evidence that the previous belief by John about the outcome of the spin "measurement" he has performed cannot even be related to an objective property of the electron after the "measurement"; in particular it cannot be used to foresee the outcomes of subsequent measurements, i.e., in general it has no predictive value. Of course, the situation would have been completely different if the particle $P$ in John 's device were a macroscopic "pointer". In that case, when the state on the right hand side of equation (12.4) obtains, also in the de Broglie–Bohm theory one can assert that things go as if wavepacket reduction had occurred, the reason for this being that it is practically impossible to completely reverse the measuring process. In such a case, therefore, the belief of the observer about the spin of the electron corresponds to an "effective determinateness" of the spin of the electron after the measurement, and all observers performing subsequent measurements of the same component of the spin find the same result which has been found by the first (macroscopic) observer[67]. In our opinion, this analysis puts into evidence that, even though in the de Broglie–Bohm scheme John can be considered to have definite perceptions as he claims, no validity, in the spirit of the theory, can be attributed to them.

We come now to the consideration of the many–minds approach. As already discussed in section 4, what a many–minds theory takes physics to be ultimately about is what sentient observers think. As a consequence, the adoption of a many–minds position implies a dualistic attitude: there are a physical universe, described by the statevector obeying an exact Schrödinger's equation, and a mental universe made up of the impressions of sentient observers (an expression which, in this context is synonymous with observers with minds). In such a theory even the beliefs of human observers are unavoidably not always "true" (i.e., reflecting objective properties of the physical universe) but have a sort of "effective validity" (in the sense that the future evolution of the mental states of such beings will proceed, in general, as if their beliefs were true). Obviously, within such a theory, the possibility to attribute effective validity to a belief requires a reliable memory recording the belief itself.

It seems to us that the fundamental criterion to judge whether the many–minds theory can accommodate other kinds of sentient beings has to be related to its be-

[67] We want to stress that this holds independently of the fact that at the instants of the subsequent measurements the brain of the first observer be still active and healthy or not.

ing able to guarantee to their beliefs the "effective validity" which is required to characterize the beliefs of human beings. Since, as already remarked, the "effective validity" criterion requires some sort of memory, one is compelled to choose between the following alternatives: either he attributes to John a macroscopic memory, but in this case the whole criticism to the dynamical reduction program breaks down; or, alternatively, he assumes that not only the formation of beliefs but also the memory storing process involves only microscopic changes, e.g., the displacement of $P$, and nothing else. In such a case, for obvious reasons (see the preceding discussion and/or take into account that wavepackets unavoidably spread), John's memory has not the level of reliability which is necessary to allow the attribution of an effective validity to his (ephemeral) beliefs.

Concluding, the above analysis has made plausible that even on the basis of quantum mechanical considerations one is led to recognize that consciousness and memory must be macroscopically supported in order to exhibit those features of reliability and/or effective validity that are necessary to make them meaningful and which are characteristic of human consciousness and memory; thus, both QMSL and CSL allow us to "close the circle".

# Part IV
# Relativistic Dynamical Reduction Models

## 13 White noise models: general framework and examples

J. Bell [32], in explicating QMSL, immediately identified two aspects of it which required further investigations:

1. The model does not respect the symmetry requirements for systems of identical particles.

2. The introduction of the localizations assigns a special role to position and requires a smearing on space, which makes it quite problematic to find a relativistic generalization of it.

The first difficulty has been overcome by CSL, in which the sudden localizations of QMSL have been replaced by a continuous stochastic evolution of the statevector. Steps toward a solution of the second problem have been made with the introduction of relativistic CSL models [67, 138, 89, 139, 140], which are the subject of this fourth part of the report.

In the present section we consider the general problem of describing relativistic dynamical reductions in terms of white–noise stochastic differential equations. In subsection 13.1 we analyze the issue of stochastic Galileian invariance within non relativistic QMSL and CSL, which guarantees that all the physical predictions of the above models are invariant under the Galilei group of transformations. We will see that stochastic invariance puts very severe limitations on the type of stochastic processes which can be used to describe spontaneous collapses.

In subsection 13.2 we set up a relativistic (and stochastically invariant under Lorentz transformation) model for decoherence (i.e. for ensemble or von Neumann reductions) which is useful as a first step towards the formulation of relativistic dynamical reduction models: these will be considered in subsection 13.3.

In subsections 13.4 and 13.5 we analyze the invariance and reduction properties of such models, respectively. In the final subsection, we discuss a specific relativistic model of spontaneous collapse which has all the desired properties of QMSL and CSL, except that it induces an infinite increase of energy per unit time and unit volume on physical system. This model, then, is not fully consistent; actually, this is a feature of all models in which quantum fields are locally coupled to white noises.

## 13.1 Stochastic Galileian invariance of QMSL and CSL

To bring out some concepts which will be useful in the following subsections, it is appropriate to consider the transformation and the invariance properties of non relativistic CSL.

Let us start by limiting our considerations to the evolution equation for the statistical operator and let us consider two observers $O$ and $O'$ related by a transformation of the Galilei group. We take the so–called passive point of view according to which the two observers look at the same physical situation. For simplicity, let us suppose that the transformation connecting $O$ and $O'$ is a translation in space of an amount $\mathbf{a}$ and a translation in time of an amount $\tau$, so that

$$\mathbf{r}' \;=\; \mathbf{r} - \mathbf{a}, \qquad\qquad t' \;=\; t - \tau \tag{13.1}$$

Let the observer $O$ describe the physical situation at his subjective time $t$ by the statistical operator $\rho(t)$. Observer $O'$, at the same objective time, i.e., at his subjective time $t' = t - \tau$, will describe the physical situation by the statistical operator

$$\rho'(t') \quad = \quad U(\mathbf{a})\,\rho(t)\,U^\dagger(\mathbf{a}) \tag{13.2}$$

where $U(\mathbf{a}) = e^{i\,\mathbf{P}\cdot\mathbf{a}}$ is the usual unitary operator inducing the space translation. The dynamical equation for the statistical operator for observer $O'$ is then

$$\frac{d\,\rho'(t')}{dt'} \quad = \quad U(\mathbf{a})\,\frac{d\,\rho(t)}{dt}\,U^\dagger(\mathbf{a}) \tag{13.3}$$

Substituting equation (7.51), describing the evolution of the statistical operator for the observer $O$, into the right hand side of equation (13.3), one gets[68]

$$\begin{aligned}\frac{d\,\rho'(t')}{dt'} \quad &= \quad -i\,U(\mathbf{a})\,[H(t),\rho(t)]\,U^\dagger(\mathbf{a}) \;+\; \gamma\sum_i U(\mathbf{a})\,A_i\rho(t)A_i\,U^\dagger(\mathbf{a}) \\ &\quad - \frac{\gamma}{2}\,U(\mathbf{a})\,\sum_i\{A_i^2,\rho(t)\}\,U^\dagger(\mathbf{a}).\end{aligned} \tag{13.4}$$

If $H$ is invariant under space and time translations

$$H'(t') \;\equiv\; U(\mathbf{a})\,H(t)\,U^\dagger(\mathbf{a}) \;=\; H(t') \tag{13.5}$$

and if, moreover

$$\sum_i U(\mathbf{a})\,A_i\,U^\dagger(\mathbf{a})\,X\,U(\mathbf{a})\,A_i\,U^\dagger(\mathbf{a}) \;=\; \sum_i A_i\,X\,A_i \tag{13.6}$$

for any bounded operator $X$, then equation (13.4) implies

$$\frac{d\,\rho'(t')}{dt'} \quad = \quad -i\,[H,\rho'(t')] \;+\; \gamma\sum_i A_i\rho'(t')A_i \;-\; \frac{\gamma}{2}\sum_i\left\{A_i^2,\rho'(t')\right\}. \tag{13.7}$$

[68] Throughout this section, we set $\hbar = 1$.

i.e., the theory is invariant for space and time translations. If the same holds for all transformations of the restricted Galilei group, we have invariance for the transformations of this group. QMSL and CSL actually possess this invariance property.

Nonetheless, it is important to stress that there is a difference between equations of the type we are considering and the usual case in which one has a purely Hamiltonian evolution, with respect to the connection between invariance and representations of the symmetry group. This key difference arises from the fact that while in the standard case one can always relate the statistical operators used by $O$ and $O'$ to describe the physical situation at the same subjective time t, in the present case this cannot be done in general, when one considers negative values of $t$ in equation (13.1). In fact, let us suppose that $O$, at his own time $t = 0$, is dealing with a physical system described by a pure state $\rho(0) = |\psi\rangle\langle\psi|$. Since the dynamical evolution transforms pure states into statistical mixtures, there is no way for $O$ to prepare a physical situation at his own time $\tau < 0$ (corresponding to $t' = 0$ for $O'$) such that it evolves into the pure state $\rho(0)$ at $t = 0$. Correspondingly, there is no way for $O'$ to prepare at his own time $t' = 0$ a statistical operator such that its evolved state at his time $-\tau > 0$ is $|\psi\rangle\langle\psi|$[69].

However, if the active point of view is taken and $O'$, at his time $t' = 0$, prepares the same state $\rho(0)$, and the above stated invariance requirements are satisfied, then $O$ and $O'$ will observe the same dynamical evolution at the ensemble level for the same (subjective) initial situation.

Coming now to the group theoretic point of view, since for the above reasons the map $\Sigma_t[\rho(0)]$, for a pure state $\rho(0)$ is not defined for negative $t$, one has to consider the proper Galilei semigroup $G_+$, with only forward time translations [142]. Any transformation $g \in G_+$ can be expressed as a transformation of the subgroup $G_0$ of $G_+$ which does not contain time translations, times a forward time translation

$$g \in G_+ \; : \; g = g_\tau g_0 \tag{13.8}$$

The map on the Banach space of the trace class operators

$$g : \rho \longrightarrow \rho_g, \qquad \rho_g \; = \; \Sigma_\tau[U(g_0)\,\rho\,U^\dagger(g_0)] \tag{13.9}$$

where $U(g_0)$ is the usual unitary representation of $G_0$ and $\Sigma_\tau$ is such that, for $\tau > 0$, $\Sigma_\tau[\rho(t)] = \rho(t+\tau)$ is the solution of equation (7.51), is then easily checked to yield a representation of $G_+$.

Up to now we have discussed the invariance properties of dynamical reduction models from the point of view of the statistical operator. However, since we are mainly interested in the evolution equation for the statevector, it is appropriate to discuss the problem of the invariance also at this level. For simplicity, we will limit ourselves to the discussion of space translations.

Let us start by considering the simpler linear Stratonovich equation (yielding only ensemble reduction, i.e. decoherence)

$$i\,\frac{d}{dt}\,|\psi_V(t)\rangle \quad = \quad V(\mathbf{r},t)\,|\psi_V(t)\rangle \tag{13.10}$$

If we denote by $O'$ an observer whose reference frame is translated by an amount $\mathbf{a}$ with respect to the frame of $O$, he will experience the potential

$$V'(\mathbf{r}',t) \quad = \quad V(\mathbf{r}'+\mathbf{a},t) \tag{13.11}$$

[69]Concerning this point, we call the attention of the reader to D. Albert's investigations [141] on the impact of dynamical reduction models on statistical mechanics and thermodynamics. He points out that, precisely due to their fundamentally irreversible nature, such models allow, in his opinion, a much more satisfactory derivation of the thermodynamical tendency to equilibrium.

so that, for a particular realization of $V$, there is no invariance.

However, since we are dealing with a fundamentally stochastic theory, the invariance requirement has to be formulated in the appropriate way. We will say that the theory is **stochastically invariant** under space translations if, for all observers $O'$, translated by any $\mathbf{a}$ with respect to $O$, the stochastic ensemble of potentials is the same and characterized by the same probabilities. This is equivalent to requiring that, if $V(\mathbf{r},t)$ is a possible sample function for $O$, then $V(\mathbf{r}-\mathbf{a},t)$, for any $\mathbf{a}$, is also a possible sample function for him, having the same probability of occurrence of $V(\mathbf{r},t)$, i.e.,

$$P_{\rm Raw}[V(\mathbf{r},t)] \quad = \quad P_{\rm Raw}[V(\mathbf{r}-\mathbf{a},t)] \tag{13.12}$$

Note that this is automatically guaranteed by the form (8.7) for the mean value and covariance function of the gaussian noise.

In the case of the model based on equation (8.8) describing Heisenberg reduction processes, a separate discussion is needed, since the stochastic invariance requirement has to be referred to the cooked probabilities which depend on the initial statevector. Let us therefore consider two observers $O$ and $O'$ and suppose they prepare the same (subjective) state $|\psi(0)\rangle$ at time $t=0$. The probability density of occurrence of the same (subjective) potential $V(\mathbf{r},t)$ is, for the two observers,

$$\begin{aligned} P^{O'}_{\rm Cook}[V(\mathbf{r},t)] &= P^{O'}_{\rm Raw}[V(\mathbf{r},t)]\,\||\psi^{O'}_V(t)\rangle\|^2 \\ P^{O}_{\rm Cook}[V(\mathbf{r},t)] &= P^{O}_{\rm Raw}[V(\mathbf{r},t)]\,\||\psi^{O}_V(t)\rangle\|^2 \end{aligned} \tag{13.13}$$

Since $|\psi^{O'}_V(t)\rangle$ and $|\psi^{O}_V(t)\rangle$ are the solutions of equation (8.8) with the same (subjective) potential and satisfy the same initial conditions, they coincide. Moreover, due to equation (13.12), $P^{O'}_{\rm Raw}[V(\mathbf{r},t)] = P^{O}_{\rm Raw}[V(\mathbf{r},t)]$, implying

$$P^{O'}_{\rm Cook}[V(\mathbf{r},t)] \quad = \quad P^{O}_{\rm Cook}[V(\mathbf{r},t)] \tag{13.14}$$

This guarantees the invariance from the active point of view, i.e., the observers cannot, by making physical experiments in their own frames, discover that they are displaced. They agree on the statistical distributions of future outcomes.

## 13.2 Quantum Field Theory with a Hermitian stochastic coupling: the case of decoherence

In trying to set up the framework for a relativistic generalization of reduction models, we adopt the quantum field theoretic point of view. We remark that the analogue of the idea of considering, within a non–relativistic framework, a stochastic potential $V(\mathbf{r},t)$ consists in assuming that the Lagrangian density for fields contains a stochastic interaction term. In this subsection we consider a model analogous to the non–relativistic ones discussed in subsection 3.2, yielding only ensemble, not individual reductions.

Let us consider, in the context of quantum field theory, the Lagrangian density

$$L(x) \quad = \quad L_0(x) \;+\; L_I(x)V(x) \tag{13.15}$$

where $L_0$ and $L_I$ are Lorentz scalar functions of the fields (for the moment we do not need to specify the fields we deal with). We assume that $L_I$ does not depend on the derivatives of the fields, and that $V(x)$ is a $c$–number stochastic process which is a scalar with respect to transformations of the restricted Poincarè group, i.e., that under the change of variables $x' = \Lambda x + b$, it transforms according to

$$V'(x') \quad = \quad V[\Lambda^{-1}(x'-b)]. \tag{13.16}$$

We will also assume that $V(x)$ is a Gaussian noise with zero mean and, to get a relativistic stochastically invariant theory, that its covariance is an invariant function

$$\langle\langle V(x)\, V(x')\rangle\rangle \quad = \quad A(x-x') \tag{13.17}$$

with $A(\Lambda^{-1}x) = A(x)$.

As discussed in the previous subsection, stochastic invariance requires different observers to agree on the unfolding of physical processes. This, in turn, is guaranteed by the condition that the family of all sample functions $V(x)$ and the probability density of occurrence of the same (subjective) sample function be the same for all observers. This is achieved by requiring that, for a single observer,

$$P_{\rm Raw}[V(x)] \quad = \quad P_{\rm Raw}[V(\Lambda(x+b))] \tag{13.18}$$

We stress that property (13.18) holds automatically if the covariance is a relativistically invariant function. In fact, from

$$P_{\rm Raw}[V(x)] \quad = \quad \frac{1}{N}\, e^{\,-\frac{1}{2}\,\int dx\, dx'\, V(x)\tilde{A}(x-x')V(x')} \tag{13.19}$$

[where we have denoted by $\tilde{A}(x-x')$ the function satisfying $\int dx'' A(x-x'')\tilde{A}(x''-x') = \delta(x-x')$], one gets immediately, using the scalar nature of $A$ and consequently of $\tilde{A}$, that

$$P_{\rm Raw}[V(\Lambda(x+b))] \quad = \quad P_{\rm Raw}[V(x)]. \tag{13.20}$$

The most natural generalization of the case discussed in the previous subsection is obtained by assuming that $V(x)$ is a white noise in all variables, i.e.,

$$\langle\langle V(x)\, V(x')\rangle\rangle \quad = \quad A(x-x') \quad = \quad \lambda\delta(x-x'). \tag{13.21}$$

We study, first of all, the physical consequences of the stochastic coupling $L_I(x)V(x)$. In Schrödinger's picture we have, for a given $V(x)$, the evolution equation:

$$i\frac{d}{dt}\,|\psi_V(t)\rangle \quad = \quad \left[H_0 \;-\; \int d^3x\, L_I(\mathbf{x},0)V(\mathbf{x},t)\right]|\psi_V(t)\rangle \tag{13.22}$$

where $H_0$ is the Hamiltonian corresponding to $L_0$. Equation (13.22) implies

$$|\psi_V(t)\rangle \quad = \quad T\, e^{\,-iH_0t \;+\; i\,\int_0^t d\tau \int d^3x\, L_i(\mathbf{x},0)V(\mathbf{x},\tau)}\,|\psi(0)\rangle \tag{13.23}$$

This equation shows how, for a given initial state $|\psi(0)\rangle$, one gets an ensemble of states $|\psi_V(t)\rangle$ at time $t$, according to the particular realization of the stochastic process. The statistical ensemble can then be described by the statistical operator obtained by averaging over the sample functions. In the case under consideration one gets a closed evolution equation for the statistical operator. In fact, we observe that, due to the fact that $L_I(x)$ does not depend on the derivatives of the fields

$$[L_I(\mathbf{x},0), L_I(\mathbf{x}',0)] \quad = \quad 0. \qquad\qquad \forall\, \mathbf{x},\, \mathbf{x}' \tag{13.24}$$

We then have:

$$\begin{aligned}
\rho(t+\epsilon) \quad = \quad & \Big\langle\!\!\Big\langle \left[1 - iH_0\epsilon + i\int_t^{t+\epsilon} d\tau \int d^3x\, L_I(\mathbf{x},0)V(\mathbf{x},\tau)\right. \\
& \left. -\frac{1}{2}\int_t^{t+\epsilon} d\tau\, d\tau' \int d^3x\, d^3x'\, L_I(\mathbf{x},0)L_I(\mathbf{x}',0)V(\mathbf{x},\tau)V(\mathbf{x}',\tau')\right] \\
& |\psi_V(t)\rangle\langle\psi_V(t)| \left[1 + iH_0\epsilon - i\int_t^{t+\epsilon} d\tau \int d^3x\, L_I(\mathbf{x},0)V(\mathbf{x},\tau)\right. \\
& \left. -\frac{1}{2}\int_t^{t+\epsilon} d\tau\, d\tau' \int d^3x\, d^3x'\, L_I(\mathbf{x},0)L_I(\mathbf{x}',0)V(\mathbf{x},\tau)V(\mathbf{x}',\tau')\right] \Big\rangle\!\!\Big\rangle
\end{aligned} \tag{13.25}$$

We recall now the properties associated with a zero mean gaussian probability distribution

$$\begin{aligned} \langle\!\langle V(\mathbf{x}_1,t_1)\dots V(\mathbf{x}_n,t_n)\rangle\!\rangle &= 0 \qquad\qquad\qquad\qquad \text{for } n \text{ odd} \\ \langle\!\langle V(\mathbf{x}_1,t_1)\dots V(\mathbf{x}_n,t_n)\rangle\!\rangle &= \sum_{\text{all pairs}} \langle\!\langle V(\mathbf{x}_i,t_i)V(\mathbf{x}_j,t_j)\rangle\!\rangle \cdot \\ &\qquad \cdot\langle\!\langle V(\mathbf{x}_k,t_k)V(\mathbf{x}_l,t_l)\rangle\!\rangle \quad \text{for } n \text{ even.} \end{aligned} \tag{13.26}$$

¿From (13.25) we then have

$$\frac{d}{dt}\,\rho(t) \;=\; -i[H_0,\rho(t)] + \lambda\int d^3x\, L_I(\mathbf{x},0)\rho(t)L_I(\mathbf{x},0) - \frac{\lambda}{2}\int d^3x\, \left\{L_I^2(\mathbf{x},0),\rho(t)\right\}. \tag{13.27}$$

Note that the obtained equation is of the Lindblad type.

