Stratification of the moduli space of four–gonal curves

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Abstract

Let X be a smooth irreducible projective curve of genus g and gonality 4. We show that the canonical model of X is contained in a uniquely defined surface, ruled by conics, whose geometry is deeply related to that of X. This surface allows us to define four invariants of X and hence to stratify the moduli space of four–gonal curves by means of closed irreducible subvarieties whose dimensions we compute.

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Contents

Introduction

Let X be a smooth irreducible curve of genus g and gonality γ, i.e. γ is the minimal degree of a base–point–free linear series on X. Let $\mathcal{M}_g$ denote the moduli space of curves of genus g and $\mathcal{M}_{g,\gamma} \subset \mathcal{M}_g$ denote the variety parametrizing the γ-gonal curves; it is well–known that $\mathcal{M}_{g,\gamma}$ is an irreducible variety of dimension $2g + 2\gamma - 5$, as far as $2 \leq \gamma \leq \frac{g}{2} + 1$ (see [**13**] and [**1**]).

The structure of $\mathcal{M}_{g,\gamma}$ is completely understood in the cases $\gamma = 2$ (hyperelliptic curves) and $\gamma = 3$ (trigonal curves). In this paper we are interested in the study of four–gonal curves. Let us briefly recall the setting in the trigonal case.

Let K denote the canonical divisor on X and $X_K \subset \mathbb{P}^{g-1}$ be the canonical model of X. From the Geometric Riemann–Roch Theorem, any trigonal divisor spans a line in $\mathbb{P}^{g-1}$, therefore X_K is contained in a rational normal ruled surface, R say. It is clear that R is of the form $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(g-2-m))$; assuming $m \leq g - 2 - m$, the integer m is uniquely determined and it is called the *Maroni invariant* of X.

Set $\mathcal{M}_{g,3}(m)$ the variety parametrizing the trigonal curves of Maroni invariant not bigger than m. The following fact holds:

Theorem. *If $\frac{g-4}{3} \leq m < \frac{g-2}{2}$ (resp. $m = \frac{g-2}{2}$) then $\mathcal{M}_{g,3}(m)$ is a locally closed subset of $\mathcal{M}_{g,3}$ of dimension $g + 2m + 4$ (resp. $2g + 1$).*

(See [**14**], Proposition 1.2).

One can see that for each curve of genus $g \geq 5$ of Maroni invariant m there exists a unique linear series g^1_λ, where λ is the minimum integer bigger than 3 and $\lambda = g - m - 1$. Hence λ is uniquely determined by m and the above filtration of $\mathcal{M}_{g,3}$ given by the varieties $\mathcal{M}_{g,3}(m)$ can be rewritten in terms of λ.

In general, it seems interesting to find "good invariants" arising from the geometric properties of γ–gonal canonical curves, in order to obtain an analogous stratification of the moduli space $\mathcal{M}_{g,\gamma}$.

As in the trigonal case, one can introduce the rational normal scroll V, whose fibres are the $(\gamma-2)$–planes spanned by the γ–gonal divisor on X. Clearly $V = \mathbb{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{\gamma-1}))$, where $a_1 + \cdots + a_{\gamma-1} = g - \gamma + 1$; in this way the integers $a_1, \ldots, a_{\gamma-2}$ play the role of the Maroni invariant m in the trigonal case.

In this paper we focus on 4–gonal curves. We show that in the volume $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ there exists an (almost always) uniquely determined "minimal" surface, ruled by conics, containing X_K.
Such a surface S gives rise to other two invariants: on one hand, one defines the number t which is the uniquely determined invariant of a suitable geometrically ruled surface birationally equivalent to S. On the other hand, analyzing the embedding of X in S, we obtain another number $\lambda > 4$ which turns out to be the minimum degree of a linear series on X different from the gonal one.
Comparing the configuration $X_K \subset S \subset V$ in the 4–gonal case with the analougous situation $X_K \subset R$ of the trigonal case, it is clear that the invariant m has been replaced, in some sense, by a, b and t. Finally, one can prove that λ is now independent of a, b and t; so a four–gonal curve is determined by the four invariants a, b, λ, t.

In Section 6 we describe the geometric meaning of λ, while, in Sections 5 and 7, we find the ranges for the above invariants λ, t and a, b, respectively.