The non–Hamiltonian terms in equation (13.27) imply a suppression of the off–diagonal elements of the statistical operator in the basis of the common eigenstates of the commuting operators $L_I(\mathbf{x},0)$. Putting

$$L_I(\mathbf{x},0)\,|\dots\nu\dots\rangle \;=\; \nu(\mathbf{x})\,|\dots\nu\dots\rangle \tag{13.28}$$

one gets, when the Hamiltonian term in (13.27) is disregarded,

$$\langle\dots\nu\dots|\rho(t)|\dots\nu'\dots\rangle \;=\; e^{-\frac{\lambda}{2}t\int d^3x\,[\nu(\mathbf{x})-\nu'(\mathbf{x})]^2}\,\langle\dots\nu\dots|\rho(0)|\dots\nu'\dots\rangle \tag{13.29}$$

As in the non–relativistic case, however, for a single realization of the stochastic potential $V(\mathbf{x},t)$, the statevector is not driven into one of the eigenmanifolds characterized by a given $\nu(\mathbf{x})$, since $|\langle\dots\nu\dots|\psi_V(t)\rangle|^2$ does not change with time. These considerations point out that, in order to have Heisenberg reductions, one has to resort to a skew–hermitian coupling with the noise.

Equation (13.27) for the statistical operator is not manifestly covariant, even though, from the procedure which has been followed to derive it, we know that the theory is stochastically invariant. To obtain a manifestly covariant description of the statistical operator evolution, we note that the model presented above is obviously equivalent to the following scheme:

1. Assume that the fields are solutions of the Heisenberg equations obtained in the standard way from the Lagrangian density $L_0(x)$ (note that we do not require $L_0(x)$ to describe free fields).

2. Assume that the evolution of the statevector is governed by the Tomonaga–Schwinger equation

$$i\,\frac{\delta|\psi_V(\sigma)\rangle}{\delta\sigma(x)} \;=\; -L_I(x)V(x)|\psi_V(\sigma)\rangle, \tag{13.30}$$

$L_I(x)$ being a function of the fields considered in 1) which does not involve their derivatives. As a consequence of the assumptions about $L_I(x)$, for any two points $x,x'\in\sigma$, $\sigma$ being a spacelike surface, $[L_I(x),L_I(x')]=0$, and consequently equation (13.30) is integrable.

Let us consider the formal solution of equation (13.30):

$$|\psi_V(\sigma)\rangle \;=\; T\,e^{\,i\int_{\sigma_0}^{\sigma} d^4x\, L_I(x)V(x)}\,|\psi(\sigma_0)\rangle. \tag{13.31}$$

Defining

$$\rho(\sigma) \quad = \quad \langle\!\langle |\psi_V(\sigma)\rangle\langle\psi_V(\sigma)| \rangle\!\rangle \tag{13.32}$$

using (13.32), and following the procedure outlined in equations (13.25)–(13.27), we get the Tomonaga–Schwinger equation for the statistical operator

$$\frac{\delta\,\rho(\sigma)}{\delta\sigma(x)} \quad = \quad \lambda\, L_I(x)\rho(\sigma)L_I(x) \;-\; \frac{\lambda}{2}\,\left\{L_I^2(x), \rho(\sigma)\right\} \tag{13.33}$$

which is manifestly covariant.

## 13.3 White noise relativistic dynamical reduction models

In this subsection we present a stochastically invariant theory yielding Heisenberg reductions. To this purpose we keep assumption 1) of the previous subsection and we replace 2) by the requirement that $|\psi_V(\sigma)\rangle$, instead of being governed by equation (13.30), obeys the following equation of the Tomonaga–Schwinger type:

$$\frac{\delta|\psi_V(\sigma)\rangle}{\delta\sigma(x)} \quad = \quad [L_I(x)V(x) \;-\; \lambda\, L_I^2(x)]|\psi_V(\sigma)\rangle. \tag{13.34}$$

The main difference between the two equations (13.30) and (13.34) derives from the skew–hermitian character of the coupling to the stochastic $c$–number field. On the right hand side of (13.34) a term guaranteeing the conservation of the average value of the square norm of the state appears. It is important to remark that equation (13.34), for a given sample potential, does not conserve the norm of the statevector.

Let $|\psi_V(\sigma)\rangle$ be the solution of equation (13.34) for a given realization of the stochastic potential

$$|\psi_V(\sigma)\rangle \quad = \quad T\, e^{\displaystyle\int_{\sigma_0}^{\sigma} d^4x\,[L_I(x)V(x) - \lambda\, L_I^2(x)]}\,|\psi_V(\sigma_0)\rangle \tag{13.35}$$

and let us define the stochastic average

$$\rho(\sigma) \quad = \quad \langle\!\langle |\psi_V(\sigma)\rangle\langle\psi_V(\sigma)| \rangle\!\rangle. \tag{13.36}$$

Following the same procedure of the previous subsection one sees that $\rho(\sigma)$ still satisfies equation (13.33) derived in the Hermitian case.

As in the non–relativistic case we have then two conceptually different dynamical evolutions for the statevector, i.e., (13.30) and (13.34), which give rise to the same dynamics for the statistical operator and therefore to the same physical predictions at the ensemble level. The very definition (13.36) of the statistical operator, when confronted with the fact that the equation for the statevector does not preserve the norm, implies the adoption of the point of view that a cooking procedure, analogous to the one discussed in section 7.1, is necessary. This means that one has to consider normalized vectors $|\psi_V(\sigma)\rangle/\||\psi_V(\sigma)\rangle\|$ and has to attribute to the considered realization $V(x)$ of the stochastic potential, having support in the spacetime region lying between the two spacelike hypersurfaces $\sigma_0$ and $\sigma$, not the probability density $P[V(x)]$ given by (13.19), but a cooked probability density $P_{\rm Cook}[V(x)]$ given by

$$P_{\rm Cook}[V(x)] \quad = \quad P_{\rm Raw}[V(x)]\,\||\psi_V(\sigma)\rangle\|^2 \tag{13.37}$$

In the above equation $|\psi_V(\sigma)\rangle$ is the solution of equation (13.34) satisfying

$$|\psi_V(\sigma_0)\rangle \quad = \quad |\psi_0\rangle. \tag{13.38}$$

Before discussing the cooking procedure, the role of the counterterm, and the relativistic invariance of the theory, an important remark is necessary. As has been discussed in [67], at the level of the statistical operator the map $\Sigma_t$ does not exist when $t < 0$. For this reason, even at the statevector level, we will only consider equation (13.34) as yielding the evolution from the statevector associated to a given spacelike surface $\sigma_0$ to spacelike surfaces lying entirely in the future of $\sigma_0$.

For what concerns the properties of the cooking procedure one can immediately see that equation (13.33) preserves the trace of $\rho$, which amounts to the statement that equation (13.34) preserves the average of the square norm of the statevector. In particular, this implies

$$\int \mathcal{D}[V]\, P_{\rm Cook}[V(x)] \quad = \quad \int \mathcal{D}[V]\, P[V(x)]\, \| |\psi_V(\sigma)\rangle \|^2 \quad = \quad 1 \tag{13.39}$$

which shows that the requirement that the cooked probability density sums to 1, is satisfied.

## 13.4 Transformation properties and invariance of the theory

We discuss now the transformation properties of the theory for a given realization of the stochastic potential, in going from a given reference frame $O$ to another one $O'$ related to it by a transformation of the restricted Poincaré group

$$(\Lambda, b) : \; x \; \rightarrow \; x' \; = \; \Lambda x \; + \; b \tag{13.40}$$

We remind the reader that in the Tomonaga–Schwinger formalism of conventional quantum field theory each reference frame $O$ is able to assign a statevector to each spacelike hypersurface. Our first concern is to demonstrate that the consistency of the composition law for Lorentz transformations remains intact when one resorts to the Tomonaga–Schwinger formalism.

Suppose that the transformation (13.40) involves a boost and consider a given spacelike surface $\sigma$ for $O$. The surface which is subjectively the same for $O'$ involves points which lie in the past of the surface $\sigma$ for $O$. Our previous discussion has pointed out that we will only use the Tomonaga–Schwinger equation to go from a given spacelike surface $\sigma$ to surfaces lying entirely in the future of $\sigma$. Therefore, contrary to the standard case, we are not allowed to raise here the following question: which state vector $|\psi'(\sigma)\rangle$ would $O'$ associate to his subjective surface $\sigma$ to describe the same physical situation described by $O$ who assigns the statevector $|\psi(\sigma)\rangle$ to his subjective surface $\sigma$?

We can, however, legitimately consider subjective surfaces $\sigma^{\sim\prime}$ for $O'$, such that they lie in the future of the surface $\sigma$ for $O$. Suppose the observer $O$ associates the statevector $|\psi_O(\sigma)\rangle$ to his subjective surface $\sigma$ to describe the physical situation. Let us denote by $\sigma^\sim$ the surface of $O$ which is objectively the same as the above–mentioned surface $\sigma^{\sim\prime}$ for $O'$. Then $O$ associates to $\sigma^\sim$ the state $|\psi_O(\sigma^\sim)\rangle$ obtained by solving equation (13.34) with the initial condition that it reduces to $|\psi_O(\sigma)\rangle$ on $\sigma$. We have

$$|\psi_O(\sigma^\sim)\rangle \quad = \quad \frac{S_V(\sigma^\sim, \sigma)|\psi_O(\sigma)\rangle}{\|S_V(\sigma^\sim, \sigma)|\psi_O(\sigma)\rangle\|} \tag{13.41}$$

with

$$S_V(\sigma^\sim, \sigma) \quad = \quad T\, e^{\displaystyle \int_\sigma^{\sigma^\sim} d^4x \, [L_I(x)V(x) - \lambda L_I^2(x)]} \tag{13.42}$$

Then the observer $O'$ will associate to his surface $\sigma^{\sim\prime}$ the statevector

$$|\psi_{O'}(\sigma^{\sim\prime})\rangle \; = \; U(\Lambda, b)|\psi_O(\sigma^\sim)\rangle \; = \; \frac{U(\Lambda, b)S_V(\sigma^\sim, \sigma)|\psi_O(\sigma)\rangle}{\|S_V(\sigma^\sim, \sigma)|\psi_O(\sigma)\rangle\|} \tag{13.43}$$

In equation (13.43), $U(\Lambda, b)$ is the unitary operator whose infinitesimal generators $P^\mu$ and $J^{\mu\nu}$ are obtained in the standard way from the Lagrangian density $L_0(x)$. Let now $\sigma$, $\sigma^\sim$, $\sigma^{\sim\sim}$ be three spacelike surfaces for $O$ each lying entirely in the future of the previous ones. Let us consider two other observers $O'$ and $O''$ related by two successive Lorentz transformations (the generalization to Poincaré transformations is straightforward): $O' = \Lambda_1 O$, $O'' = \Lambda_2 O'$, and let us denote by $\sigma'$, $\sigma^{\sim\prime}$, $\sigma^{\sim\sim\prime}$ and $\sigma''$, $\sigma^{\sim\prime\prime}$, $\sigma^{\sim\sim\prime\prime}$ the above surfaces as seen by $O'$ and $O''$, respectively.

The map (13.42), for a given realization of the stochastic potential, has the following property. Suppose $O$ assigns the state $|\psi_O(\sigma)\rangle$ to the surface $\sigma$. Then $O'$ assigns the state (13.43) to the surface $\sigma^{\sim\prime}$. For $O'$ this state evolves according to the Tomonaga–Schwinger equation (13.34) with $V'(x') = V(\Lambda_1^{-1}x')$ from $\sigma^{\sim\prime}$ to $\sigma^{\sim\sim\prime}$:

$$|\psi_{O'}(\sigma^{\sim\sim\prime})\rangle \quad = \quad \frac{S'_{V'}(\sigma^{\sim\sim\prime}, \sigma^{\sim\prime})|\psi_{O'}(\sigma^{\sim\prime})\rangle}{\|S'_{V'}(\sigma^{\sim\sim\prime}, \sigma^{\sim\prime})|\psi_{O'}(\sigma^{\sim\prime})\rangle\|}. \tag{13.44}$$

The observer $O''$ will describe the final situation by assigning the statevector

$$|\psi_{O''}(\sigma^{\sim\sim\prime\prime})\rangle \quad = \quad U(\Lambda_2)|\psi_{O'}(\sigma^{\sim\sim\prime})\rangle \tag{13.45}$$

to the surface $\sigma^{\sim\sim\prime\prime}$. On the other hand, one can consider the evolution from $\sigma$ to $\sigma^{\sim\sim}$ as seen from $O$,

$$|\psi_O(\sigma^{\sim\sim})\rangle \quad = \quad \frac{S_V(\sigma^{\sim\sim}, \sigma)|\psi_O(\sigma)\rangle}{\|S_V(\sigma^{\sim\sim}, \sigma)|\psi_O(\sigma)\rangle\|}. \tag{13.46}$$

and then look at it from $O'' = \Lambda_2\Lambda_1 O$, getting the state

$$|\psi^*_{O''}(\sigma^{\sim\sim\prime})\rangle \quad = \quad U(\Lambda_2\Lambda_1)|\psi_O(\sigma^{\sim\sim})\rangle. \tag{13.47}$$

For consistency, $|\psi^*_{O''}(\sigma^{\sim\sim\prime})\rangle$ must coincide with $|\psi_{O''}(\sigma^{\sim\sim\prime})\rangle$. This can be easily proved to hold.

Although we have just seen that the theory implies an assignment of a statevector to a hypersurface by any observer that fulfills the Lorentz (also Poincaré) group requirements, this does not mean that the description is Lorentz invariant. In fact, because a particular realization of the stochastic potential $V$ looks different from two different reference frames, the map $S_V(\sigma^\sim, \sigma)$ obviously depends upon the reference frame $O$. This shows that, at the individual level, the theory does not posses the property of standard (i.e., non–stochastic) Lorentz invariance. However, for stochastic Lorentz invariance, one must consider the ensemble of possible sample potentials. When one takes into account the Lorentz invariance of the requirement (13.17) for the correlation function $\langle\langle V(x)\, V(x')\rangle\rangle$, and the invariance of the cooking procedure that must be performed to get the physics of the problem, one can easily prove, along the same lines as in the non relativistic case, that there is stochastic invariance in the statevector language, i.e., the stochastic ensemble of evolution operators $S_V(\sigma^\sim, \sigma)$ is the same in each reference frame.

In the language of the statistical operator, invariance is evident from the manifestly covariant Tomonaga–Schwinger form (13.33) of the evolution equation.

## 13.5 Reduction properties

Once we have guaranteed the invariance of the formalism by using its Tomonaga–Schwinger formulation, in order to discuss specific features of the process, we can consider $t = \text{const}$ hyperplanes in the Schrödinger picture. In so doing, the equation corresponding to (13.34) is

$$\frac{d\,|\psi_V(t)\rangle}{dt} \;=\; \left[-iH_0 \,+\, \int d^3x\, \left(L_I(\mathbf{x}, 0)V(\mathbf{x}, t) \,-\, \lambda L_I^2(\mathbf{x}, 0)\right)\right] |\psi_V(t)\rangle \tag{13.48}$$

This is a Stratonovich equation for the statevector. By standard procedures one can consider the corresponding Itô stochastic dynamical equation

$$d\,|\psi_V(t)\rangle \;=\; \left[\left(-iH_0 \,-\, \frac{\lambda}{2}\int d^3x\, L_I^2(\mathbf{x},0)\right)dt \,+\, \int d^3x\, L_I(\mathbf{x},0)dV(\mathbf{x})\right]|\psi_V(t)\rangle, \tag{13.49}$$

where $dV(\mathbf{x})$ is a real Wiener process satisfying

$$\langle\!\langle dV(\mathbf{x})\rangle\!\rangle = 0 \qquad\qquad \langle\!\langle dV(\mathbf{x})dV(\mathbf{y})\rangle\!\rangle = \lambda\delta(\mathbf{x}-\mathbf{y})dt. \tag{13.50}$$

Note that both equations (13.48) and (13.49) do not conserve the norm of the statevector but they conserve the average of its squared norm.

As discussed in section 7 one can take two equivalent attitudes to describe the physics of the process. One can solve equation (13.48) or (13.49) for a given initial condition, and then one can consider the normalized vectors $|\psi_V(t)\rangle/||\psi_V(t)\rangle\|$ at time $t$ and assume that the probability of their occurrence is obtained by cooking the probability density of occurrence of $V(x)$, i.e., by multiplying it by $\||\psi_V(t)\rangle\|^2$. Alternatively, one can consider the nonlinear stochastic dynamical equation

$$\begin{aligned} d\,|\psi_V(t)\rangle \;=\; & \left[\left(-iH_0 \,-\, \frac{\lambda}{2}\int d^3x\,[L_I^2(\mathbf{x},0) \,-\, \langle L_I(\mathbf{x},0)\rangle]^2\right)dt \,+\right. \\ & \left. +\int d^3x\,(L_I(\mathbf{x},0) \,-\, \langle L_I(\mathbf{x},0)\rangle)\,dV(\mathbf{x})\right]|\psi_V(t)\rangle, \end{aligned} \tag{13.51}$$

(where $\langle L_I(\mathbf{x},0)\rangle \,=\, \langle\psi_V(t)|L_I(\mathbf{x},0)|\psi_V(t)\rangle$), without cooking, i.e., using just the probability weighting of $V(x)$.

As we know from the discussion of the previous sections, when one disregards the Hamiltonian term in (13.51), the evolution leads the state vector to enter one of the common eigenmanifolds of the commuting operators $L_I(\mathbf{x},0)$. The theory induces therefore Heisenberg reductions, as required.

## 13.6 The models so far proposed

In this subsection we will consider some specific choices for the Lagrangian densities $L_0$ and $L_I$ which, when used in connection with the formalism presented in the previous subsections, yield stochastically invariant relativistic reduction models. The goal is to build up a framework leading to localization in position of the basic constituents of all matter.

The simplest and most immediate idea would be to introduce a fermion field for particles of mass $M$ and to choose for the Lagrangian density the expressions

$$L_0(x) \;=\; \overline{\psi}(x)\,(i\gamma^\mu\partial_\mu - M)\,\psi(x), \qquad\qquad L_I(x) \;=\; \overline{\psi}(x)\,\psi(x) \tag{13.52}$$

However, in the non relativistic limit, the dynamics induced by the above choice would lead to infinitely sharp position localizations for the fermions, and this, as is well known [143], is unacceptable.

We have then to enrich the formalism. This can be done by following the proposal put forward in reference [89]. One considers a fermion field coupled to a real scalar meson field and chooses

$$\begin{aligned} L_0(x) \;&=\; \frac{1}{2}\left[\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x)\right] \,+\, \overline{\psi}(x)\,(i\gamma^\mu\partial_\mu - M)\,\psi(x) \,+ \\ &\quad\; \eta\overline{\psi}(x)\,\psi(x)\,\phi(x) \\ L_I(x) \;&=\; \phi(x). \end{aligned} \tag{13.53}$$

The introduction of the meson field coupled to the fermion field allows one to overcome the difficulty of the infinitely sharp localization for fermions met by the previous model. In the Schrödinger representation the evolution equation for the state vector corresponding to the choice (13.53) is

$$\frac{d\,|\psi_V(t)\rangle}{dt} \;=\; \left[-iH_0 \,+\, \int d^3z\, \left(\phi(\mathbf{z},0)V(\mathbf{x},t) \,-\, \lambda\phi^2(\mathbf{z},0)\right)\right] |\psi_V(t)\rangle \qquad (13.54)$$

Let us consider now the non–relativistic infinite mass limit for fermions and let us confine our discussion to the sector containing one fermion (note that in the limit the fermion number is a conserved quantity). The state of a fermion at position $\mathbf{q}$ is the "dressed" state

$$|1\mathbf{q}\rangle \quad = \quad a^\dagger(\mathbf{q})\, A(\mathbf{q})\, |0\rangle \qquad (13.55)$$

where $a^\dagger(\mathbf{q})$ is the creation operator for a fermion at $\mathbf{q}$ and $A(\mathbf{q})\,|0\rangle = |m_\mathbf{q}\rangle$ is a coherent state which can be characterized as either the common eigenstate of the annihilation operators of physical mesons with eigenvalue zero or as the common eigenstate of the annihilation operators $b(\mathbf{k})$ of bare mesons with momentum $\mathbf{k}$, with eigenvalues $(\eta/\sqrt{2})e^{-i\mathbf{k}\cdot\mathbf{q}}/(2\pi k_0)^{3/2}$.