If $t = 0$ the cited ranges become:

$$\frac{g+3}{3} \leq \lambda \leq \frac{g+3}{2} \tag{R_1}$$

$$a_{\min} \leq a \leq \frac{g-3}{3} \tag{R_2}$$

$$g - \lambda - 1 \leq a + b \leq \frac{2(g-3)}{3} \tag{R_3}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda - 4}{2} \right\rceil & \text{if} \quad \lambda \geq \frac{2g+6}{5} \\ g - 2\lambda + 1 & \text{if} \quad \lambda \leq \frac{2g+6}{5} \end{cases}$$

In Section 8 (see Theorem 8.5) we then show that, if $(R_1), (R_2), (R_3)$ are satisfied, there exists a 4–gonal curve of genus g and invariants a, b, λ and $t = 0$.

Finally, in Section 10 we study the moduli spaces $\mathcal{M}_{g,4}$ of 4–gonal curves with $t = 0$. Set $\mathcal{M}_g^\lambda \subset \mathcal{M}_{g,4}$ be the variety parametrizing the 4–gonal curves of invariant λ and $\mathcal{M}_g^\lambda(a,b) \subset \mathcal{M}_g^\lambda$ the subvariety parametrizing the curves of further invariants a and b. We prove the following:

Main Theorem. *Let g, λ, a, b be positive integers satisfying (R_1), (R_2), (R_3) and $g \geq 10$. Then:*

i) There exists a stratification of the moduli space $\mathcal{M}_{g,4}$ of 4–gonal curves given by:

$$\mathcal{M}_{g,4} = \overline{\mathcal{M}}_g^{\left\lceil \frac{g+2}{2} \right\rceil} \supset \overline{\mathcal{M}}_g^{\left\lceil \frac{g}{2} \right\rceil} \supset \cdots \supset \overline{\mathcal{M}}_g^{\lambda} \supset \cdots \supset \overline{\mathcal{M}}_g^{\left\lceil \frac{g+3}{3} \right\rceil}$$

and $\overline{\mathcal{M}}_g^\lambda$ are irreducible locally closed subsets of dimension $g + 2\lambda + 1$, if $\lambda < \left\lceil \frac{g+2}{2} \right\rceil$.

ii) For each admissible λ, we can write:

$$\overline{\mathcal{M}}_g^\lambda = \bigcup_{a,b} \overline{\mathcal{M}}_g^\lambda(a,b)$$

where $\overline{\mathcal{M}}_g^\lambda(a,b)$ is a non–empty, irreducible subvariety whose dimension is :

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \begin{cases} 2(2a + b + \lambda) + 10 - g - \epsilon - \tau - \xi, & \text{if } a \geq \frac{g-\lambda-1}{2} \\ 2(a+b) + \lambda + 8 - \epsilon - \xi, & \text{if } a < \frac{g-\lambda-1}{2} \end{cases}$$

where

$$\epsilon := \begin{cases} 0, & \text{if } b < c \\ 1, & \text{if } a < b = c \\ 2, & \text{if } a = b = c \end{cases}, \quad \tau := \begin{cases} 0, & \text{if } a < b \\ 1, & \text{if } a = b \end{cases} \quad \text{and} \quad \xi := \begin{cases} 1, & \text{if } \lambda = \frac{g+3}{2} \\ 0, & \text{otherwise} \end{cases}.$$

In Section 11 we briefly describe the moduli space of four–gonal curves of invariant $t \geq 1$.

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0. Preliminaries

We say that a curve is 4–gonal if it has a linear series g^1_4 but no g^1_d, for any $d \leq 3$. We also assume that such curve is not bi–hyperelliptic (i.e. the degree four map on $\mathbb{P}^1$ does not factorize through a hyperelliptic curve), in particular that is not bielliptic.

Let X be a 4–gonal curve of genus g. In order to have a unique g^1_4 on X, we assume $g \geq 10$.

Denote by $\varphi_K : X \to X_K \subset \mathbb{P}^{g-1}$ the canonical map associated to X and by X_K the canonical model of X. In general, if Y is a variety and D is a divisor on Y, we denote by $\varphi_D : Y \to \varphi_D(Y) \subset \mathbb{P}(H^0(Y, \mathcal{O}_Y(D)))$ the morphism associated to D.

If $\Phi \in g^1_4$ is a 4–gonal divisor, by the Geometric Riemann–Roch Theorem (see [**2**], Ch. I, Sect. 2) we have that: $\dim\langle\varphi_K(\Phi)\rangle = \deg(\Phi) - h^0(\mathcal{O}_X(\Phi)) = 2$; therefore

$$V := \bigcup_{\Phi \in g^1_4} \langle\varphi_K(\Phi)\rangle \subset \mathbb{P}^{g-1}$$

is a scroll, ruled by planes on $\mathbb{P}^1$, containing X_K. Denote $\pi : V \longrightarrow \mathbb{P}^1$ the natural projection.

Recall that a non degenerate variety $W \subset \mathbb{P}^r$ is said to be *projectively normal* if it is normal and, for any $k \in \mathbb{N}$, the homomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \longrightarrow H^0(W, \mathcal{O}_W(k))$$

induced by the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_W \longrightarrow 0$$

is surjective.
We say that W is *linearly normal* if the homomorphism above is surjective for $k = 1$. In particular, if W is a non degenerate curve, then it is linearly normal if and only if $h^0(W, \mathcal{O}_W(1)) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r + 1$.

It is well–known that X_K is projectively normal; so V is a rational normal scroll (hence projectively normal as well). We then set $V = \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a vector bundle of rank 3 on $\mathbb{P}^1$ i.e.

$$\mathcal{F} = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c),$$

for suitable non–negative integers $a \leq b \leq c$. It is also well–known that, for any k, it holds:

$$h^0(V, \mathcal{O}_V(k)) = h^0(\mathbb{P}^1, \pi_*\mathcal{O}_V(k)) = h^0(\mathbb{P}^1, \mathrm{Sym}^k\mathcal{F}) \tag{1}$$

and that the Riemann – Roch Theorem for any vector bundle $\mathcal{G}$ on $\mathbb{P}^1$ with non–negative splitting type gives:

$$h^0(\mathbb{P}^1, \mathcal{G}) = \deg(\mathcal{G}) + \mathrm{rk}(\mathcal{G}). \tag{RR}$$

From the two above relations, since $a, b, c \geq 0$, we then have: $h^0(V, \mathcal{O}_V(1)) = h^0(\mathbb{P}^1, \mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})$. Taking into account that $h^0(V, \mathcal{O}_V(1)) = g$, we finally obtain:

$$a + b + c = g - 3. \tag{2}$$

In the following we will need some basic notations and facts about ruled surfaces.
We denote by $\mathbb{F}_t$ (where $t \geq 0$) the *Hirzebruch surface* of *invariant* t, i.e. the $\mathbb{P}^1$–bundle over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}(-t) \oplus \mathcal{O}$ (here $\mathcal{O}$ means $\mathcal{O}_{\mathbb{P}^1}$).

If $1 \leq a \leq b$, a *rational ruled surface* $R_{a,b}$ is $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$, naturally embedded in $\mathbb{P}^{a+b+1}$. Clearly, setting $t := b - a$, we have $R_{a,b} \cong \mathbb{F}_t$, so t is the *invariant* of $R_{a,b}$.

Let us recall the following well–known facts (see [**11**], Ch. V, 2.9, 2.17 and 2.3):

Lemma 0.1. *Let $\mathbb{F}_t$ be as before, f its generic fibre and $C_0 = \mathbb{P}(\mathcal{O}(-t)) \subset \mathbb{F}_t$. Then:*

i) $C_0^2 = -t$;

ii) *if U is any directrix (i.e. an irreducible unisecant curve) of $\mathbb{F}_t$, different from C_0, then $U^2 \geq t$;*

iii) *if there exists a directrix U of R such that $U^2 = 0$ then $t = 0$, i.e. $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$.*
Moreover, $t > 0$ if and only if $\mathbb{F}_t$ has exactly one unisecant curve (namely C_0) having negative self–intersection.

iv) $Num(\mathbb{F}_t) = \mathbb{Z}\langle C_0\rangle \times \mathbb{Z}\langle f\rangle$.

Finally let us recall three classical formulas concerning ruled surfaces and scrolls, due to C. Segre.

Unisecants Formula. *Let $R \subset \mathbb{P}^{r+1}$ be a ruled surface R of degree r and invariant t and let $Un^d(R)$ be the variety of the unisecant curves on R having degree d and self–intersection bigger than t. Then the general element of $Un^d(R)$ is irreducible and*

$$\dim(Un^d(R)) = 2d + 1 - r. \tag{UF}$$

Proof. Recall that, if $U \sim C_0 + nf$ is a unisecant curve on R, where $U^2 > t$, then

$$h^0(R, \mathcal{O}_R(U)) = 2n - t + 2 \tag{3}$$

(see [**11**], Ch. V, 2.19). By appliying the equality (3) to the hyperplane section H of R, we get $H \sim C_0 + \frac{r+t}{2} f$. Take $D \in Un^d(R)$; since $D \cdot H = d$, then $D \sim C_0 + (d - \frac{r-t}{2})f$. Therefore, since $D^2 > t$ by assumption, we can apply (3) and obtain the required formula. $\diamond$

The following Genus Formula (GF) is a consequence of the Adjuction Formula.