To be rigorous, in the three–dimensional case, one should introduce an ultraviolet cut–off on the momentum of mesons in the interaction term to avoid ultraviolet singularities. In the limit in which the cut–off is removed the meson states $|m_\mathbf{q}\rangle$, $|m_{\mathbf{q}'}\rangle$ tend to become orthogonal for $\mathbf{q} \neq \mathbf{q}'$. In this way, due to the coupling of the fermion field to the meson field, the "position" of one fermion turns out to be strictly correlated to states of the meson field which are approximately orthogonal.

We note that the mean value of $\phi(\mathbf{z},0)$ in the state $|m_\mathbf{q}\rangle$ turns out to be

$$\langle m_\mathbf{q}|\phi(\mathbf{z},0)|m_\mathbf{q}\rangle \quad = \quad f(\mathbf{z}-\mathbf{q}) \quad = \quad \frac{\eta e^{-m|\mathbf{z}-\mathbf{q}|}}{4\pi|\mathbf{z}-\mathbf{q}|}. \qquad (13.56)$$

In what follows, in order to discuss the localization properties of the model for physical fermions, we make a gross simplification (which coincides with the non relativistic approximation), i.e., we treat the states $|m_\mathbf{q}\rangle$ as eigenstates of $\phi(\mathbf{z},0)$ pertaining to the eigenvalue $f(\mathbf{z}-\mathbf{q})$. Let us then consider the physical state for one fermion

$$|\psi(t)\rangle \quad = \quad \int d^3q\, \psi(\mathbf{q},t)|1\mathbf{q}\rangle. \qquad (13.57)$$

By substituting (13.57) into equation (13.54) and disregarding the standard Hamiltonian $H_0$, we get the equation for $\psi(\mathbf{q},t)$:

$$\frac{\partial\psi_V(\mathbf{q},t)}{\partial t} \;=\; \int d^3z\, f(\mathbf{z}-\mathbf{q})V(\mathbf{z},t)\,\psi_V(\mathbf{q},t) \,-\, \frac{\lambda\eta^2}{8\pi m}\,\psi_V(\mathbf{q},t), \qquad (13.58)$$

i.e.,

$$\frac{\partial\psi_V(\mathbf{q},t)}{\partial t} \;=\; V^\sim(\mathbf{q},t)\,\psi_V(\mathbf{q},t) \,-\, \frac{\lambda\eta^2}{8\pi m}\,\psi_V(\mathbf{q},t), \qquad (13.59)$$

with $V^\sim(\mathbf{q},t)$ a Gaussian noise with zero mean and covariance

$$\langle\!\langle V^\sim(\mathbf{q},t)\, V^\sim(\mathbf{q}',t')\rangle\!\rangle \quad = \quad \frac{\lambda\eta}{8\pi m}\, e^{-m|\mathbf{q}-\mathbf{q}'|}\delta(t-t'). \qquad (13.60)$$

Equation (13.59) is essentially the same as equation (8.8) of CSL for the case of a single particle. If one considers the sector with $N$ fermions, in the above approximations, one gets an equation of the CSL type [see equations (7.9) and (8.1)] with the operator

$$D(\mathbf{x}) \quad = \quad \frac{m^2}{4\pi}\int d^3z\, \frac{e^{-m|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|}\, a^\dagger(\mathbf{z})a(\mathbf{z}) \qquad (13.61)$$

taking the place of $N(\mathbf{x})$, and $(\lambda\eta^2)/m^4$ taking the place of $\gamma$.

Thus it appears reasonable that the model (13.53) possesses the desired localizing features. However, it also presents a serious difficulty. The evolution equation (13.27) for the statistical operator, specialized to the Lagrangian (13.53), is:

$$\frac{d}{dt}\,\rho(t) \;=\; -i[H_0,\rho(t)] + \lambda\int d^3z\,\phi(\mathbf{z},0)\rho(t)\phi(\mathbf{z},0) - \frac{\lambda}{2}\int d^3z\,\left\{\phi^2(\mathbf{z},0),\rho(t)\right\}. \tag{13.62}$$

Let us consider the Hamiltonian $H$ for the free meson field; by using (13.62) one can evaluate the increase per unit time of the mean value of $H$, getting

$$\frac{d\,\langle H\rangle}{dt} \quad = \quad -\frac{\lambda}{2}\int d^3z\,\langle[\phi(\mathbf{z},0),[\phi(\mathbf{z},0),H]]\rangle \tag{13.63}$$

i.e.,

$$\frac{d\,\langle H\rangle}{dt} \quad = \quad \frac{\lambda}{2}\int d^3z\,\delta(0) \tag{13.64}$$

Therefore, the increase per unit time and per unit volume of the mean value of the energy of the meson field turns out to be infinite. So, in addition to the desired reduction behaviour, the model displays an undesidered additional behaviour: because the white noise source is locally coupled to the meson field, it copiously produces mesons out of the vacuum[70]. We note that the now outlined difficulty does not show up in the non–relativistic approximation of the model discussed above [equations (13.58)–(13.61)] due to the gross simplification of treating the states $|m_{\mathbf{q}}\rangle$ as eigenstates of $\phi(\mathbf{z},0)$.

Since the divergences originate from the local coupling between the quantum fields and the white noise, the natural way to cure the infinite vacuum fluctuations is to replace the white noise with a more general Gaussian stochastic process; this possibility has been explored by P. Pearle [90, 91]. He considers an evolution equation for the statevector which is the straightforward generalization of the white noise equation (13.35). He then proves that — in the lowest order in perturbation theory — only the time–like components of the spectrum of the noise are responsible for the vacuum excitations; accordingly, he chooses as the spectrum of the noise that of a tachion of mass $\mu = \hbar/\alpha c \sim 1$ eV, where $\alpha$ is the QMSL localization parameter. Anyway, at higher orders in perturbation theory there are still vacuum excitations; to avoid such excitations, Pearle proposes to remove the time–ordering product from Feyman diagrams.

Recently, Nicrosini and Rimini [92] have proposed a different solution to the problem arising from the appearance of divergences. They couple the noise not to the quantum fields, but to "macroscopic operators" which are defined as the integral of the usual quantum fields over appropriately chosen (Lorenz invariant) spacetime surfaces. In other words, the coupling between the quantum fields and the white noise is not local anymore.

Both the attempts by Pearle and by Nicrosini and Rimini are promising; anyway, they still have to be studied in detail. In particular, it is not clear yet whether the evolution can be expressed in terms of a integrable Tomonaga–Schwinger equations: this is a necessary requirement in order to put on solid grounds any relativistic theory of dynamical reductions.

[70] A similar conclusion has been reached also by Adler and Brun in recent investigations on relativistic statevector collapse models [144].

# 14 Local and nonlocal features of relativistic CSL

As is well known, the quantum theory of measurement, in addition to the difficulties discussed in section 2 which constitute the main motivation for the consideration of dynamical reduction models, presents some further difficulties arising specifically from the assumed instantaneous nature of the collapse of the wavefunction. In particular, at the individual level of description, nonlocal features as well as odd aspects (from the relativistic point of view) emerge. Such problems have already been extensively discussed in the literature [13, 145, 146, 147], in the case of standard quantum mechanics: we will review them in subsection 14.1. It is interesting to look at them from the perspective of the relativistic dynamical reduction models, analyzed in the previous section: this will be the subject of subsection 14.2.

In the final section we discuss the problem of parameter dependence in dynamical reduction models. We will show that in the non linear model there is a set of realizations of the stochastic process for which parameter independence is violated for parallel spin components in a EPR–Bohm–like setup. Such a set has an appreciable probability of occurrence ($\simeq 1/2$). On the other hand, we will prove that the linear model exhibits only extremely small parameter dependence effects.

## 14.1 Quantum theory with the reduction postulate

### 14.1.1 Objective properties of individual systems

Suppose one accepts it as meaningful, within standard quantum theory, to consider an individual level of description with the possibility of attributing objective properties to a quantum system. As discussed in section 2.4, a natural attitude corresponding to the one first introduced in the celebrated EPR paper [5] is to assume the following. If an individual physical system $S$ is associated to a definite statevector $|\psi\rangle$ which is an eigenstate of an observable $A$ pertaining to the eigenvalue $a$, then one can state that "$S$ has the property $A = a$" or that "there exists an element of physical reality" referring to the considered observable. We remark that if we denote by $P_a$ the projection operator on the closed linear manifold of the eigenstates of $A$ belonging to the eigenvalue $a$, then

$$\langle\psi|P_a|\psi\rangle \quad = \quad 1 \tag{14.1}$$

We want to stress, however, that even within non–relativistic standard quantum mechanics, one is compelled to take the attitude of attributing objective properties to a system even when condition (14.1) is valid only to an extremely high degree of accuracy. To clarify this statement, we can think, for example, of the spin measurement of a spin–1/2 particle by a Stern–Gerlach apparatus. In such a case, the two spin values are strictly correlated to two states $\psi_1$ and $\psi_2$ describing the spatial degrees of freedom. Even though these wavefunctions are appreciably different from zero in two extremely narrow and distant regions, their supports cannot have a void intersection. As a consequence even an arbitrarily precise measurement of the position cannot reduce the statevector exactly to an eigenstate of the spin component. The final state unavoidably exhibits an (extremely slight) entanglement of position with spin variables and as such cannot be an eigenstate of a spin component operator.

Incidentally we remark that the above considerations are even more appropriate in the case of dynamical reduction models. In fact, on the one hand, such models, with the requirement that they induce Heisenberg reductions, are introduced just with the purpose of implying, at the individual level, the emergence of objective properties for macroscopic objects (in particular the property of being in one place rather than in another). Correspondingly, they induce indirectly the appearance of objective properties also for microscopic systems, at least when they interact with

macroscopic measuring–like devices. On the other hand, as is well known and as has been repeatedly stressed in references [67, 148, 89], within dynamical reduction models, the unavoidable persistence of the tails, the tiny but nonzero terms corresponding to the parts of a linear superposition which have been suppressed by the spontaneous localization process, prevents us from asserting with absolute certainty that the "macroscopic pointers" are in a definite space region, if one adopts the standard probability interpretation.

The conclusion is the same we have drawn in Section 11 concerning the tails problem, i.e., that within the dynamical reduction program is perfectly consistent to accept that it makes sense to attribute appropriate objective properties to individual systems even when the mean value of the projection operator on the eigenmanifold associated to the eigenvalue corresponding to the attributed property is not exactly equal to 1, but is extremely close to it.

### 14.1.2 Non locality

Nonlocal features[71] of quantum mechanics arise from the fact that, due to the instantaneous nature of the collapse of the wavefunction, possible actions performed in a certain space region can, under specific circumstances, induce immediate changes in distant regions. In this connection two important questions arise: first, do these changes correspond to some modifications of the physical situation in the distant region? Secondly, are these modifications detectable, so that one can take advantage of them to send faster than light signals?

To make precise and unambiguous these questions, it is necessary to specify the level of description of physical processes one is considering. In particular, it is important to make a clear distinction between the ensemble and the individual levels of description.

To understand the above situation, one can make reference either to the well–known EPR–Bohm type setup for an "entangled" state of a composite system $S = S_1 + S_2$, the components being far apart and non–interacting, or to the position measurement of a particle whose state is the linear superposition of two distant packets. In the first case, as is well known, at the level of the individual members of the ensemble, the far away system (let us say $S_2$) is [150] "steered or piloted into one or the other type of state" according to the measurement which is performed on $S_1$ and the specific result which is obtained. In the second case, let us write $\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$, where the states $\psi_1(x,t)$ and $\psi_2(x,t)$ have equal norms and are appreciably different from zero only in two far apart regions $\alpha_1$ and $\alpha_2$, respectively. Then, a measurement aimed to test whether the particle is in $\alpha_1$ and yielding, for example, the answer "no" ("yes"), instantaneously collapses $\psi(x,t)$ to $\psi_2(x,t)$ ($\psi_1(x,t)$). Correspondingly the quantity $\int_{\alpha_2} dx\, |\psi(x,t)|^2$ (i.e., "the mean value"[72] of the projection operator on region $\alpha_2$), changes from 1/2 to either 1 or 0 according to the outcome of the position measurement at $\alpha_1$. This puts into evidence how, if interpreted as a theory describing individual systems, quantum mechanics exhibits nonlocal features.

The situation is quite different when looked at from the ensemble point of view. In fact, as is well known [71, 72, 73, 74], no measurement procedure in a given region can change the statistical distribution of prospective measurement results in a distant region. Of course, this does not mean that, from the ensemble point of view, Quantum Mechanics displays a local character. The theory is still highly nonlocal (for example, Bell's inequalities hold), but this non locality cannot be used in any way to send faster that light signals to distant observers.

[71] For an exhaustive discussion, the reader is referred to the excellent book by M. Redhead [149].

[72] We are using the common phrase "mean value" to represent diagonal matrix elements like (14.1), even though the statistical connotation of this phrase has no meaning in our discussion.

These remarks, although made in the context of ordinary quantum theory with the reduction postulate, are not essentially modified (i.e., the word "instantaneously" must be changed to "in a split second" [32]) in the case of the CSL theory with its reduction dynamics.

### 14.1.3 Relativistic oddities with observations

In the above analysis we have discussed a measurement process in a given reference frame $O$. The consideration of the instantaneous change of the statevector induced by a measurement raises interesting questions when looked at by different observers. Since the distance between the two space regions $\alpha_1$ and $\alpha_2$ mentioned above can be arbitrarily large, even the passage to a reference frame which is moving with respect to $O$ with an arbitrarily small velocity can change the time order of simultaneous (for $O$) events occurring in the two regions.

To illustrate briefly the main points of the problem, we consider the observer $O$ looking at a system of one particle in the state $\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$ which is a superposition of two well–localized wavepackets propagating in opposite directions with respect to the origin $x = 0$. Disregarding the extension and the spreading of the wavepackets, we can represent the situation by the spacetime diagram of figure 5,

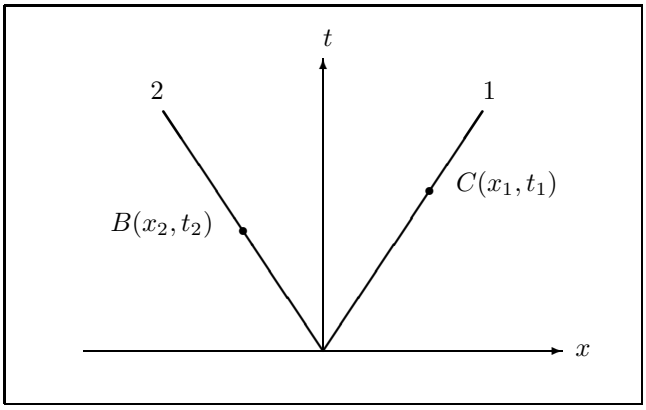



Figure 5: World lines of two well–localized wavepackets 1 and 2, belonging to a single particle which is detected at event $C$.

in which the two world lines 1 and 2 are associated with $\psi_1$ and $\psi_1$, respectively. Suppose that, at the spacetime point $C = (x_1, t_1)$ there is a device designed to test whether the particle is there, and let us suppose that, in the specific individual case we are considering, the result of the test is "yes". This is a covariant statement on which all observers must agree. If one adopts the wavepacket reduction postulate of standard quantum theory and one assumes that the collapse occurs for each reference frame along the hyperplane $t' = \text{const}$, where $t'$ is the subjective time of the event $C$ for such a frame, one meets a puzzling situation. Let us in fact consider an objective point $B$ on world line 2, which is spacelike separated from $C$ and which is labeled by $(x_2, t_2)$ (see figure 5). For $O$, $t_2 < t_1$ and, by the above assumption, no reduction has occurred at time $t_2$ and the statevector is $\psi(x, t_2)$. If one considers the projection operator $P_2$ on the space region around $x_2$, one has $\langle\psi|P_2|\psi\rangle \simeq 1/2$. Accordingly, we could say that the situation is such that, at time $t_2$, $O$ cannot attribute to the particle the "property" of being or not being in the region around $x_2$.

However, there exists an observer $O'$ such that $t'_2 > t'_1$, where $t'_2$ and $t'_1$ are the time labels attributed by $O'$ to the events $B$ and $C$, respectively. For $O'$ the particle has triggered the detector in $C$ at $t'_1$. Therefore at $t'_2$ the state of the system is $\psi_1$.

Then, for $O'$, the mean value of the projection operator $P_2$ at $t'_2$ is zero[73]. Observer $O'$ can then state that the particle has the property of "not being around $B$". Thus, $O$ and $O'$ do not agree on a statement referring to a local property at an objective spacetime point.

It is useful to note that this ambiguity occurs only for the points of the world line 2 which are spacelike with respect to $C$; for a point $B$ in the past of $C$ all observers agree in stating that the particle has no definite location while for a point $B$ in the future of $C$ all observers agree in saying that the particle "is not around $B$".

The above discussion follows essentially the one given in reference [145]. The consideration of these kinds of difficulties have led various authors to take different attitudes. Bloch [145] and Aharonov and Albert [147] derive from this the conclusion that one cannot attach an objective meaning to wavefunctions for individual systems. Hellwig and Kraus [146] have tried to solve the ambiguity about the wavefunction at a given objective spacetime point by requiring that the collapse of the state vector due to the measurement at $C$ takes place along the past light cone originating from $C$. Thus, at points outside the past light cone the statevector is reduced, while at points inside the past light cone the statevector is unreduced. This is a covariant statement and leads the authors to the identification of a unique statevector to be associated to any given spacetime point. However, such a prescription implies that there are spacelike surfaces (those crossing the past cone of $C$) to which it is not possible to associate a definite state vector. This, as nicely illustrated by Aharonov and Albert [147], forbids the consideration of nonlocal observables on these hypersurfaces; for example it does not allow one to speak consistently of the total charge of the system. Moreover, the assumption that the reduction occurs on the hypersurface delimiting the past light cone raises conceptual difficulties with the cause–effect relation.

## 14.2 Relativistic reduction models

We discuss here the local and nonlocal features of reduction models in the relativistic case. In order to investigate whether the dynamics presented in section 13.4 has nonlocal effects, we make reference to the procedure outlined in reference [151], i.e., we consider whether a modification of the Lagrangian density in a spacetime region $C$ can have effects in a region $B$ which is spacelike separated from it (this will be discussed in subsections 14.2.1 and 14.2.2). In particular, since we want to study the possibility of nonlocal effects due to the reducing character of the dynamics, we will take into account modifications of the Lagrangian density $L_I$ coupled to the noise.

The problems which we want to discuss require the consideration of "local observables." By this expression we mean the integral of a function of the fields and their derivatives in the interaction picture:

$$A_I(\sigma) \quad = \quad \int_\sigma dx'\, f_\alpha(x')\, F[\phi_I(x'), \partial_\mu \phi_I(x')] \tag{14.2}$$

with $f_\alpha(x')$ a function of class $C^\infty$ with compact support $\alpha$ on the spacelike surface $\sigma$. The physically interesting quantities, for our analysis, are the mean values of such local observables. As usual it is necessary to make precise the level at which the nonlocality problem is discussed. We will consider it, as before, both at the ensemble and at the individual level.

At this last level, we will discuss also questions analogous to those considered in subsection 14.1.3 which originate from looking at the wavepacket reduction postulate, taking the point of view of relativity theory. In the present context, they emerge

[73] Obviously, to be rigorous, both the statement that the state is $\psi_1$ or $\psi_2$, as well the consideration of the projection operators $P_1$ and $P_2$, are not correct, because one should consider a relativistic description of the system and of the observables. However, since $O'$ is moving with a very small velocity $v \ll c$ with respect to $O$, the above approximations are appropriate.

naturally from the relativistic dynamics described by the Tomonaga–Schwinger equation. In particular, it turns out that, for all Tomonaga–Schwinger surfaces coinciding on $\alpha$, the mean value of the local observable depends upon the specific Tomonaga–Schwinger surface on which it is evaluated (see subsection 14.2.3). This is not the case with the Tomonaga–Schwinger description of an ordinary relativistic quantum field theory, and such a difference gives rise to interesting questions about the possibility of attributing objective properties to the systems which we will discuss in subsection 14.2.4.

### 14.2.1 Ensemble level

As already emphasized, at the ensemble level, the statistical operator and therefore the physics of the two models considered in sections 13.2 and 13.3 coincide. Thus, to investigate properties referring to the statistical ensemble, one can make reference to the stochastic dynamics with hermitian coupling, which can be easily handled by familiar methods.

With reference to the model of section 13.3, we consider the mean value of a local observable $A_I(\sigma)$:

$$\langle A_I(\sigma)\rangle \quad = \quad \mathrm{Tr}\,[A_I(\sigma)\rho_I(\sigma)] \tag{14.3}$$

Let us denote by $U_V(\sigma,\sigma_0)$ the evolution operator

$$U_V(\sigma,\sigma_0) \quad = \quad T\, e^{\,i\int_{\sigma_0}^{\sigma} dx\, L_I(x)V(x)} \tag{14.4}$$

and by $A_{HV}(\sigma) = U_V^\dagger(\sigma,\sigma_0)A_I(\sigma)U_V(\sigma,\sigma_0)$ the observable in the Heisenberg picture which corresponds to $A_I(\sigma)$ when the realization $V$ of the stochastic potential occurs. Let $A_H(\sigma)$ be the stochastic average over V of $A_{HV}(\sigma)$

$$A_H(\sigma) \quad = \quad \int \mathcal{D}[V]\, P_{\mathrm{Raw}}[V]\, A_{HV}(\sigma). \tag{14.5}$$

We then have

$$\langle A_I(\sigma)\rangle \quad = \quad \mathrm{Tr}\,[A_H(\sigma)\,\rho(\sigma_0)]. \tag{14.6}$$

The support of $A_I(\sigma)$ defines a partition of spacetime into three regions: the future, the past, and the set of points which are spacelike separated from all points belonging to this support (see figure 6).

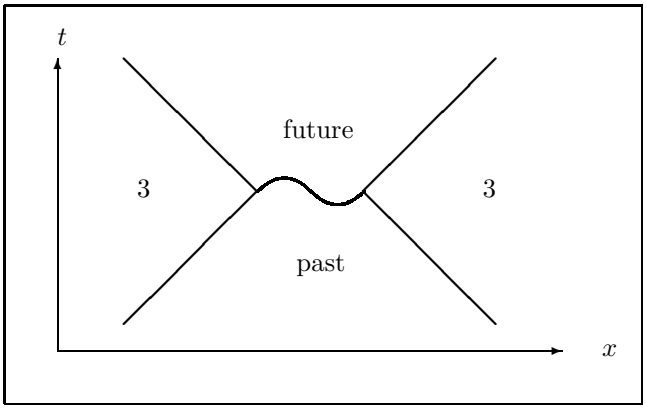



Figure 6: The support of the local observable $A_I(\sigma)$, and the set (3) of points bearing a spacelike relation to this support.

We choose now a spacetime region $C$ entirely contained in region 3 and we consider a modification of the Lagrangian density $L_I(x)$ coupled to the noise. We replace $L_I(x)$ with a new density $L_I^{\sim}(x) = L_I(x) + \Delta L_I(x)$, with $\Delta L_I(x)$ different from zero only for $x \in C$. If $A_{HV}^{\sim}(\sigma)$ denotes the local observable in the Heisenberg picture, when we replace $L_I(x)$ with $L_I^{\sim}(x)$, we have

$$A_{HV}^{\sim}(\sigma) = \left[T\, e^{\,i\int_{\sigma_0}^{\sigma} dx\, \Delta L_I(\phi_{HV}(x))V(x)}\right]^{\dagger} A_{HV}(\sigma) \left[T\, e^{\,i\int_{\sigma_0}^{\sigma} dx\, \Delta L_I(\phi_{HV}(x))V(x)}\right]. \tag{14.7}$$

The fields $\phi_{HV}(x)$ which appear in $\Delta L_I(x)$ are the fields in Heisenberg picture for the original Lagrangian density $L_0(x) + L_I(x)V(x)$. The appearance of $\Delta L_I(x)$ actually restricts the integration in the exponential to the spacelike region $C$, which is spacelike separated with respect to the support of $A_{HV}(\sigma)$. It follows that the exponential commutes with $A_{HV}(\sigma)$, and therefore

$$A_{HV}^{\sim}(\sigma) \quad = \quad A_{HV}(\sigma) \tag{14.8}$$

for any given realization of the stochastic potential. One then has

$$A_{H}^{\sim}(\sigma) \quad = \quad A_{H}(\sigma) \tag{14.9}$$

i.e., due to equation (14.6), at the level of the statistical ensemble any modification of $L_I(x)$ in a spacetime region $C$ cannot cause physical changes in regions which are spacelike separated from it. We stress that this conclusion is true for the case of non hermitian coupling as well as for the case of hermitian coupling, even though the argument was carried out in terms of the hermitian coupling alone, as it depends solely upon the consideration of the statistical operators which are identical for both cases.

### 14.2.2 Individual Level

¿From the result (14.8) it is also evident that, in the case of hermitian coupling [i.e. for (13.30)] a variation of the Lagrangian density $L_I(x)$ in a region $C$ has no effect on the mean value of any local observable with support which is spacelike separated from $C$, even at the level of an individual system (i.e., for any realization of the stochastic potential). This property is related to the fact that, in this case, no Heisenberg reduction takes place.

The situation is quite different in the case of a non hermitian coupling. In fact, let us consider equation (13.34) and the operator $S_V(\sigma, \sigma_0)$ given by (13.42). The mean value of a local observable $A_I(\sigma)$ is then

$$\begin{aligned} \langle A_I(\sigma)\rangle \quad &= \quad \frac{\langle \psi_V(\sigma)|A_I(\sigma)|\psi_V(\sigma)\rangle}{\||\psi_V(\sigma)\rangle\|^2} \\ &= \quad \frac{\langle \psi(\sigma_0)|S_V^{\dagger}(\sigma,\sigma_0)A_I(\sigma)S_V(\sigma,\sigma_0)|\psi(\sigma_0)\rangle}{\|S_V(\sigma,\sigma_0)|\psi(\sigma_0)\rangle\|^2} \end{aligned} \tag{14.10}$$

We now replace in (13.34) $L_I(x)$ by $L_I(x)+\Delta L_I(x)$, $\Delta L_I(x)$ being different from zero only for $x \in C$, and we denote by $S_V^{\Delta}(\sigma, \sigma_0)$ the corresponding evolution operator. The mean value $\langle A_I^{\Delta}(\sigma)\rangle$ of the same local observable, for the same initial condition, is now

$$\langle A_I^{\Delta}(\sigma)\rangle = \frac{\langle \psi(\sigma_0)|S_V^{\Delta\dagger}(\sigma,\sigma_0)A_I(\sigma)S_V^{\Delta}(\sigma,\sigma_0)|\psi(\sigma_0)\rangle}{\|S_V^{\Delta}(\sigma,\sigma_0)|\psi(\sigma_0)\rangle\|^2}. \tag{14.11}$$

Note that in general

$$\langle A_I^{\Delta}(\sigma)\rangle \quad \neq \quad \langle A_I(\sigma)\rangle \tag{14.12}$$

in spite of the fact that $[\Delta L_I(x), A_I(\sigma)] = 0$, $\forall\, x$.

### 14.2.3 Mean values of local observables and oddities in relativistic reduction models

Let us consider a physical system satisfying the initial condition $|\psi(\sigma_0)\rangle = |\psi_0\rangle$ on the spacelike surface $\sigma_0$, the local observable $A$, and two arbitrary spacelike surfaces $\sigma_1$ and $\sigma_2$ coinciding on the support $\alpha$ of $A$ (see figure 7).

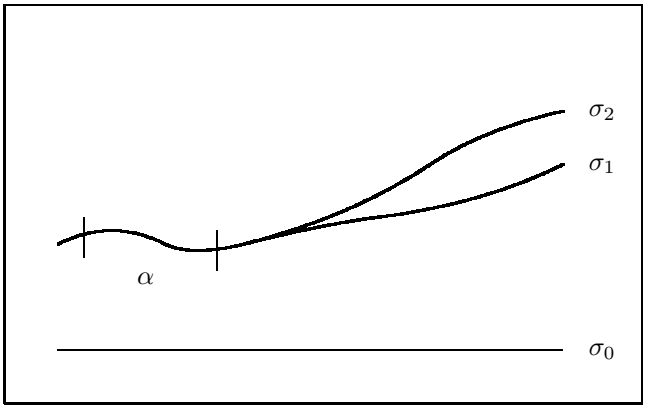



Figure 7: The spacelike surfaces $\sigma_1$ and $\sigma_2$ coinciding on the support $\alpha$ of local observable $A$.

When the dynamics (13.30) ruled by a hermitian interaction is considered, for any given realization of the stochastic potential, as is well known, the mean value of $A$ in the state $|\psi(\sigma_1)\rangle$ coincides with the one in the state $|\psi(\sigma_2)\rangle$. It follows, at the individual level and for the case of hermitian coupling and, as a consequence, at the ensemble level for both cases of hermitian and skew–hermitian coupling, that the mean value of a local observable does not depend on the particular spacelike surface which one chooses among all those coinciding on its support (and therefore on the specific statevector $|\psi(\sigma_1)\rangle$ or $|\psi(\sigma_2)\rangle$ which describes the physical situation concerning the two considered surfaces). Incidentally, this represents a different proof that also in the case of dynamical reduction models, at the ensemble level, one can consistently define, as in standard quantum field theory, local observables.

Again, the situation at the individual level is quite different in the skew–hermitian case. In fact, for a given realization of the stochastic potential, one has

$$\frac{\langle\psi_V(\sigma_2)|A|\psi_V(\sigma_2)\rangle}{\||\psi_V(\sigma_2)\rangle\|^2} = \frac{\langle\psi_V(\sigma_1)|S_V^\dagger(\sigma_2,\sigma_1)AS_V(\sigma_2,\sigma_1)|\psi_V(\sigma_1)\rangle}{\|S_V(\sigma_2,\sigma_1)|\psi_V(\sigma_1)\rangle\|^2} \tag{14.13}$$

which, in general, is different from $\langle\psi_V(\sigma_1)|A|\psi_V(\sigma_1)\rangle/\||\psi_V(\sigma_1)\rangle\|^2$ even though the spacetime region spanned in tilting $\sigma_1$ into $\sigma_2$ is spacelike separated from the support $\alpha$ of $A$, and, consequently

$$[A, S_V(\sigma_2,\sigma_1)] = 0. \tag{14.14}$$

This dependence, at the individual level, of the mean value of a local observable upon the spacelike surface (among those coinciding on the support) over which it is evaluated is not *per se* a difficulty of the theory. It becomes, however, a difficulty if one wishes to claim that such a mean value corresponds to an objective property of an individual system.

Before facing this problem (see next subsection), a deeper analysis of the implications of relativistic reduction models for microscopic [case (a) below] and macroscopic [case (b)] systems is necessary.

**Case (a).** Let us start by reconsidering the case (subsection 14.1.3) of a microscopic system coupled to a macroscopic one which acts as a "measuring apparatus" in the sense of dynamical reduction models. Let $A_1$ and $A_2$ be two local observables of the microsystem whose supports $\alpha_1$ and $\alpha_2$ are spacelike separated, and suppose the macroscopic system is devised to measure $A_1$. For our purposes we can ignore the hamiltonian evolution for the operators and we consider the Tomonaga–Schwinger evolution equation of the statevector, for a specific realization of the stochastic potential

$$\frac{\delta|\psi_V(\sigma)\rangle}{\delta\sigma(x)} \quad = \quad [iL_{1-S}(x) \; + \; L_I(x)V(x) \; - \; \lambda\, L_I^2(x)]|\psi_V(\sigma)\rangle. \tag{14.15}$$

Here $L_{1-S}(x)$ (describing the local system–apparatus interaction) and $L_I(x)$ may be taken as different from zero only in a spacetime region $C$ which is spacelike with respect to $\alpha_2$ (see figure 8).

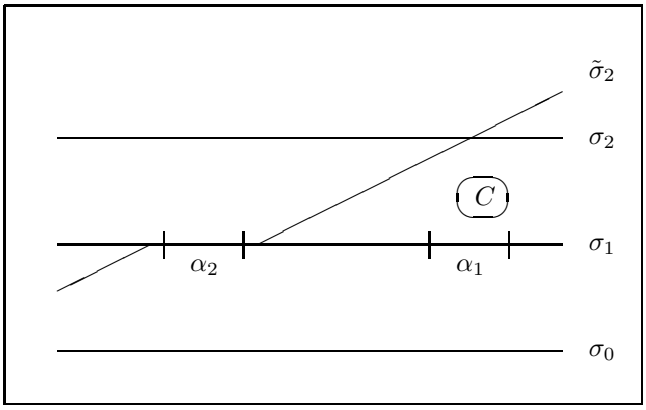



Figure 8: A macroscopic apparatus measures local observable $A_1$ in spacetime region $C$. $A_1$'s support $\alpha_1$ is spacelike separated with respect to $\alpha_2$, the support of another local observable $A_2$.

Let us assume that the local observables $A_1$ and $A_2$ have a purely point spectrum with eigenvalues 0 and 1, and let us consider the initial state

$$|\psi(\sigma_0)\rangle \quad = \quad \frac{1}{\sqrt{2}}\,[|\psi_1\rangle \; + \; |\psi_2\rangle]\,|\chi_1\rangle \tag{14.16}$$

with

$$A_i\,|\psi_j\rangle \quad = \quad \delta_{ij}\,|\psi_j\rangle, \qquad\qquad i,j = 1,2 \tag{14.17}$$

$|\chi_1\rangle$ being the untriggered apparatus state. Let us furthermore assume that the particular realization of the stochastic potential $V(x)$ is one of those "yielding the result 1 for the measurement of $A_1$". The situation is then the following:

1. The state associated to $\sigma_0$ and $\sigma_1$ is $|\psi(\sigma_0)\rangle$.

2. The state associated to $\sigma_2$ is ($N$ being a normalization factor)

$$|\psi_V(\sigma_2)\rangle \; = \; \frac{1}{N}\, e^{\displaystyle\int_{\sigma_1}^{\sigma_2} dx\,[iL_{1-S}(x) \; + \; L_I(x)V(x) \; - \; \lambda\, L_I^2(x)]}\,|\psi(\sigma_0)\rangle \tag{14.18}$$

which, under the assumptions which have been made, is approximately an eigenstate of $A_2$ pertaining to the eigenvalue zero.

3. The state associated to $\sigma_2^\sim$ is also $|\psi_V(\sigma_2)\rangle$.

Indeed, the relativistic CSL dynamics considered in section 13.6 is such that, when a spacelike hypersurface crosses the region $C$ toward the future, no matter what is the behaviour in regions far apart from $C$, the statevector associated to this hypersurface collapses to the eigenstate of $A_1$ corresponding to the eigenvalue which has been found.

Looking at the problem from the point of view of the evolution from $\sigma_1$ to $\sigma_2$, one could be tempted to say that, since the mean value of $A_2$ has become practically zero as a consequence of the "measurement" in the spacetime region $C$, an element of physical reality associated with $A_2$ has emerged. This is a nonlocal effect of the type of those occurring in an EPR setup.

However, one must realize that the same change of the mean value of $A_2$ occurs when one considers the Tomonaga–Schwinger evolution from $\sigma_1$ to $\sigma_2^\sim$, in accordance with point 3. This gives rise to an ambiguity in the mean value of $A_2$, i.e., in a quantity that, when the support $\alpha_2$ shrinks to zero, refers to a unique objective spacetime point. This is not surprising; it corresponds simply to the emergence, within the relativistic reducing dynamics, of the aspects discussed in subsection 14.1.3 for the standard quantum theory with the reduction postulate. In fact, one can remark that $\sigma_1^\sim$ can be approximately identified with a $t' = \text{const}$ hyperplane for a boosted observer for which the interaction with the macro–object has already taken place[74].

**Case (b).** Let us discuss now the same problem for macroscopic systems. We consider a situation analogous to the previous one but in which there are two macroscopic systems performing measurements of the observables $A_1$ and $A_2$. The initial condition is given by assigning to the surface $\sigma_0$ the state:

$$|\psi(\sigma_0)\rangle \quad = \quad \frac{1}{\sqrt{2}}\,[|\psi_1\rangle \;+\; |\psi_2\rangle]\,|\chi_1\rangle\,|\chi_2\rangle \tag{14.19}$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ refer to the untriggered apparatuses. The evolution equation, with the usual approximation, is now

$$\begin{aligned}\frac{\delta|\psi_V(\sigma)\rangle}{\delta\sigma(x)} \quad &= \quad [iL_{1-S}(x) \;+\; iL_{2-S}(x) \;+\; L_{I1}(x)V(x) \;+\; L_{I2}(x)V(x) \\ &\quad - \lambda\, L_{I1}^2(x) \;-\; \lambda\, L_{I2}^2(x)]|\psi_V(\sigma)\rangle. \end{aligned}\tag{14.20}$$

where the meaning of the symbols is obvious. To clearly define the situation from the physical point of view, we assume that the time which is necessary in order that the microsystem triggers the apparatus is sensibly shorter than the typical reduction time for the apparatus. This means that in the above equation we can consider $L_{1-S}(x)$ and $L_{2-S}(x)$ to be different from zero only in the regions $C_1$ and $B_1$, respectively, and $L_{I1}(x)$ and $L_{I2}(x)$ in the regions $C_2$ and $B_2$, respectively, as shown in figure 9.

[74]The bending of the surface at the left of $\alpha_2$ shown in figure 5 is allowed since, under the assumptions we have made, $L_I(x) = 0$ in that region.

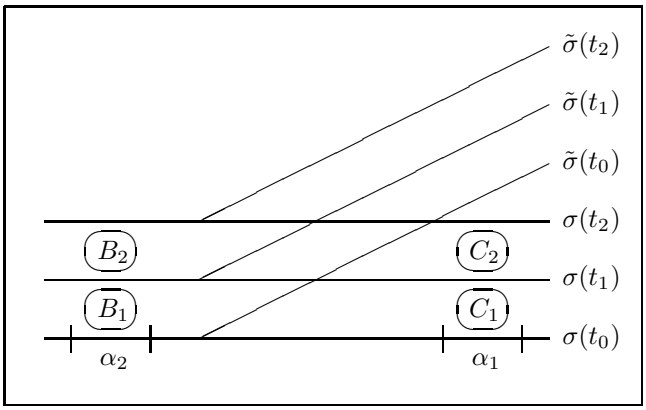



Figure 9: Measurements take place in $C_1$ and $B_1$, followed by reduction dynamics in $C_2$ and $B_2$, of local observables $A_1$ and $A_2$, respectively.

Let us also assume that the specific realization of the stochastic potential is one leading to the value 1 for $A_1$. We are interested in discussing the states of the macro–system used to measure $A_2$ and the mean values of its observables on various hypersurfaces. In particular, let $A_2^\sim$ be the observable of the apparatus corresponding to the yes–no experiment asking whether the result 0 has been found in a measurement of $A_2$. We consider $t = \text{const}$ hypersurfaces $\sigma(t)$ and also the bent hypersurfaces $\sigma^\sim(t)$ containing the spatial support of $A_2^\sim$ at time $t$ (see figure 9). The situation can now be summarized as follows:

1. For $t < t_0$ the state associated to any surface $\sigma(t)$ or $\sigma^\sim(t)$ has always the form of a factorized state; one of the factors refers to the apparatus 2 and it is $|\chi_2\rangle$. Note that what changes in going from $\sigma(t)$ to $\sigma^\sim(t)$ is the state of the system + apparatus 1.