Genus Formula. *If Y is a q-secant curve on a ruled surface $R \subset \mathbb{P}^r$, then*

$$p_a(Y) = \frac{q-1}{2}\,\big[2(\deg(Y) - 1) - q\deg(R)\big]. \tag{GF}$$

The following relation (IF), generalizing the analogous property for ruled surfaces, comes from the Intersection Law on a scroll ([**8**], 8.3.14):

Intersection Formula. *Let W be a rational scroll ruled by n–planes and let C_1 and C_2 be two subschemes of W meeting properly and such that C_i is m_i–secant, for $i = 1, 2$ (i.e. C_i meets the general fibre of W in a variety of degree m_i). Then the following equality holds:*

$$\deg(C_1 \cdot C_2) = m_1 \deg(C_2) + m_2 \deg(C_1) - m_1 m_2 \deg(W). \tag{IF}$$

Let us also recall the following notions:

Definition. Let D be a very ample bisecant divisor on a Hirzebruch surface $\mathbb{F}$; then the surface $S_0 := \varphi_D(\mathbb{F})$ is said *geometrically ruled by conics* (over $\mathbb{P}^1$). Equivalently, a projective surface $S_0 \subset \mathbb{P}^N$ is geometrically ruled by conics if there exists a surjective morphism $\pi: S_0 \longrightarrow \mathbb{P}^1$ such that the fibre $\pi^{-1}(y)$ is a smooth rational curve of degree 2 for every point $y \in \mathbb{P}^1$ and π admits a section.
We say that a projective surface $S \subset \mathbb{P}^N$ is *ruled by conics* (over $\mathbb{P}^1$) if it is birational to a surface geometrically ruled by conics. Equivalently, if there exists a surjective morphism $\pi: S \longrightarrow \mathbb{P}^1$ and an open subset $U \subseteq \mathbb{P}^1$ such that:

- the fibre $\pi^{-1}(y)$ is a curve of degree 2 and arithmetic genus 0 for every point $y \in \mathbb{P}^1$;
- the fibre $\pi^{-1}(y)$ is smooth for every point $y \in U$;
- π admits a section.

The following classification of the degenerate fibres of a surface ruled by conics is Thm. 2.4 (see also 1.13), [**6**].

Theorem 0.2. *Let $S \subset \mathbb{P}^N$ be a projective surface ruled by conics over a smooth irreducible curve. Then the degenerate fibres of S are of one of the following types (where n is an integer ≥ 3 in the last two statements):*

- *F_1 is the union of two distinct lines and S is smooth along F_1;*
- *$F_2(A)$ is the union of two distinct lines, whose common point is an ordinary double point of S;*
- *$F_2(D)$ is the union of two coincident lines, containing exactly two ordinary double points of S;*
- *$F_n(A)$ is the union of two distinct lines, whose common point is a rational double point of type (A_{n-1});*
- *$F_n(D)$ is the union of two coincident lines, containing exactly one rational double points of S; in particular, this point is of type (A_3), if $n = 3$, and of type (D_n), if $n \geq 4$.*

Since any surface S ruled by conics is birational to a surface S_0, geometrically ruled by conics, then S can be obtained from a suitable S_0 by a finite number of monoidal transformations. In particular, each singular fibre of S (as described in 0.2) arises in this way. Again in [**6**] we have studied this situation, as summarized below.