2. For $t = t_1$ the state associated to $\sigma(t_1)$ is

$$|\psi_V(\sigma(t_1))\rangle \;=\; \frac{1}{\sqrt{2}}\,[|\psi_1\rangle\,|\chi_1^1\rangle\,|\chi_2^0\rangle \;+\; |\psi_2\rangle\,|\chi_1^0\rangle\,|\chi_2^1\rangle], \tag{14.21}$$

   where, obviously, the superscripts identify the states of the macroscopic apparatuses which have been triggered by the interaction with the microsystem, these states being labeled by the eigenvalues which have been found.

   ¿From (14.21) one sees that the state $|\psi_V(\sigma(t_1))\rangle$ is not a factorized state and as a consequence it cannot be an eigenstate of the relevant observable of apparatus 2. In particular, the mean value of $A_2^\sim$ in the state (14.21) is 1/2.

   However, it is important to remark that the state to be associated to the surface $\sigma^\sim(t_1)$ of figure 6 is, for the particular realization of the stochastic potential,

$$|\psi_V(\sigma^\sim(t_1))\rangle \quad \simeq |\psi_1\rangle\,|\chi_1^1\rangle\,|\chi_2^0\rangle. \tag{14.22}$$

   This state is factorized and it is an eigenstate of $A_2^\sim$.

3. The state to be associated with any surface $\sigma(t)$ and $\sigma^\sim(t)$ when $t > t_2$, is once more a factorized state with the factor $|\chi_2^0\rangle$ for the apparatus 2.

The conclusion is that, even though the dependence of the mean value of a local observable upon the spacelike surface on which it is evaluated is present also in the case of macro–objects, this dependence occurs only for a time interval of the order of the one which is necessary for the reduction to take place[75].

[75]Here the argument has been presented with reference to the evolution from one space-like surface lying below the regions $C_1$ and/or $C_2$ to one which has "crossed" these regions. Obviously analogous

### 14.2.4 Objective properties of micro and macroscopic systems

We started this subsection 14.2 by relating the possibility of attributing objective properties to individual systems to requirement (14.1) being satisfied to an extremely high degree of accuracy. In the relativistic case, however, as shown with great detail in the previous subsection, the mean value of a projection operator associated to a local observable is affected by an ambiguity depending on the spacelike surface used to evaluate it, and, under specific circumstances, by changing the surface its value can vary from, for example, 1/2 to almost exactly 1. This shows that the above definition of objective properties for individual systems is inadequate, and must be made more precise.

In accordance with the discussion of Section 10 we think that the appropriate attitude is the following: when considering a local observable $A$ with its associated support we say that an individual system has the accessible property $a$ ($a$ being an eigenvalue of $A$), only when the mean value of $P_a$ is extremely close to one, when evaluated on all spacelike hypersurfaces containing the support of $A$.

Thus, according to this prescription, one cannot attribute an objective property to an individual system when there is an appreciable dependence of the mean value of the local observable upon the surface used to evaluate it.

Let us analyze the implications of this attitude in the cases of microscopic and macroscopic systems. For a microsystem, with reference to case (a) of the previous subsection, we observe that no objective property corresponding to a local observable can emerge as a consequence of a "measurement process" performed in a region which is spacelike separated from the support of the considered observable. This does not mean that microsystems cannot acquire objective local properties as a consequence of a measurement performed in another spacetime region; in fact, with reference to the discussion in (a) and to an EPR–Bohm–like setup one can remark that if one considers the spin component of particle 2, when the particle is in the future of the region in which the spin of particle 1 has been measured, then one can attribute to particle 2 the objective local property of having its spin "up" or "down" and that such a property has emerged just due to the measurement which has been performed.

We wish to emphasize again that in the case of macrosystems the discussion under (b) has shown that the impossibility of associating local properties to them lasts only for a time interval of the order of the one which is necessary for the "spontaneous dynamical reduction" to take place. In fact, before macroapparatus 2 interacts with the microsystem the state of the apparatus is obviously well defined and corresponds to the untriggered state, independently of the considered surface. After the reduction ensuing from the interaction of the microsystem with it, apparatus 2 is again in a well–defined state, corresponding to the result which it has registered. Moreover, this result is "correctly" correlated to the result registered by apparatus 1[76].

considerations hold with reference to the regions $B_1$ and $B_2$ and to the apparatus which is present there.

[76] Perhaps it is worth noticing that it would be possible to give another covariant prescription for the attribution of objective local properties to physical systems. More precisely one could, for any local observable $A$, consider the mean value of the projection operator $P_a$ on one of $A$'s eigenmanifolds evaluated for the state vector associated to the surface which delimits the future light cone of the support of $A$. Then, if this mean value is extremely close to 1, one asserts that the system has the objective property $a$. This is quite different from the previously considered criterion (i.e., that the mean value be extremely close to one on all hypersurfaces containing the support of $A$) and would, in case (a) of the previous subsection, lead to the assignment of the objective property corresponding to the value zero for the observable $A_2$ to the microsystem, contrary to what would occur by the adoption of the previous criterion.

This attitude would correspond to the following particular interpretation, at the relativistic level, of the EPR criterion for elements of physical reality: "if there exists *at least one* observer who can *predict*, almost (in the above specified sense) with certainty and without disturbing a system in any way, the value of a physical quantity, then there exists an element of physical reality corresponding

In conclusion, relativistic dynamical reduction models, together with the prescription for the attribution of objective properties to physical systems proposed in this section, allows one to overcome the difficulties discussed in subsection 14.1.3. The theory assigns a statevector to any spacelike hypersurface, and the dependence, at the individual level, of the mean value of a local observable upon the specific spacelike surface used to evaluate it, does not constitute a difficulty. It simply requires a precise and appropriate criterion for relating the objective properties of a physical system to the mean values of local observables: in particular, this criterion permits practically always the attribution of objective local properties to macro–objects, at the individual level. In a sense, the above analysis has proven once more that dynamical reduction models meet the requirement put forward by J. S. Bell [122] for an exact and serious formulation of quantum mechanics, i.e., that it should "allow electrons to enjoy the cloudiness of waves, while allowing tables and chairs, and ourselves, and black marks on photographs, to be rather definitely in one place rather than another, and to be described in classical terms".

## 14.3 Parameter dependence in dynamical reduction models

As is well known, the locality assumption needed to prove Bell's theorem [152] is equivalent to the conjunction of two other assumptions, viz., in Shimony's terminology, parameter independence and outcome independence [153, 154, 155, 74]; in view of the experimental violation of the Bell inequality, one has to give up either or both of these assumptions. We now analyze these issues within the framework of dynamical reduction models.

To start with, let us fix our notation. We will denote by $\lambda$ all parameters (which may include the quantum mechanical statevector or even reduce to it alone) that completely specify the state of an individual physical system. For simplicity we will refer to a standard EPR–Bohm setup and we will denote by

$$p_\lambda^{LR}(x, y; \mathbf{n}, \mathbf{m}) \tag{14.23}$$

the joint probability of getting the outcome $x$ ($x = \pm 1$) in a measurement of the spin component along $\mathbf{n}$ at the left ($L$) and $y$ ($y = \pm 1$) in a measurement of the spin component along $\mathbf{m}$ at the right ($R$) wing of the apparatus. We assume that the experimenter at $L$ can make a free–will choice of the direction $\mathbf{n}$; and similarly for the experimenter at $R$ and the direction $\mathbf{m}$. Both experimenters can also choose not to perform the measurement. Finally, we assume that the micro–macro interactions taking place at $L$ and $R$ that trigger the reduction are governed by appropriate coupling constants $g_L$ and $g_R$; in particular, the situations in which one of the coupling constants is made equal to zero corresponds to no measurement being performed.

Bell's locality assumption can be expressed as

$$p_\lambda^{LR}(x, y; \mathbf{n}, \mathbf{m}) \quad = \quad p_\lambda^{L}(x; \mathbf{n}, *)\, p_\lambda^{R}(y; *, \mathbf{m}) \tag{14.24}$$

where the symbol $*$ appearing on the right hand side denotes that the corresponding measurement is not performed. Condition (14.24) has been shown [155, 74] to be equivalent to the conjunction of two logically independent conditions:

$$p_\lambda^{L}(x; \mathbf{n}, \mathbf{m}) \quad = \quad p_\lambda^{L}(x; \mathbf{n}, *)$$

to that quantity".

We do not want to enter here into a detailed discussion of the conceptual implications involved in adopting the above prescription. We believe that they lead to some conceptual difficulties in connection with the cause–effect relation. This is not surprising since the considered prescription is analogous, in the present context, to the Hellwig–Kraus [146] postulate about wavepacket reduction. For these reasons we drop the criterion considered in this footnote.

$$p^R_\lambda(y;\mathbf{n},\mathbf{m}) \quad = \quad p^R_\lambda(y;*,\mathbf{m}) \tag{14.25}$$

and

$$p^{LR}_\lambda(x,y;\mathbf{n},\mathbf{m}) \;=\; p^L_\lambda(x;\mathbf{n},\mathbf{m})\, p^R_\lambda(y;\mathbf{n},\mathbf{m}) \tag{14.26}$$

where we have denoted, e.g., by the symbol $p^L_\lambda(x;\mathbf{n},\mathbf{m})$ the probability of getting, for the given settings $\mathbf{n}$, $\mathbf{m}$, the outcome $x$ at $L$.

Conditions (14.25) express **parameter independence**, i.e., the requirement that the probability of getting an outcome at $L$ ($R$) is independent of the setting chosen at $R$ ($L$), while equation (14.26) (**outcome independence**) expresses the requirement that the probability of an outcome at one wing does not depend on the outcome obtained at the other wing.

### 14.3.1 The case of the nonlinear CSL Model

To simplify the discussion, we assume that the initial state $|\psi(0)\rangle$ is the singlet state and we confine our attention to the case in which both spin measurements are in the same direction, i.e., $\mathbf{n} = \mathbf{m}$. We assume that the measurement at $R$, if it takes place (i.e., if $g_R \neq 0$), occurs at an earlier time than the one at $L$.

Consider now the realizations $\tilde{w}_L(\mathbf{x},t)$ of $w_L(\mathbf{x},t)$ that give rise to the outcome $+1$ for the left apparatus when it is triggered by $|\psi(0)\rangle$. The probability of occurrence of such processes is $1/2$. We will denote by $p^L_{|\psi(0)\rangle}(-1;g_R=0|w_L)$ and $p^L_{|\psi(0)\rangle}(-1;g_R\neq 0|w_L)$ the conditional probability, given $w_L$, of the outcome $-1$ at left when the initial state is $|\psi(0)\rangle$ and the $R$ apparatus is switched off or on, respectively. We then have

$$p^L_{|\psi(0)\rangle}(-1;g_R=0|\tilde{w}_L) \quad = \quad 0. \tag{14.27}$$

We now evaluate the probability $p^L_{|\psi(0)\rangle}(-1;g_R\neq 0|\tilde{w}_L)$. Since $g_R \neq 0$ and the measurement at $R$ occurs before the one at $L$, we have to take into account the possible realizations of the stochastic process at $R$. Let us consider the realizations $\tilde{w}_R(\mathbf{x},t)$ of $w_R(\mathbf{x},t)$ that, when triggered by the singlet state, yield the outcome $+1$ at $R$. When one of these processes $\tilde{w}_R$ occurs, the outcome at $L$ turns out to be $-1$ irrespective of the particular realization of the stochastic process $w_L$ and therefore also for all processes $\tilde{w}_L$ considered above. To understand this, recall that within the nonlinear model, the same stochastic process at $L$ can give rise to different outcomes, depending on the statevector which triggers the apparatus at $L$. As a consequence one has

$$p^L_{|\psi(0)\rangle}(-1;g_R\neq 0|\tilde{w}_L\&\tilde{w}_R) \quad = \quad 1. \tag{14.28}$$

Since the probability of occurrence of a process $\tilde{w}_R$ is equal to $1/2$ and is independent of the particular realization $\tilde{w}_L$, and since if $w_R$ is not one of the $\tilde{w}_R$, then outcome $-1$ on the left cannot occur (barring improbable exceptions), one has

$$p^L_{|\psi(0)\rangle}(-1;g_R\neq 0|\tilde{w}_L) \quad = \quad 1/2. \tag{14.29}$$

We stress that the difference of the probabilities is appreciable,

$$0 \;=\; p^L_{|\psi(0)\rangle}(-1;g_R=0|\tilde{w}_L) \;\neq\; p^L_{|\psi(0)\rangle}(-1;g_R\neq 0|\tilde{w}_L) \;=\; 1/2 \tag{14.30}$$

and that the probability of occurrence of these realizations $\tilde{w}_L$ is also appreciable ($= 1/2$). Thus the nonlinear CSL model exhibits parameter dependence.

### 14.3.2 The case of the linear CSL model

For the linear model, we can easily solve the evolution equation, and thereby show parameter independence in the $t \to \infty$ limit, once we simplify the description by considering only the spin Hilbert space.

Thus one has, in the case in which both apparatuses are switched on ($g_R \neq 0$ and $g_L \neq 0$), a linear dynamical equation analogous to (7.42):

$$\frac{d|\psi_{w_L,w_R}(t)\rangle}{dt} \;=\; \left\{[(\sigma^L \cdot \mathbf{n})w_L(t) - \gamma] + [(\sigma^R \cdot \mathbf{m})w_R(t) - \gamma]\right\} |\psi_{w_L,w_R}(t)\rangle \quad (14.31)$$

with

$$\langle\!\langle w_{L,R}(t)\rangle\!\rangle \;=\; 0 \qquad \langle\!\langle w_L(t)\, w_R(t')\rangle\!\rangle \;=\; \gamma\, \delta_{L,R}\, \delta(t-t'). \quad (14.32)$$

The probability distribution of the stochastic processes is obtained through the cooking procedure. To compare this case with the one in which $g_R = 0$, one has to consider another stochastic equation, i.e.,

$$\frac{d|\psi_{w_L}(t)\rangle}{dt} \;=\; \left\{(\sigma^L \cdot \mathbf{n})w_L(t) - \gamma\right\} |\psi_{w_L}(t)\rangle \quad (14.33)$$

The solutions of equations (14.31) and (14.33) at time $t$ for the same initial conditions are

$$|\psi_{B_L,B_R}(t)\rangle \quad = \quad e^{F_{L\,B_L}(t)} e^{F_{R\,B_R}(t)} |\psi(0)\rangle \quad (14.34)$$

and

$$|\psi_{B_L}(t)\rangle \quad = \quad e^{F_{L\,B_L}(t)} |\psi(0)\rangle \quad (14.35)$$

respectively[77]. In equations (14.34) and (14.35) we have put

$$F_{L\,B_L}(t) \;=\; \sigma^L \cdot \mathbf{n}\, B_L(t) - \gamma t, \qquad F_{R\,B_R}(t) \;=\; \sigma^R \cdot \mathbf{n}\, B_R(t) - \gamma t, \quad (14.36)$$

where

$$B_L(t) \;=\; \int_0^t d\tau\, w_L(\tau), \qquad B_R(t) \;=\; \int_0^t d\tau\, w_R(\tau). \quad (14.37)$$

We come back now to equation (14.31) and we evaluate the cooked probability density of occurrence of the Brownian processes $B_L(t)$ and $B_R(t)$ by multiplying the raw probability density by the square of the norm of the statevector (14.34). As usual we have

$$P_{\text{Cook}}[B_L(t)\,\&\,B_R(t)] \;=\; P_{\text{Raw}}[B_L(t)\,\&\,B_R(t)]\, \||\psi_{B_L,B_R}(t)\rangle\|^2 \quad (14.38)$$

and

$$P_{\text{Raw}}[B_L(t)\,\&\,B_R(t)] \quad = \quad P_{\text{Raw}}[B_L(t)]\, P_{\text{Raw}}[B_R(t)]. \quad (14.39)$$

Taking into account equation (14.34), one then gets from (14.38)

$$\begin{aligned} P_{\text{Cook}}[B_L(t)\,\&\,B_R(t)] \;&=\; P_{\text{Raw}}[B_L(t)]\, P_{\text{Raw}}[B_R(t)]\||\psi_{B_L,B_R}(t)\rangle\|^2 \;= \\ &=\; P_{\text{Raw}}[B_L(t)]\, \|e^{F_{L\,B_L}(t)}|\psi(0)\rangle\|^2 \cdot \\ &\quad\; P_{\text{Raw}}[B_R(t)] \left\| \frac{e^{F_{R\,B_R}(t)} e^{F_{L\,B_L}(t)}|\psi(0)\rangle}{\|e^{F_{L\,B_L}(t)}|\psi(0)\rangle\|} \right\|^2 \end{aligned} \quad (14.40)$$

[77] In equation (14.34) and following, we change notation for the same reason as we did in equation (7.45).

Let us consider the marginal cooked probability density of $B_L(t)$

$$\begin{aligned} P^{\#}_{\rm Cook}[B_L(t)] &= \int \mathcal{D}[B_R(t)] P_{\rm Cook}[B_L(t)\,\&\,B_R(t)] \\ &= P_{\rm Raw}[B_L(t)]\,\|e^{F_L\,B_L(t)}|\psi(0)\rangle\|^2 \times \\ &\quad \int \mathcal{D}[B_R(t)] P_{\rm Raw}[B_R(t)] \left\| \frac{e^{F_R\,B_R(t)} e^{F_L\,B_L(t)}|\psi(0)\rangle}{\|e^{F_L\,B_L(t)}|\psi(0)\rangle\|} \right\|^2 . \end{aligned} \tag{14.41}$$

Since the equation

$$\frac{d|\psi_{w_R}(t)\rangle}{dt} = \left\{(\sigma^R \cdot \mathbf{n}) w_R(t) - \gamma\right\} |\psi_{w_R}(t)\rangle \tag{14.42}$$

preserves the stochastic average of the square of the norm of the statevector, the last integral in equation (14.41) takes the value 1. This means that $P^{\#}_{\rm Cook}[B_L(t)]$ turns out to equal $P_{\rm Cook}[B_L(t); *]$, i.e., the cooked probability density of occurrence of the Brownian process $B_L(t)$ for the same initial condition if the process were described by equation (14.33), i.e., if the apparatus at $R$ were switched off.

But now recall from section 7.4 that within linear CSL there is a one–to–one correspondence between the outcome at left (right) at $t = \infty$ and the specific value taken by the Brownian process $B_L(t)$ $[B_R(t)]$ for $t \to \infty$. So the above proof that $P^{\#}_{\rm Cook}[B_L(t)]$ equals $P_{\rm Cook}[B_L(t); *]$ amounts to a proof that linear CSL exhibits parameter independence at the $t = \infty$ limit.

When one considers a *finite* time $t$ of the order of or greater than the characteristic reduction time $\Delta t$, the situation is more complicated: the one–to–one correspondence between the outcomes and the values taken by the Brownian process is only approximate (though valid to an extremely high degree of accuracy). As a consequence, linear CSL does not enjoy strict parameter independence at finite times. To clarify this point, consider the values $B_L(t) = 2\gamma t$ and $B_R(t) = 4\gamma t$ for the Brownian processes at time $t$. The cooked probabi1ity density of occurrence of such values at the *finite* time $t$, though extremely small, is not exactly zero. One can show [156] that these values lead, through equation (14.34), to a statevector at $t$ which corresponds to the outcomes $+1$ at right and $-1$ at left, respectively. On the other hand, for the case in which $g_R = 0$, the substitution of $B_L(t) = 2\gamma t$ in equation (14.35) leads, at time $t$, to a statevector corresponding to the outcome $+1$ at left. Thus, there are values of the Brownian process $B_L(t)$ for which the outcome at left depends on whether $g_R$ is equal to zero or not. Accordingly, there is parameter dependence at the level of individual $B(t)$'s. However, given $B_L(t)$, this happens only for values $B_R(t)$ of the Brownian process at right such that the cooked conditional probability $P_{\rm Cook}[B_R(t)\,|\,B_L(t)]$ is extremely small. This in turn implies that the model exhibits only negligibly small parameter dependence effects.

To conclude, although the linear CSL model exhibits parameter dependence at finite times, these effects are at any rate extremely small with respect to those of the nonlinear CSL model[78].

[78] Actually, explicit evaluations of such effects show that they are characterized by probabilities which are smaller, e.g., than those of classical thermodynamical processes which violate the second law of thermodynamics.

# Part V
# Dynamical Reduction Models and Experiments

## 15 Decoherence, quantum telegraph, proton decay, and superconducting devices

Dynamical reduction models require a precise change of quantum dynamics, so that they constitute a theory genuinely different from standard quantum mechanics. It becomes then interesting to analyze the conceptual and practical possibility of testing them versus quantum mechanics.

In subsection 15.1, we analyze the role of decoherence in experiments, and how it can mask the physical consequences of the localization mechanism of dynamical reduction models. In subsections 15.2, 15.3 and 15.4 we discuss three specific experiments, the quantum telegraph, the nucleon decay and dissipation in superconducting devices respectively, and the role played by dynamical reductions.

### 15.1 Decoherence and the possibility of testing dynamical reductions

We have discussed in sections 3.2 and 5 that decoherence — i.e., the interaction between a given physical system and the surrounding environment — by itself does not constitute a solution of the macro–objectification problem of Quantum Mechanics, since it yields only an apparent collapse of the wavefunction, not a real one. Nonetheless, decoherence effects on quantum measurements are very important and often pose serious limitations to the possibility of measuring specific properties of physical systems, in particular to put into evidence the superposition of different states of mesoscopic and macroscopic systems, i.e. systems whose interaction with the environment is more difficult to control.

As regards the possibility of testing dynamical reduction models versus standard quantum mechanics, the role played by decoherence is very tricky. In fact, in order to observe dynamical reductions, experiments must be performed on quantum systems containing a sufficiently large number of particles — this is the case of mesoscopic or macroscopic systems — otherwise the reduction mechanism would be ineffective for too a long time. On the other hand, mesoscopic and macroscopic systems are very rapidly affected by decoherence in such a way that, given a superposition of different states, what would appear to be a spontaneous reduction into one of such states might be attributed only to the interactions with the surrounding environment. It is then important to compare the "reduction rates" and the physical consequences of specific examples of decoherence mechanisms with those of QMSL and CSL, in order to understand whether there are situations in which a possibly observed reduction process is *real* — thus confirming the predictions of dynamical reduction models — or only *apparent*, i.e. it is a result of decoherence. Such a comparison has been presented in an interesting paper by Tegmark [157], which we are going to discuss.

In the just quoted paper the environment is felt by the physical system of interest as a background noise due to the (instantaneous) scattering of photons, neutrinos or air molecules off a system, the effect on the compound initial state $\rho_{S+E}(t_i)$ being determined by a transition matrix $T$

$$\rho_{S+E}(t_i) \quad \longrightarrow \quad \rho_{S+E}(t_f) \;=\; T\,\rho_{S+E}(t_i)\,T^{\dagger}. \tag{15.1}$$

Let $\mathbf{p}$, $\mathbf{k}$ be the momenta of the system and of a background incident particle, respectively, and $a_{\mathbf{pk}}(\mathbf{q})$ the probability amplitude that the momentum transferred to the system is $\mathbf{q}$. The author makes the following reasonable assumptions and approximations about the nature of the scattering processes:

1. Conservation of energy and momentum:

$$\langle \mathbf{p}', \mathbf{k}'|T|\mathbf{p}, \mathbf{k}\rangle \quad = \quad \delta(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k})\, a_{\mathbf{pk}}(\mathbf{p}' - \mathbf{p}).$$

2. Independence of $a_{\mathbf{pk}}(\mathbf{q})$ from the motion of the system due to the high velocity of the incident particle:

$$a_{\mathbf{pk}}(\mathbf{q}) \quad = \quad a_{\mathbf{k}}(\mathbf{q}).$$

3. The system is supposed to be exposed to a constant particle flux $\Phi$ per unit area and unit time scattered off with a total scattering cross section $\sigma$ and a temporal distribution modeled by a Poisson process with intensity $\Lambda = \sigma\Phi$. Furthermore, the background incident particles are supposed to be in momentum eigenstates or incoherent mixtures of them with probability momentum distribution $\mu(\mathbf{k})$.

4. The momentum of the incident particles is isotropically distributed with

$$\mu(\mathbf{k}) \quad = \quad \frac{1}{4\pi\mathbf{k}^2}\, \lambda_0 \nu(\lambda_0|\mathbf{k}|), \tag{15.2}$$

   where $\lambda_0$ is a typical wavelength of the hitting particle and $\nu(x)$ is a probability distribution on the positive real axis.

If $\rho_S(t_i)$ and $\rho_S(t_f)$are the initial (before) and final (after a scattering process) states of the system obtained by tracing out the environmental degrees of freedom, and

$$P_{\mathbf{k}}(\mathbf{q}) \quad = \quad |a_{\mathbf{k}}(\mathbf{q})|^2 \tag{15.3}$$

$$\hat{P}_{\mathbf{k}}(\mathbf{x}) \quad = \quad \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} d^3q\, e^{-i\mathbf{q}\mathbf{x}} P_{\mathbf{k}}(\mathbf{q}) \tag{15.4}$$

are the probability distributions of momentum transfer and its Fourier transform, respectively,it follows that, in coordinate representation,

$$\langle \mathbf{x}|\rho_S(t_f)|\mathbf{y}\rangle \quad = \quad \hat{P}(\mathbf{x} - \mathbf{y})\, \langle \mathbf{x}|\rho_S(t_i)|\mathbf{y}\rangle \tag{15.5}$$

$$\hat{P}(\mathbf{x} - \mathbf{y}) \quad = \quad \int_{\mathbf{R}^3} d^3k\, \mu(\mathbf{k})\, \hat{P}_{\mathbf{k}}(\mathbf{x} - \mathbf{y}). \tag{15.6}$$

Taking into account assumption 3 above and denoting by $T_{\text{Sca}}[\rho]$ the $\rho_S(t_f)$ in (15.5), in the time interval $[t, t+dt]$ a scattering induced process occurs with probability $\Lambda dt$. Consequently the unitary Schrödinger evolution becomes a master equation very much similar to (6.8):

$$\rho(t + \delta t) \quad = \quad -\frac{i}{\hbar}\, [H, \rho(t)]\delta t + (1 - \Lambda\delta t)\rho(t) + \Lambda\delta t T_{\text{Sca}}[\rho(t)]. \tag{15.7}$$

A comparison of $T_{\text{Sca}}[\rho]$ and $T_{\text{GRW}}[\rho]$ (i.e. the reduction operator given by equation (6.8)) is possible by looking at the expansions of the Gaussian damping factor that appears in (6.9) and the factor $\hat{P}(\mathbf{x})$, which plays an analogous role in (15.5) ($|\hat{P}(\mathbf{x})| \leq 1$), in powers of $\mathbf{x} = (x_1, x_2, x_3)$:

$$\langle \mathbf{x}|T_{\text{GRW}}[\rho]|\mathbf{y}\rangle \quad \simeq \quad \Big[1 - \frac{\alpha}{4}\, |\mathbf{x} - \mathbf{y}|^2 + \ldots\Big]\, \langle \mathbf{x}|\rho|\mathbf{y}\rangle, \tag{15.8}$$

$$\langle \mathbf{x}|T_{\rm Sca}[\rho]|\mathbf{y}\rangle \;\simeq\; \Bigg[1 - i\sum_{j=1}^{3}(x_j - y_j)\int_{\mathbf{R}^3} d^3q\, q_j\, P(\mathbf{q}) - \tag{15.9}$$

$$-\frac{1}{2}\sum_{j,k=1}^{3}(x_j - y_j)(x_k - y_k)\int_{\mathbf{R}^3} d^3q\, q_j q_k\, P(\mathbf{q}) + \ldots\Bigg]\,\langle \mathbf{x}|\rho|\mathbf{y}\rangle$$

According to assumption 4, the linear term in (15.9) vanishes and the quadratic one (covariance matrix) is completely determined by

$$l_{\rm eff}^{-2} \;=\; \int_{\mathbf{R}^3} d^3q\, q_i^2\, P(\mathbf{q}) \;=\; \frac{\partial}{\partial x_i^2}\hat{P}(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{0}} \qquad i = 1,2,3, \tag{15.10}$$

which has dimension $\mathrm{cm}^{-2}$ and defines a characteristic length $l_{\rm eff}$ of the reduction processes which has to be compared with $(\alpha/2)^{-1/2}$ of QMSL. Moreover, a natural time scale is given by $\tau = \Lambda^{-1}$. Solving (15.7) for short times $\Lambda\delta t \gg 1$, yields:

$$\langle \mathbf{x}|\rho(t+\delta t)|\mathbf{y}\rangle \;\simeq\; e^{-\Lambda\delta t(1-\hat{P}(\mathbf{x}-\mathbf{y}))}\langle \mathbf{x}|\rho(t)|\mathbf{y}\rangle. \tag{15.11}$$

Off the diagonal ($|\mathbf{x}-\mathbf{y}|l_{\rm eff}^{-1} \gg 1$) the damping is dominated by $e^{-\Lambda\delta t}$, whereas near the diagonal ($|\mathbf{x}-\mathbf{y}|l_{\rm eff}^{-1} \ll 1$) the damping goes as $e^{-\Lambda\delta t|\mathbf{x}-\mathbf{y}|^2/2l_{\rm eff}^2}$. We can then introduce:

$$\textbf{Decoherence time:} \qquad \tau \;=\; \Lambda^{-1} \tag{15.12}$$

$$\textbf{Decoherence rate:} \qquad \Delta \;=\; \frac{\Lambda}{l_{\rm eff}^2}. \tag{15.13}$$

The decoherence time $\tau$ is fixed by the total scattering cross section $\sigma$ and the flux of incident particles per unit area and unit time, while the decoherence rate $\Delta$ by the differential cross section that enters the expression of $P_{\mathbf{k}}(\mathbf{q})$ in (15.3). Tegmark also calculates the values of $l_{\rm eff}$ and $\tau$ for a microsystem (electron) in different physical backgrounds which we report in Table 1.

| Cause of collapse | $l_{\rm eff}$ [cm] | $\Phi$ [$\mathrm{cm}^2$ $\mathrm{sec}^{-1}$] | $\tau_{\rm electron}$ [sec] |
|---|---|---|---|
| 300K air at 1 atm | $10^{-9}$ | $10^{24}$ | $10^{-13}$ |
| 300K air in lab vacuum | $10^{-9}$ | $10^{11}$ | 1 |
| Sunlight on earth | $9\times10^{-5}$ | $10^{17}$ | $10^{7}$ |
| 300K photons | $2\times10^{-3}$ | $10^{19}$ | $10^{5}$ |
| Background radioactivity | $10^{-12}$ | $10^{-4}$ | $10^{18}$ |
| Quantum gravity | $10^{5}$ – $10^{12}$ | $10^{109}$ | 30 |
| GRW effect | $10^{-5}$ | | $10^{16}$ |
| Cosmic microwave background | $2\times10^{-1}$ | $10^{13}$ | $10^{11}$ |
| Solar neutrinos | $10^{-9}$ | $10^{11}$ | $10^{33}$ |
| Comic background neutrinos | $3\times10^{-1}$ | $10^{13}$ | $10^{51}$ |

**Table** 1: $l_{\rm eff}$ for various scattering processes and $\tau_{\rm electron}$.

The absence of a second figure in the row describing the GRW situation emphasizes the deep conceptual difference between collapse and decoherence: there is no particle flux scattered off the system inducing the localization mechanism on the length scale $l_{\rm eff} = (\alpha/2)^{-1/2}$, the effect being due to a completely new dynamics and not to the environment. Analogously, Tegmark gives estimates of $\Delta$ for various objects in different backgrounds.

| Cause of apparent wavefunction collapse | Free electron | Dust particle | Bowling ball |
|---|---|---|---|
| 300K air at 1 atm | $10^{31}$ | $10^{37}$ | $10^{45}$ |
| 300K air in lab vacuum | $10^{18}$ | $10^{23}$ | $10^{31}$ |
| Sunlight on earth | 10 | $10^{20}$ | $10^{28}$ |
| 300K photons | 1 | $10^{19}$ | $10^{27}$ |
| Background radioactivity | $10^{-4}$ | $10^{15}$ | $10^{23}$ |
| Quantum gravity | $10^{-25}$ | $10^{10}$ | $10^{22}$ |
| GRW effect | $10^{-6}$ | $10^{9}$ | $10^{21}$ |
| Cosmic microwave background | $10^{-10}$ | $10^{6}$ | $10^{17}$ |
| Solar neutrinos | $10^{-15}$ | 10 | $10^{13}$ |

**Table** 2: $\Delta$ in $\mathrm{cm}^{-2}\mathrm{sec}^{-1}$ for various scattering processes.

As is evident from table 2, the GRW effect, e.g. for a free electron, is weaker than those of air molecules in lab vacuum or photons on earth by a factor in between $10^{26}$ and $10^{6}$, respectively, hence masked by them. There follows that, to put into evidence effects due to spontaneous localization mechanisms one should isolate the physical system of interest from the environment to a presently hardly attainable degree of accuracy. However, the figures in table 2 might lead to erroneous conclusions, if not correctly understood. In fact, e.g. for bound electrons, QMSL can, as we shall discuss later, induce, as a result of the localization mechanism, transitions (which are not considered in the preceding analysis of environment induced decoherence) leading to excitations or dissociations of the composed systems to which the electrons belong.

As an example, in a recent paper [158], it is argued that the Lyman–$\alpha$ ultraviolet radiation emitted by hydrogen atoms (about $1 - 10$ photons per second per mole) as a consequence of the spontaneous localizations suffered by electrons could be detected by an appropriate experimental setup. On the other hand, on the basis of energy balance considerations, it can be shown that an analogous effect due to collisions of an atom with 300 Kelvin air molecules at 1 atmosphere is, in comparison, much smaller .

## 15.2 Dynamical reduction and the quantum telegraph

In this subsection we do not examine a suggested experimental test for the dynamical reduction program, but, rather, some recent claims that there exists some already available empirical evidence that spontaneous collapses, but of a nature different from either QMSL or CS, are necessary. We quote directly from A. Shimony [125]:

> A great weakness of the investigations carried out so far in search of modifications of quantum dynamics is the absence of empirical heuristic. To be sure there is a grand body of empirical fact which motivates all the advocates of nonlinear modifications: that is, the occurrence of definite events, and, in particular, the achievement of definite outcomes of measurement. But this body of fact is singularly unsuggestive of the details of a reasonable modification of Quantum Mechanics. What is needed are phenomena which are suggesting and revelatory ...
>
> No more promising phenomena for this purpose have been found than the intermittency of resonant fluorescence of a three–level atom.

### 15.2.1 The phenomenology of the quantum telegraph

The physical system consists of two laser beams of intensities $I_1$, $I_2$ scattered off a single trapped atomic system which can be treated as a three–level system with a

ground state $|0\rangle$ and two excited states: a higher level $|1\rangle$ and a metastable lower level $|2\rangle$, with mean lives

$$\beta_1^{-1} \simeq 10^{-8}\,\text{sec} \ll \beta_2^{-1} \simeq 1\,\text{sec}. \tag{15.14}$$

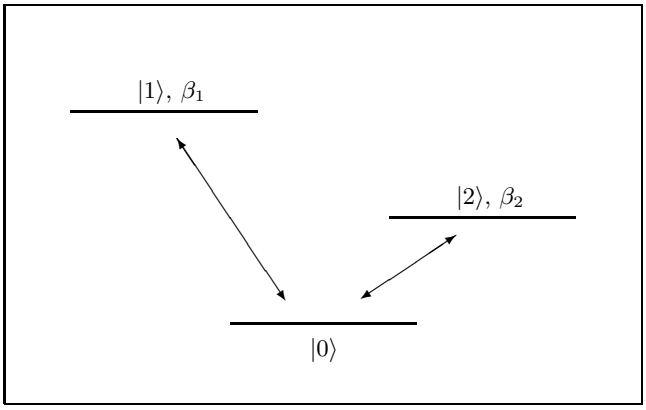



Figure 10: Atomic system in the quantum telegraph phenomenon.

The laser beams are tuned so that the one of intensity $I_1$ excites the atom from the ground state $|0\rangle$ to $|1\rangle$, that of intensity $I_2$ provokes the transition $|0\rangle \rightarrow |2\rangle$, followed by emissions of blue, respectively, red photons and return to $|0\rangle$.

The emission pattern of an experiment conducted with $I_1 \gg I_2$, nearly $10^8$ vs. 10 photons per second, reveals an intermittent blue fluorescence randomly interrupted by periods of darkness.

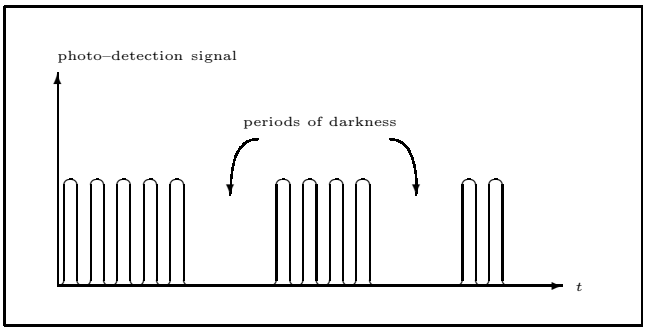



Figure 11: Emission pattern in the quantum telegraph phenomenon.

### 15.2.2 Quantum mechanical interpretations

At first glance, the experimental evidence seems to be explainable by a naive argument based on the concepts of photons and of transitions among energy levels.

**A.** Because of the higher intensity of the beam $I_1$, the atom is most of the time excited to the short–lived level $|1\rangle$ from which it jumps down to the ground state in approximately $10^{-8}$ seconds with the emission a blue photon. But, every now and then, a red photon from the beam $I_2$ sneaks in and the atom is excited to the metastable state $|2\rangle$, where it gets shelved for approximately 1 second before emitting a red photon and starting again a period of blue fluorescence.

However, these conclusions are far too classical (à la Bohr) and underestimate a relevant quantum effect, namely the interference between blue and red photons in

the laser beams which is propagated to the atom by the linearity of the quantum evolution and results in the emergence of linear superpositions of the atomic levels.

**B.** After interacting with the laser beams, the atom, initially in its ground state $|0\rangle$, evolves in $t$ seconds into a new state $|\phi(t)\rangle$ which is the coherent superposition of the three levels:

$$|0\rangle \longrightarrow |\phi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle + c_2(t)|2\rangle, \tag{15.15}$$

the probability $P_i$ of a spontaneous emission corresponding to the jump $|i\rangle \rightarrow |0\rangle$ being $P_i = \beta_i |c_i(t)|^2$.

Since the amplitude $|c_2(t)|^2 \ll |c_1(t)|^2$ for almost all $t$, one expects an emission pattern consisting in continuous fluorescence and, sometimes, the emission of a red photon, periods of darkness resulting extremely unlikely.

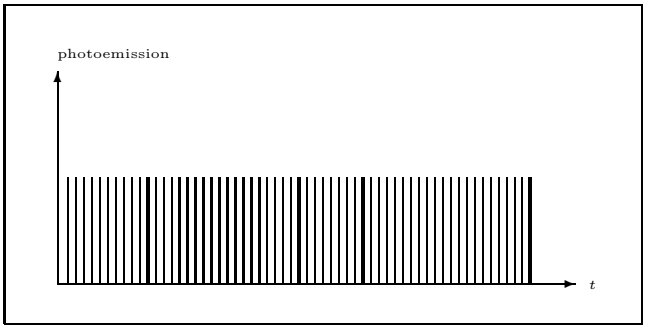



Figure 12: Emission pattern according to argument B. Thick lines correspond to *red* photons.

The above attempt to embody linearity does not explain the occurrence of intermittency in the emission pattern and in reference [159] (compare also corresponding references in [125]) it is suggested that an explanation is only possible if a reduction mechanism corresponding to null measurements (seeing no photons) is introduced into the game.

In view of these facts A. Shimony concludes [125]:

> Two propositions seem to me to suggest themselves quite strongly. The first is that a stochastic modification of quantum dynamics is a natural way to accommodate the jumps from a period of darkness to a period of fluorescence. The second is that the natural locus of the jumps is the interaction of a physical system with the electromagnetic vacuum. Whether stochasticity is exhibited when the system in question is simple and microscopic like a single atom, or only when it is macroscopic and complex like the phosphor of a photo–detector, is not suggested preferentially by the quantum telegraph, for the simple reason that the single trapped atom and the photo–detector are both essential ingredients in the phenomenon ...
>
> But, whichever choice is made points to a stochastic modification of quantum dynamics which has little to do with spontaneous localization.