Let $\mathbb{F}$ and D be as before and $S_0 = \varphi_D(\mathbb{F})$ be a surface geometrically ruled by conics via the morphism $\pi : S_0 \longrightarrow \mathbb{P}^1$. Consider a point $P_1 \in S_0$ and let $f_0 := \pi^{-1}(y)$ be the fibre of S_0 containing P_1. Consider the blow–up $\sigma_{P_1}$of S_0 at P_1 and the corresponding projection on $\mathbb{P}^1$, π_1 say:

$$\begin{array}{cccc} Bl_{P_1}(S_0) := & S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \\ & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ & \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

Denote also by $f_1 := \pi_1^{-1}(y)$ the total transform of f_0 via σ_{P_1}.
Take now $P_2 \in f_1$ and consider the corresponding blow–up $\sigma_{P_2} : \ S_2 \longrightarrow S_1$. With obvious notations, we can iterate this construction and obtain a sequence of blow–ups:

$$\begin{array}{ccccccccc} \widetilde{S}_0 := S_n & \xrightarrow{\sigma_{P_n}} & \cdots & \longrightarrow & S_2 & \xrightarrow{\sigma_{P_2}} & S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \\ \cup & & & & \cup & & \cup & & \cup \\ \widetilde{f}_0 := f_n & & & & f_2 & & P_2 \in f_1 & & P_1 \in f_0 \end{array}$$

where, for any $i = 1, \ldots, n$, we define $P_i \in f_{i-1}$, $f_i := \pi_i^{-1}(y)$ and $\pi_i : S_i := Bl_{P_i}(S_{i-1}) \longrightarrow \mathbb{P}^1$ is the natural projection.

Definition. With the above notation, we say that $f_n = \widetilde{f}_0 \subset \widetilde{S}_0$ is a fibre of *level n over f_0*.

Denoting by σ the sequence of blowing–ups of S_0 defined above, setting $\widetilde{D}$ to be the strict transform of D (very ample bisecant divisor on S_0) via σ and B the base locus of $\widetilde{D}$, then S can be obtained in this way:

$$\begin{array}{ccc} \widetilde{S}_0 & \xrightarrow{\sigma} & S_0 \\ {\scriptstyle \varphi_{\widetilde{D}-B}}\downarrow & \nearrow_{\rho} & \\ S & & \end{array}$$

where ρ is defined as the birational map such that the diagram is commutative.

Definition. We say that the fibre $f \subset S$ is an *embedded fibre of level* n if

$$n = \min_i \{\text{there exists a blow–up } \sigma : \widetilde{S}_0 \to S_0 \text{ and a fibre } f_i \subset \widetilde{S}_0 \text{ of level } i \text{ such that } f = \varphi_{\widetilde{D}-B}(f_i)\}.$$

Again in [**6**], we noted that each fibre $f \subset S$ of type $F_n(A)$ or $F_n(D)$ is an embedded fibre of level n. There we also gave the following:

Definition. Let $f^{(1)}, \ldots, f^{(p)}$ be the degenerate fibres of S and let l_i be the level of $f^{(i)}$, for $i = 1, \ldots, p$. If $\sum_{i=1}^{p} l_i = L$, we say that S is of *level* L.

Moreover, we proved that all the surfaces geometrically ruled by conics (briefly g.r.c.) and giving rise – by a minimal number of elementary transformations – to a surface S ruled by conics of level L, are exactly the elements of the following set:

$$\begin{aligned}\mathbf{GRC}_L(S) :=\{S_0 \mid\ & S_0 \text{ is a g.r.c. surface and } S \text{ can be obtained from it}\\ & \text{by a sequence of } L \text{ blow–ups and contractions}\}.\end{aligned}$$

1. The surface S of minimum degree, ruled by conics and containing X_K

Starting from the situation $X_K \subset V \subset \mathbb{P}^{g-1}$, described at the beginning of the previous section, we will try to "canonically" define a surface (ruled by conics) containing X_K and contained in V.

Notation. As usual, if n is a rational number, $[n]$ denotes the greatest integer smaller or equal than n, while $\lceil n \rceil$ denotes the smallest integer bigger or equal than n.

Theorem 1.1. *There exists a surface S ruled by conics such that $X_K \subset S \subset V$ and* $\deg(S) \leq \left\lceil \frac{3g-8}{2} \right\rceil$. *Moreover, S is unique unless* $\deg(S) = \frac{3g-7}{2}$*; in this case, S varies in a pencil.*