### 15.2.3 The correct quantum argument

Concerning the alleged impossibility of explaining the intermittent fluorescence of the quantum telegraph by resorting to a dynamical reduction model with localization, it must be stressed that

**C.** The presence of periods of darkness in the emission pattern can be deduced within a purely unitary quantum mechanical scheme [160], by taking into account the whole system

ATOM + RADIATION FIELD

without any need of invoking reduction processes induced by detecting no photons.

To be correct, the analysis must consider states $|\psi(t)\rangle$ of the form

$$|\psi(t)\rangle \quad = \quad \sum_{i=0}^{2} \sum_{\{n\}} c_{i,\{n\}}(t)\, |i\rangle \otimes |\{n\}\rangle, \tag{15.16}$$

where $|\{n\}\rangle$ is a state with $n$ scattered photons. Then, with

$$P \quad = \quad \left[ \sum_{i=0}^{2} |i\rangle\langle i| \right] \otimes |\{0\}\rangle\langle\{0\}| \tag{15.17}$$

the orthogonal projection onto the Fock sector with no scattered photons, the probability $P(t)$ of periods of darkness extending in the interval of time $[0, t]$ when, initially, $|\psi(0)\rangle = |0\rangle \otimes |\{0\}\rangle$, is:

$$P(t) \quad = \quad \|P|\psi(t)\rangle\|^2 \quad = \quad \sum_{i=0}^{2} |c_{i,\{0\}}(t)|^2. \tag{15.18}$$

The study of $P(t)$ leads to the correct prediction of periods of darkness in a purely quantum dynamical context. Moreover, during a period of darkness the state of the system ATOM + RADIATION FIELD is

$$[c_0(t)|0\rangle + c_1(t)|1\rangle + c_2(t)|2\rangle] \otimes |\{0\}\rangle, \tag{15.19}$$

so that periods of darkness can end with the emission of both red and blue photons with an emission pattern like the one of the following picture:

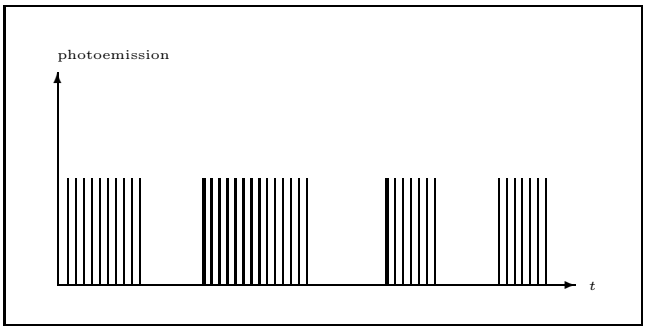



Figure 13: Emission pattern according to argument C.

As a further remark, it must be noted that a complete account of the quantum telegraph experiment ought to include the macroscopic detectors that are involved in the measurement of the emission pattern. Consequently, the physical system to be dealt with is

ATOM + RADIATION FIELD + MACROSCOPIC DETECTORS

Within the dynamical reduction program the actualization of the different macrostates of the detecting apparatuses is accounted for by the new dynamics and the

corresponding objectification of macroproperties is thus obtained. One could raise the question whether this can have any appreciable influence on the quantum telegraph phenomenon.

It is sufficient to observe that, according to the analysis of the previous subsection, the effects of the reduction mechanism are comparable with those of environment–induced reductions that occur at the detectors level. Indeed, on the basis of the agreement of the correct quantum mechanical computations of the probability of occurrence of periods of darkness and of their duration with the experimental results, one can safely conclude that:

1. QMSL and CSL dynamics play, for the process under consideration, exactly the same role as for any macroscopic detection process, namely they objectify macro–properties.

2. There is no need to require that new nonlinear and stochastic modifications of standard quantum mechanics become effective at the microscopic level to account for the quantum telegraph phenomenology.

3. In particular, nothing, in the considered experiments, suggests that reductions take place with respect to an "energy" rather than to a "position" preferred basis.

## 15.3 Dynamical reduction and the nucleon decay

The presence in nature of a mechanism that localizes particles would be accompanied by a corresponding spreading in their momenta. It is thus interesting to study its effect on the stability of atoms and nuclei. It is possible to get a rough idea of the consequences of QMSL by modeling atoms and nuclei as one dimensional systems moving in a harmonic potential so that their ground states can be approximated by Gaussian wavefunctions $\psi_G(q)$ of appropriate width $\gamma^{-1}$:

$$\psi_G(q) \;=\; \left[\frac{2\gamma^2}{\pi}\right]^{1/4} e^{-\gamma^2 q^2}, \qquad \begin{array}{ll} \gamma \simeq 10^8 \mathrm{cm}^{-1} & \text{for an atom,} \\ \gamma \simeq 10^{12} \mathrm{cm}^{-1} & \text{for a nucleus.} \end{array} \tag{15.20}$$

If the particle undergoes a localization around $x$ as in as in (6.1), $\psi_G(q)$ changes into:

$$\frac{\phi_x(q)}{\|\phi_x(q)\|}, \qquad \phi_x(q) \;=\; \left[\frac{\alpha}{\pi}\right]^{1/4} e^{-\frac{\alpha}{2}(x-q)^2} \psi_G(q). \tag{15.21}$$

¿From (6.2), the probability density that a localization takes place at $x$ is given by $\|\phi_x\|^2$. Accordingly, the probability that, if a hitting process occurs, the state of the system is still $\psi_G(q)$ is given by:

$$P_{ND} \;=\; \int d^3x \, |\langle \psi_G | \phi_x \rangle|^2 \;=\; \frac{1}{\sqrt{1+\alpha/4\gamma^2}} \;\simeq\; \left\{ \begin{array}{ll} 1-10^{-7} & \text{for an atom,} \\ 1-10^{-15} & \text{for a nucleus.} \end{array} \right. \tag{15.22}$$

Since microsystems are supposed to undergo one localization every $10^{16}$ seconds, the transition rate $Q_{E+D}$ to an excited or dissociated state is:

$$Q_{E+D} \;=\; \lambda(1-P_{ND}) \;\simeq\; \frac{\lambda\alpha}{8\gamma^2} \;=\; \left\{ \begin{array}{ll} 10^{-23} & \text{for an atom,} \\ 10^{-31} & \text{for a nucleus.} \end{array} \right. \tag{15.23}$$

In [158] Pearle has considered the case of the hydrogen atom and has compared the flux of Lyman–$\alpha$ ultraviolet photons emitted by intergalactic hydrogen with the one expected if a GRW mechanism were at work, the latter turning out to be much weaker than the one observed.

Applying the same argument to the quark model of a proton one would get a decay time of the same order of magnitude as the one of a nucleus ($10^{31}$ sec), whereas the proton lifetime is estimated longer than $10^{31}$ years, that is $10^{38}$ sec. This fact seems to indicate that the reduction rate $\Delta$ should be decreased by a factor $10^7$.

However, the consequences of $\Delta \simeq 10^{-13}$ cm$^{-2}$ sec$^{-1}$ would be unacceptable. In fact, since $\alpha^{-1/2} = 10^{-5}$ cm is a reasonable value for the localization length ($l_{\text{eff}}$ in Table 1), it would yield the value $\tau_N \simeq 10^{23}/N$ sec for the macroscopic decoherence time in (15.12). Thus, linear superpositions of spatially separated states of any reasonable macroscopic "pointer" ($N \simeq 10^{23}$) would typically take times of the order of the second to be suppressed.

### 15.3.1 Reconsidering the argument within the CSL approach

In [107] the authors have considered an initial bound state $\rho_B = |\psi_B\rangle\langle\psi_B|$ which evolves into $\rho(t)$ according to the dynamics (8.12) of CSL and have studied the transition probability $P(t)$ to an excited orthogonal state $|\Psi_E\rangle$:

$$\frac{d}{dt}P(t)\Big|_{t=0} \quad = \quad \frac{\partial}{\partial_t}\langle\psi_E|\rho(t)|\psi_E\rangle\Big|_{t=0}. \tag{15.24}$$

The only contribution is that from the reducing term in (8.12). By developing up to the first order in $\alpha$ one gets:

$$\frac{d}{dt}P(t)\Big|_{t=0} \simeq \sum_j \frac{\alpha\lambda N_j^2}{2}\,|\langle\psi_E|\mathbf{Q_j}|\psi_B\rangle|^2, \qquad \mathbf{Q}_j \;=\; \frac{1}{N_j}\sum_{i=1}^{N_j}\mathbf{q}_{ij}, \tag{15.25}$$

$j$ numbering the species of identical particles making up the system, $\mathbf{q}_{ij}$ being the position operators of the particles of type $j$ and $\mathbf{Q}_j$ their center of mass.

For just one nucleon the result agrees with that of QMSL, while, for macro–systems, due to the more efficient decoupling rate, it appears that a correction of $\Delta = \alpha\lambda/2$ which would lead to no conflict with the proton lifetime is possible. However, such a change of the value of $\Delta$ would lead to the limit of acceptability for small objects: the dynamics will not reduce within the perception time an object containing $10^{15}$ particles like a carbon particle of radius $10^{-3}$ cm. Similarly, the argument of [121] discussed in section 12 about the perception mechanism, would no longer be correct.

### 15.3.2 The Pearle and Squires argument

Considering a macroscopic body of total mass $M$ made up of different types of identical particles of mass $m_j$, one may think of relating the reduction process to the mass density operator

$$M(\mathbf{x}) \quad = \quad \sum_k m_k \left[\frac{\alpha}{2\pi}\right]^{3/2}\int_{\mathbf{R}^3} d^3y\, e^{-\frac{\alpha}{2}(\mathbf{y}-\mathbf{x})^2}\, a_k^\dagger(\mathbf{y})\, a_k(\mathbf{y}), \tag{15.26}$$

(see the discussion of section 8.6) rather than considering independent stochastic processes for the various kinds of particles.

The first order analysis that has led to (15.25) can be similarly carried out in this case; one merely has to consider the total center of mass $\mathbf{Q}$ of the system and not only the centers of mass $\mathbf{Q}_j$ of the various species

$$\mathbf{Q} \;=\; \frac{1}{M}\sum_j m_j \sum_{i=1}^{N_j}\mathbf{q}_{ij}. \tag{15.27}$$

as a consequence the rate of internal excitation and/or dissociation appears as a second order effect. Indeed, the total center of mass $\mathbf{Q}$ cannot excite any internal degree of freedom:

$$\frac{d}{dt}\,P(t)\Big|_{t=0} \;\simeq\; \sum_j \frac{M^2\alpha\lambda}{2m_0^2}\,|\langle\psi_E|\mathbf{Q_j}|\psi_B\rangle|^2 \;=\; 0 \tag{15.28}$$

If one takes the QMSL value of $\lambda$ for nucleons, that is if the reference mass $m_0$ is identified with the nucleon mass, one has a remarkable decrease of the QMSL rate $Q_{D+E}$ in (15.23) for atoms: from one atom per mole being either excited or dissociated every second to every $10^{12}$ seconds ($10^5$ years). Nevertheless, all the important features of QMSL are preserved:

- The decoupling of macroscopically distinguishable states is taken care of by the nucleons of the macroscopic bodies.
- The energy increase is almost identical to acceptable one which is implied by standard QMSL.
- The collapse induced decay probability of a proton is depressed by a factor $10^{-16}$ making the CSL predicted lifetime well compatible with the experimental data.

Some concluding comments are in order at this point: the above analysis has appropriately pointed out the nice features deriving from relating reduction to the mass of the elementary particles. However one cannot avoid mentioning that:

1. The dynamical reduction program has made clear that one can try to follow a new line to solve the conceptual problems that Quantum Mechanics meets with macro–objects and measurement processes, namely, modifying the dynamics so that the physics of microsystems remains unaltered, while macro–systems exhibit an acceptable behaviour. However one cannot forget that for the time being the program still requires many improvements, in particular the crucial problem to be faced is to work out reasonable relativistic generalizations of it. Being the quark dynamics fundamentally relativistic, and due to the great difficulties haunting the so called "non–relativistic quark models", applying directly the specific models of QMSL and CSL to nucleons seems a little bit too hazardous.
2. Other collapse theories are still under consideration. The model presented in section 8.7 tries to relate the decoherence mechanism to gravity and to reduce the number of new fundamental constants characterizing CSL. In particular, the reduction mechanism is linked to the mass of the systems involved, already meeting the basic request of [107].
3. The difficulties connected with nucleon decay might also be avoided by slight modifications of the standard CSL, for instance, by using a higher power $N^{4/3}(\mathbf{x})$ of the smeared number operator appearing in (8.1).

## 15.4 Spontaneous localizations in superconducting devices

We conclude the analysis of the experimental implications of dynamical reduction models mentioning the work of Rae [161], of Rimini [109] and of Buffa, Nicrosini and Rimini [162] on the effects of spontaneous localizations on superconducting devices. The argument of Rae [161] goes as follows: consider the BCS wavefunction [163] of a superconducting state

$$\psi \;=\; \psi_{\mathbf{k_1}}\,\psi_{\mathbf{k_2}}\,\ldots\psi_{\mathbf{k_n}}\,e^{iS(\mathbf{x})}, \tag{15.29}$$

where $\psi_{\mathbf{k_i}}$ represents the wavefunction of a Cooper pair of electrons with wavevectors $+\mathbf{k_i}$ and $-\mathbf{k_i}$, and $S(\mathbf{x})$ is the macroscopic phase associated with the supercurrent. The most relevant effect of spontaneous reductions of an electron in a superconducting device is to break one of the Cooper pairs, which would result in the supercurrent being reduced by about one part in $10^{20}$. Assuming that a reduction happens every $10^{-5}$ sec, the resulting decay would remain well below the experimental detection limits which are of the order of one part in $10^{13}$ per second.

A more realistic model in which the possibility of re–creation of Cooper particles is taken into account (a phenomenon which lowers the effects of spontaneous localizations) shows that dynamical reduction models are even more compatible with the existence of superconductivity, something which is not at all trivial.

A much more detailed and mathematical precise analysis of the effect of spontaneous localizations on superconducting devices has been performed in refs. [109, 162], within the framework of CSL: the conclusion is that, by taking into account the indistinguishability of electrons, the effects are even smaller than those predicted by Rae. We refer the reader to the above cited papers for the complete analysis of the problem.

# 16 Conclusions

In this paper we have analyzed in detail a quite radical proposal which, at the non relativistic level, allows one to circumvent the conceptual difficulties that standard quantum mechanics meets with the macro–objectification problem. Obviously, even though the theory is not a reinterpretation (as many of the attempts we have discussed in part I) of the standard theory, but qualifies itself as a rival theory of it, up to the moment in which technological improvements will make experimental tests actually feasible, to accept it is, to a large extent, a matter of taste and of the attitude one has with respect to the foundational problems of quantum theory. The theory we have reviewed in this report in its present formulation still has a phenomenological character and necessitates further hard work before it can be taken as a fundamental theory of natural processes. As we have already remarked, the real (and, we believe, relevant) merits which characterize it derive from the fact that it represents a conceptually new proposal for overcoming the embarrassing questions raised by the macro–objectification problem. However, finding a consistent relativistic generalization of the dynamical reduction theories, remains, as Bell has stressed, *the big problem* to be faced.

Having made clear the perspective we consider appropriate for the dynamical reduction program and its limitations, it seems useful to conclude this report by recalling the nice features which characterize it.

First of all, according to the theory all natural processes, the microscopic and the macroscopic ones, as well as those involving interactions between micro and macro systems, are governed by the universal evolution equation of the theory. Such an equation has never to be disregarded, contrary to what happens for Schrödinger's evolution of the standard scheme, on the basis of supplementary, imprecise, verbal prescriptions. All the embarrassing ambiguities of the standard theory concerning macro processes are only momentary in the new scheme. Again, in Bell's words [32], within the GRW theory *the cat is not both dead and alive for more that a split second*.

Another feature of the theory which deserves to be stressed is its structural difference from the (in our opinion) unique other consistent and fully worked out proposal to solve the measurement problem, i.e. Bohmian mechanics. The GRW theory is a genuine Hilbert space theory and does not add any kind of variables to standard quantum mechanics. However, by introducing mathematically precise modifications to it, it allows one to answer in an unambiguous way to all the fundamental questions

which characterize the debate on quantum mechanics since its birth: which systems and processes must be treated as classical and which ones as quantum, which are essentially reversible and for which ones irreversibility plays a fundamental role, and so on. Moreover the definite mathematical description of reductions makes also precise the action at a distance of ordinary quantum mechanics, throwing a new light on EPR–like situations and on quantum non locality. The nice features of the proposal we have reviewed have been summarized by Bell in a very concise sentence [32] *for myself, I see the GRW model as a very nice illustration of how quantum mechanics, to become rational, requires only a change which is very small (on some measures!).*

Coming to the relativistic aspect we recall that the theory, even though no consistent relativistic generalization of it has been fully worked out, presents some nice aspects which, once more, can be taken as interesting hints for the elaboration of a relativistic theory inducing reductions, an old problem which, as we have discussed in this report, has drawn a lot of attention. In this respect, it is useful to stress the different conceptual status of the dynamical reduction theories with respect, e.g. hidden variable theories. We are making reference here to the fact that the locality requirement can be split in the two conditions of parameter and outcome independence and that the linear version of dynamical reduction theories exhibits only outcome dependence, a fact that conflicts less than parameter dependence with a relativistic point of view. Actually, what J.S. Bell has has proved in ref.[32] is equivalent to checking that the GRW theory does not present parameter dependence. This analysis led him to state: *the model is as Lorentz invariant as it could be in the non relativistic version. It takes away the grounds of my fear that any exact formulation of quantum mechanics must conflict with fundamental Lorentz invariance.* Finally, we would like to conclude by stressing that the natural interpretation of the theories we have reviewed implies that they do not deal, as does the standard theory, with the probabilities of something occurring provided some specific action (a measurement) is performed by a conscious observer, i.e, with *what we would find*, but they speak directly of *what is* (i.e. an objective mass distribution), at the appropriate macroscopic level.

Concerning the philosophical implications of these approaches, if one is interested also in these aspects of scientific knowledge, it has to be remarked that they allow one to *close the circle* in the precise sense of Shimony [1], i.e., to build up a coherent worldview which can accommodate at the same time what we know about the peculiar behaviour of microscopic systems and the definiteness of the macroscopic world and of our perceptions about it. In particular, the theory makes fully legitimate a macro–realistic position about nature and has no need whatsoever to attribute any peculiar role to conscious observers, an unavoidable fact within the standard formalism and most of the proposed interpretations aiming to solve its conceptual difficulties.

## Acknowledgements

As the reader would have certainly grasped, the cooperation of many researchers has been necessary for the elaboration and the full exploitation of the ideas which are the subject of this review. In particular one of us (G.C.G.) wants to express his deep gratitute to his collegue A. Rimini, the collaboration with whom has been so fundamental for the working out of the dynamical reduction program. G.C.G. wants to express similar feelings to his colleagues T. Weber, P. Pearle and to his student R. Grassi who has been an essential collaborator in furthering the program, in particular for the elaboration of its interpretation and for the attempts at a relativistic generalization. Finally he is deeply indebted to J.S. Bell, whose interest in the dynamical reduction program, whose stimulating remarks and whose friendship has meant much more for him than words can say. Finally, both the authors are grateful to Prof. S. Goldstein

who has made an incredible job acting as a referee for this report and has contributed in a fundamental way in improving it with his many deep and stimulating remarks.

# References

[1] A. Shimony, in: *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem*, J.T. Cushing and E. McMullin eds., University of Notre Dame Press, Notre Dame, Indiana (1989).

[2] J.S. Bell, *Physics World*, August issue, 33 (1990), and in: *Sixty–Two Years of Uncertainity*, A. Miller ed., Plenum Press, New York (1990).

[3] A. Einstein, in: *Albert Einstein: Philosoper–Scientist*, P.A. Schilpp ed., Tudor Publishing Co., New York (1949).

[4] A. Shimony, in: *The New Physics*, P. Davies ed., Cambridge University Press, Cambridge (1989).

[5] A. Einstein, B. Podolski and N. Rosen, *Phys. Rev.* **47**, 777 (1935).

[6] G.C. Ghirardi, L. Marinatto and T. Weber, *Journ. Stat. Phys.* 108, 49 (2002).

[7] J. von Neumann: *Mathematische Grundlagen der Quantummechanik*, Springer–Verlag (1932). English translsation: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955).

[8] H. Primas, in: *Symposium on the Foundations of Modern Physics 1990*, Joensuu, Finland. Discussion Sections edited by K.V. Laurikainen and J. Viiri.

[9] W.H. Zurek *Phys. Rev.* **D 24**, 1516 (1981). *Ibid* **D 26**, 1862 (1982).

[10] W.H. Zurek: *Physics Today*, October, 36 (1991).

[11] D. Giulini, E. Joos, C. Kiefer, J. Kumpsch, I.–O. Stamatescu, and H.D. Zeh: *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer–Verlag (1996).

[12] A. Bassi and G.C. Ghirardi, *Phys. Lett.* **A 275**, 373 (2000).

[13] B. d'Espagnat: *Conceptual foundations of quantum mechanics*, 2nd ed., Addison Wesley Publishing Company, Reading, MA (1989).

[14] G.C. Ghirardi, in: *The Interpretation of Quantum Theory: where do we stand?*, L. Accardi ed., Istituto dell'Enciclopedia Italiana (1994).

[15] D. Bohm, *Phys. Rev.* **85**, 166 (1952); *ibid*, 180 (1952).

[16] D. Bohm and B.J. Hiley, *Phys. Reports* **172**, 93 (1989).

[17] P.R. Holland, *The Quantum Theory of Motion*, Cambridge University Press (1993).

[18] D. Dürr, S. Goldstein and N. Zanghì, *Journ. Stat. Phys.* **67**, 843 (1992).

[19] D. Dürr, S. Goldstein and N. Zanghì, in: *Bohmian Mechanics and Quantum Theory, an Appraisal*, J.T. Coushing, A. Fine and S. Goldstein eds., Kluwer Academic Publishers, Dordrecht (1996).

[20] M. Daumer, D. Dürr, S. Goldstein and N. Zanghí: *Erkenntnis* **45**, 379 (1997).

[21] A.M. Gleason, *Journ. Math. and Mech.* **6**, 885 (1957).

[22] S. Kochen and E.P. Specker, *Journ. Math. and Mech.* **17**, 59 (1967).

[23] J.S. Bell, in: *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, 52 (1987).

[24] J.S. Bell, in: *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, 173 (1987).

[25] E. Deotto and G.C. Ghirardi, *Found. Phys.* **28**, 1 (1998).

[26] A. Rimini, in: *Chance in Phyics: Foundations and Perspectives*, J. Bricmont, D. Dürr, M.C. Gallavotti, G.C. Ghirardi, F. Petruccione and N. Zanghì eds., Springer (2001).

[27] J.M. Jauch, *Helv. Phys. Acta* **37**, 293 (1964).

[28] A. Daneri, A. Loinger and G.M. Prosperi, *Nuclear Phys.* **33**, 297 (1962).

[29] K. Gottfried, *Quantum Mechanics: Vol I*, Benjamin, New York (1966).

[30] E. Joos and H.D. Zeh, *Zeit. Phys.* **B 59**, 223 (1985).

[31] P. Bush, P.J. Lathi and P. Mittelstaedt, *The Quantum Theory of Measurement*, Lectures Notes in Physics m2, Springer Verlag (1991).

[32] J.S. Bell, in: *Schrödinger: Centanary Celebration of a Polymath*, C.W. Kilmister ed., Cambridge University Press, Cambridge (1987). Also in: *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, 201 (1987).

[33] B. d'Espagnat, *Found. Phys.* **20**, 1147 (1990).

[34] E. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics*, Addison–Wesley, Reading, Massachussetts (1981).

[35] B.C. van Frassen, in: *Problems in the Foundations of Physics*, G. Toraldo di Francia ed., North Holland, Amsterdam (1979). In: *Current Issues in Quantum Logic*, E. Beltrametti and B.C. van Frassen eds., Plenum Press, New York, 180 (1981). *Quantum Mechanics: an Empiricist View*, Clarendon Press, Oxford (1991).

[36] S. Kochen, in: *Symposium on the Foundations of Modern Physics 1985*, P. Lathi and P. Mittlelstaedt eds., World Scientific, Singapore, 151 (1985).

[37] D. Dieks, *Ann. Phys.* **7**, 174 (1988). *Found. Phys.* **19**, 1387 (1989).

[38] R. Healey, *The Philosophy of Quantum Mechanics: an Interactive Interpretation*, Cambridge University Press (1989).

[39] R. Clifton, *Int. Journ. Theor. Phys.* **34**, 1303 (1995). *Brit. Journ. Phil. Sci.* **46**, 33 (1995).

[40] P.E. Vermaas, *Stu. His. and Phil. Mod. Scie.* **27** 133 (1996). *Phys. Rev. Lett.* **78**, 2033 (1997).

[41] P.E. Vermaas and D. Dieks, *Found. Phys.* **25**, 145 (1995); *The Modal Interpretation of Quantum Mechanics*, Kluwer Academic Publishers, Dordrecht (1998).

[42] G. Bacciagluppi, in: *The Modal Interpretation of Quantum Mechanics*, D. Dieks and P.E. Vermaas eds, Kluwer Academic Publishing, Dordrecht (1998).

[43] D. Albert, in: *New Techniques and Ideas in Quantum Measurement Theory*, D.M. Greenberger ed., New York Academy of Science (1986).

[44] E.J. Squires, *Eur. Journ. Phys.* **8**, 171 (1987).

[45] D. Albert and B. Loewer, in: *Proceedings of the 1990 Biennial Meeting of the Philosophy of Science Association. Vol. 1*, A. Fine, M. Forbes and L. Wessels eds., Philosophy of Science Association, East Leasing, 277 (1990).

[46] R.B. Griffiths, *Journ. Stat. Phys.* **36**, 219 (1984).

[47] R.B. Griffiths, *Phys. Rev.* **A 54**, 2759 (1996).

[48] R. Omnès, *Journ. Stat. Phys.* **53**, 893 (1988). *Ibid* **53**, 933 (1988). *Ibid* **57** 956 (1988).

[49] R. Omnès: *The Interpretation of Quantum Mechanics*, Princeton University Press, Princeton (1994).

[50] M. Gell–Mann and J.B. Hartle, in: *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology*, S. Kobayashi et al. eds., Physical Society of Japan, Tokio, 321 (1990).

[51] A. Bassi and G.C. Ghirardi, *Phys. Lett.* **A 257**, 247 (1999).

[52] A. Bassi and G.C. Ghirardi, *Journ. Stat. Phys.* **98**, 457 (2000).

[53] R. Griffiths, *Phys. Lett.* **A 265**, 12 (2000).

[54] A. Bassi and G.C. Ghirardi, *Phys. Lett.* **A 265**, 153 (2000).

[55] I. Giardina and A. Rimini, *Found. Phys.* **26**, 973 (1996).

[56] B. d'Espagnat, *Phys. Lett.* **A 124**, 204 (1987). *Journ. Stat. Phys.* **56**, 747 (1989).

[57] H. Everett III: *Rev. Mod. Phys.* **29**, 454 (1857).

[58] B.S. DeWitt, in: *Foundations of Quantum Mechanics*, B. d'Espagnat ed., Academic Press, New York (1971).

[59] D.S. DeWitt and N. Graham, *The Many–Worlds Interpretation of Quantum Mechanics*, Princeton University Press (1973).

[60] D. Deutsch, *Int. Journ. Theor. Phys.* **24**, 1 (1985).

[61] D. Albert and B. Loewer, *Synthese* **77**, 195 (1988).

[62] D. Albert, *Quantum Mechanics and Experience*, Harward University Press, Cambridge, MA (1992).

[63] E.P. Wigner, *The Scientist Speculates*, E.J. Good ed., Heinemann, London (1961).

[64] G.C. Ghirardi, A. Rimini and T. Weber, in *Quantum Probability and Applications II*, L. Accardi and W. von Wandelfels eds., Lecture Notes in Mathematics vol. 1136, Springer, Berlin (1985).

[65] G.C. Ghirardi, A. Rimini and T. Weber, *Phys. Rev.* **D 34**, 470 (1986).

[66] G.C. Ghirardi, P. Pearle and A. Rimini, *Phys. Rev.* **A 42**, 78 (1990).

[67] G.C. Ghirardi, R. Grassi and P. Pearle, *Found. Phys.* **20**, 1271 (1990).

[68] G.C. Ghirardi, R. Grassi and F. Benatti, *Found. Phys.* **25**, 5 (1995).

[69] H.P. Stapp, preprint LBL–26968.

[70] G.C. Ghirardi and R. Grassi, in: *Atti del Congresso Nuovi problemi della logica e della filosofia della scienza*, CLUEB, Bologna (1991).

[71] P.H. Eberard, *Nuovo Cimento* **B 46**, 392 (1978).

[72] G.C. Ghirardi, A. Rimini and T. Weber, *Lett. Nuovo Cimento* **27**, 293 (1980).

[73] G.C. Ghirardi and T. Weber, *Nuovo Cimento* **B 78**, 9 (1983).

[74] A. Shimony, in: *Proceedings of the International Symposium on the Foundations of Quantum Mechanics*, S. Kamefuchi *et al* eds., Physical Society of Japan, Tokio, 225 (1983).

[75] N. Gisin, *Hel. Phys. Acta* **62**, 363 (1989).

[76] A. Bassi and G.C. Ghirardi, *Phys. Lett.* **A 309**, 4 (2003).

[77] A. Kent, *Modern Physics Letters* **A 4**, 1839 (1989).

[78] E.B. Davies, *Quantum Theory of Open Systems*, Academic Press, London (1976).

[79] N. Gisin, *Phy. Lett.* **A 143**, 1 (1990).

[80] S. Weinberg, *Phys. Rev. Lett.* **62**, 485 (1989). *Ann. Phys.* **194**, 336 (1989). *Nuclear Physics* **B 6**, 67 (1989).

[81] L. Fonda, G.C. Ghirardi, A. Rimini and T. Weber, *Nuovo Cimento* **15 A**, 689 (1973).

[82] L. Fonda, G.C. Ghirardi, A. Rimini, *Reports on Progress in Physics* **41**, 587 (1978).

[83] L. Fonda, G.C. Ghirardi, A. Rimini, *Nuovo Cimento* **18 B**, 1 (1973).

[84] G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).

[85] F. Benatti, G.C. Ghirardi, A. Rimini and T. Weber, *Nuovo Cimento* **101 B**, 333 (1988).

[86] G.C. Ghirardi, A. Rimini and T. Weber, *Phys. Rev.* **D 36**, 3287 (1987).

[87] J.S. Bell, in: *Quantum Gravity 2*, C. Isham, R. Pensrose and D. Sciama eds., Clarendon Press, Oxford (1981). Also in: *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, 117 (1987).

[88] P. Pearle, *Phys. Rev.* **A 39**, 2277 (1989).

[89] P. Pearle, in: *Sixty–Two Years of Uncertainity: Historical Philosophical, and Physics Inquires into the Foundations of Quantum Physics*, A.I. Miller ed., Plenum Press, New York (1990).

[90] P. Pearle, *Phys. Rev.* **A 59**, 80 (1999).

[91] P. Pearle, in: *Open Systems and Measurement in Relativistic Quantum Theory*, H.P. Breuer and F. Petruccione eds., Springer Verlag (1999).

[92] O. Nicrosini and A. Rimini, "Relativistic Spontaneous Localizations: a Proposal", preprint *quant–ph/0207145*.

[93] K. Kraus, *States, Effects and Operations*, Springer, Berlin (1983).

[94] W.F. Stinespring, *Proc. Am. Math. Soc.* **6**, 211 (1955).

[95] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Wiley & Sons, New York (1971).

[96] N. Gisin, *Phys. Rev. Lett.* **52**, 1657 (1984). *Ibid*, **53**, 1776, (1984).

[97] P. Pearle, *Phys. Rev. Lett.* **53**, 1775 (1984).

[98] L. Diósi, *Phys. Lett.* **A 129**, 419 (1988).

[99] L. Diósi, *Phys. Rev.* **A 40**, 1165 (1989).

[100] P. Pearle, *Phys. Rev.* **D 13**, 857 (1976).

[101] P. Pearle, *Found. Phys.* **12**, 249 (1982).

[102] F. Karolyhazy, *Nuovo Cimento*, **A 42**, 390 (1966).

[103] A.B. Komar, *Int. Journ. Theor. Phys.* **2**, 157 (1969).

[104] R. Penrose, in: *Quantum Comcepts in Space and Time*, R. Penrose and C.J. Isham eds., Clarendon, Oxford (1986).

[105] J. Ellis, S. Mohanty and D.V. Nanopoulos *Phys. Lett.* **B 221**, 113 (1989).

[106] A. Frenkel, *Found. Phys.* **20**, 159 (1990).

[107] P. Pearle and E. Squires, *Phys. Rev. Lett.* **73**, 1 (1994).

[108] F. Benatti, G.C. Ghirardi, and R. Grassi, in: *Advances in Quantum Phenomena*, E. Beltrametti and J.M. Levy–Leblond eds., Plenum Press, New York (1995).

[109] A. Rimini, in: *Advances in Quantum Phenomena*, E. Beltrametti and J.M. Levy–Leblond eds., Plenum Press, New York (1995).

[110] G.C. Ghirardi, R. Grassi and A. Rimini, *Phys. Rev.* **A 42**, 1057 (1990).

[111] A. Bassi and G.C. Ghirardi, *Phys. Rev.* **A 65**, 042114 (2002).

[112] P. Pearle, in: "*Perspectives on Quantum Reality*", R. Clifton ed., Kluwer Dordrecht, p. 93 (1996).

[113] L. Diósi and W. T. Strunz, *Phys. Lett.* **A 235**, 569 (1997).

[114] L. Diósi, N. Gisin and W. T. Strunz, *Phys. Rev.* **A 58**, 1699 (1998).

[115] K. Sobczyk: "*Stochastic Wave Propagation*", Elsevier, Amsterdam (1985).

[116] A.A. Budini, *Phys. Rev.* **A 63**, 012106 (2000).

[117] L.P. Hughston, *Proc. R. Soc. London*, **A 447**, 953 (1996).

[118] S. Adler and P. Howritz: *Journ. Math. Phys.* **41**, 2485 (2000).

[119] S. Adler, *Journ. Phys.* **A 35**, 841 (2002).

[120] J.S. Bell, in: *Themes in Contemporary Physics II, Essay in Honor of Julian Schwinger's 70th birthday*, S. Deser and R.J. Finkelstein eds., World Scientific, Singapore (1989).

[121] F. Aicardi, A. Borsellino, G.C. Ghirardi and R. Grassi, *Found. Phys. Lett.* **4**, 109 (1991).

[122] J.S. Bell, in: *Proceedings of the Nobel Symposium 65: Possible Worlds in Art and Science*, S. Allen ed., Nobel Foundation, Stockholm (1986).

[123] A.J. Leggett, *Suppl. Prog. Theor. Phys.* **69**, 80 (1980).

[124] J.S. Bell, CERN–TH 4035/84, (1984).

[125] A. Shimony, in *Proceedings of the 1990 Biennial Meeting of the Philosophy of Science Association. Vol. 2*, A. Fine, M. Forbes and L. Wesseles eds., Philosophy of Science Association, East Leasing, (1990).

[126] P. Lewis, *Brit. Journ. Phil. Scie.* **48**, 313 (1997).

[127] R. Clifton and B. Monton, *Brit. Journ. Phil. Scie.* **50**, 697 (1999).

[128] R. Clifton and B. Monton, *Brit. Journ. Phil. Scie.* **51**, 155 (1999).

[129] G.C. Ghirardi and A. Bassi, *Brit. Journ. Phil. Sci.* **50**, 49 (1999).

[130] A. Bassi and G.C. Ghirardi, *Brit. Journ. Phil. Sci.* **50**, 719 (1999).

[131] A. Bassi and G.C. Ghirardi, *Brit. Journ. Phil. Sci.* **52**, 125 (2001).

[132] D.Z. Albert and L. Vaidman, *Phys. Lett.* **A 139**, 1 (1989).

[133] D.Z. Albert, in: *Sixty–Two Years of Uncertainity: Historical, Philosophical and Physics Inquires into the Foundations of Quantum Mechanics*, A.I. Miller ed., Plenum Press, New York (1990).

[134] W. Bialek, *Ann. Rev. Biophys. Chem.* **16**, 455 (1987).

[135] F. Bornacin, C. Pfister and M. Charbe, in: *Sensory Transduction*, A. Borsellino, L. Cervetto and V. Torre eds., Plenum Press, New York (1990).

[136] H. Hubel, *Eye, Brain and Vision*, Freeman, San Francisco (1988).

[137] R. Penrose, *The Emperor's New Mind*, Oxford University Press, Oxford (1989).

[138] G.C. Ghirardi and P. Pearle, in: *Proceedings of the 1990 Biennial Meeting of the Philosophy of Science Association. Vol. 2*, A. Fine, M. Forbes and L. Wessels eds., Philosophy of Science Association, East Leasing, (1990).

[139] J. Butterfield, G.N. Fleming, G.C. Ghirardi and R. Grassi, *Int. Journ. Theor. Phys.* **32**, 2287 (1993).

[140] G.C. Ghirardi and R. Grassi, *Stud. Hist. Phil. Sci.* **25**, 397 (1994).

[141] D. Albert, *Time and chance*, Harvard University Press, (2000)

[142] G.C. Ghirardi, A. Rimini and T. Weber, in: *Proceedings of the XVIIth International Colloquium on Group Theoretical Methods in Physics*, Sainte–Adele, Canada, Y. Saint–Aubin and L. Vinet eds., World Scientific, Singapore (1989).

[143] G.C. Ghirardi and A. Rimini, in: *Sixty–Two Years of Uncertainity: Historical Philosophical, and Physics Inquires into the Foundations of Quantum Physics*, A.I. Miller ed., Plenum Press, New York (1990).

[144] S. Adler and T. Brun, *Journ. Phys.* **A 34**, 4797 (2001).

[145] I. Bloch, *Phys. Rev.* **156**, 1377 (1967).

[146] K.E. Hellwing and K. Kraus, *Phys. Rev.* **D 1**, 566 (1970).

[147] Y. Aharonov and D.Z. Albert, *Phys. Rev.* **D 21**, 3316 (1980). *Ibid.* **D 24**, 359 (1981).

[148] G.C. Ghirardi, P. Pearle and A. Rimini, in: *Proceedings of the International Sympoium on the Foundations of Quanum Mechanics (Tokyo 1989)*, S. Kamefuchi et al. eds., Physical Socienty of Japan, Tokyo (1990).

[149] M. Redhead, *Incompleteness, Nonlocality and Realism: a Prolegomenon to the Philosophy of Quantum Mechanics*, Clarendon Press, Oxford (1987).

[150] E. Schrödinger, *Proc. Cambridge Philos. Soc.* **31**, 555 (1935).

[151] P.H. Eberhard and R.R. Ross, *Found. Phys. Lett.* **2**, 127 (1989).

[152] J.S. Bell, in: *Foundations of Quantum Mechanics*, B. d'Epagnat ed., Academic Press, New York (1971). See also *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press (1987).

[153] P. Suppes and M. Zanotti, *On the determinism of hidden variable theories with strict correlations and conditional statistical independence of observables*, Reidel, Dordrecht, 445 (1976).

[154] B. van Fraassen, *Synthese* **52**, 25 (1982).

[155] J. Jarrett, *Nous* **18**, 569 (1984).

[156] J. Butterfield, G.N. Fleming, G.C. Ghirardi and R. Grassi, *Int. Journ. Theor. Phys.* **32**, 2287 (1993).

[157] M. Tegmark, *Found. Phys. Lett.* **6**, 571 (1993).

[158] P. Pearle, in: *The Interpretation of Quantum Theory: Where do we stand?*, L. Accardi ed., Enciclopedia Italiana, Roma (1994).

[159] M. Poratti and S. Putterman, *Phys. Rev.* **A 39**, 3010 (1989).

[160] C. Cohen–Tannoudji and J. Dalibard, *Europhys. Lett.* **1**, 441 (1986).

[161] A.I.M. Rae, *Journ. Phys.* **A 23**, L57 (1990).

[162] M. Buffa, O. Nicrosini and A. Rimini, *Found. Phys. Lett.* **8**, 105 (1995).

[163] D.R. Tilley and J. Tilley: *Superfluidity and Superconductivity*, 2nd ed. Adam Hilger, Bristol (1986).