Proof. Let us consider the vector space $\mathcal{H} := H^0(\mathbb{P}^{g-1}, \mathcal{I}_{X_K}(2))/H^0(\mathbb{P}^{g-1}, \mathcal{I}_V(2))$ and set $N := \dim(\mathcal{H})$; clearly, $\Sigma := \mathbb{P}(\mathcal{H})$ parametrizes the hyperquadrics of $\mathbb{P}^{g-1}$ containing X_K but not containing V.
Let us recall that, if W is a projectively normal subvariety of $\mathbb{P}^{g-1}$, then we get the cohomology exact sequence (see Section 0)

$$0 \longrightarrow H^0(\mathcal{I}_W(2)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) \longrightarrow H^0(\mathcal{O}_W(2)) \longrightarrow 0$$

hence $h^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) = h^0(\mathcal{I}_W(2)) + h^0(\mathcal{O}_W(2))$. Rewriting this equality for both X_K and V, we get $h^0(\mathcal{I}_{X_K}(2)) + h^0(\mathcal{O}_{X_K}(2)) = h^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) = h^0(\mathcal{I}_V(2)) + h^0(\mathcal{O}_V(2))$, so

$$N = h^0(\mathcal{I}_{X_K}(2)) - h^0(\mathcal{I}_V(2)) = h^0(\mathcal{O}_V(2)) - h^0(\mathcal{O}_{X_K}(2)).$$

In order to compute N, recall the relations (1) and (RR) on the scroll $V = \mathbb{P}(\mathcal{F})$:

$$h^0(V, \mathcal{O}_V(2)) = h^0(\mathbb{P}^1, \mathrm{Sym}^2(\mathcal{F})) = \deg(\mathrm{Sym}^2(\mathcal{F})) + \mathrm{rk}(\mathrm{Sym}^2(\mathcal{F})).$$

Clearly, $\mathrm{Sym}^2(\mathcal{F})$ is a free bundle of degree $4(a+b+c)$ and rank 6; therefore, from (2) we get: $h^0(\mathcal{O}_V(2)) = 4g-6$.
On the other hand, by the Riemann–Roch Theorem $h^0(\mathcal{O}_{X_K}(2)) = 3(g-1)$. Hence the above space Σ of hyperquadrics is a projective space of dimension

$$N - 1 = h^0(\mathcal{O}_V(2)) - h^0(\mathcal{O}_{X_K}(2)) - 1 = g - 4.$$

For each $Q \in \Sigma \cong \mathbb{P}^{g-4}$, consider the scheme–theoretic intersection

$$Q \cdot V = \Big(\bigcup_{i=1,\ldots,h_Q} F_i \Big) \cup S_Q$$

where the F_i's are the fibres of V entirely contained in Q, $h_Q \geq 0$ and S_Q is a surface, which is ruled in conics (since Q intersects the general fibre F of V in a conic passing through the four points of the divisor $\Phi \subset F$) and contains X_K.
Note that S_Q is irreducible; if not $S_Q = S_1 \cup S_2$, where the S_i's were ruled surfaces; but $X_K \subset S_Q$ and it cannot be contained in a ruled surface since each 4–gonal divisor spans a plane.
In order to find a quadric $\overline{Q} \in \Sigma$ such that $\deg(S_{\overline{Q}})$ is minimum, it is enough to require that the number $h_{\overline{Q}}$ is maximum. Note that a fibre F is contained in a quadric $Q \in \Sigma$ if Q contains two points, say P_1 and P_2,

belonging to F and such that the 0-cycle of V of degree 6 given by $\Phi + P_1 + P_2$ does not lie on a conic.
Since $\dim(\Sigma) = g-4$, we can impose that the space Σ contains $\left[\frac{g-4}{2}\right]$ pairs of points. If each such a pair of points belongs to the same fibre (and satisfies the above conditions), then we can find a $\overline{Q} \in \Sigma$ containing $\left[\frac{g-4}{2}\right]$ fibres.
Clearly $\overline{Q}$ could contain further fibres, hence

$$\deg(S_{\overline{Q}}) \leq \deg(\overline{Q} \cap V) - \left[\frac{g-4}{2}\right] \leq 2(g-3) - \left[\frac{g-4}{2}\right] = \left[\frac{3g-8}{2}\right].$$

This proves the existence of the required surface $S := S_{\overline{Q}}$.
Concerning the uniqueness, let us assume that there are two such surfaces, say S_1 and S_2.
Since $X_K \subset (S_1 \cap S_2)$, from (IF) we get:

$$2g-2 = \deg(X_K) \leq \int (S_1 \cdot S_2) = 2\deg(S_1) + 2\deg(S_2) - 4\deg(V).$$

This relation is verified if and only if $\deg(S_1) = \deg(S_2) = (3g-7)/2$. To complete the proof, just observe that the linear system of the quadrics $\overline{Q} \in \Sigma$ containing $\left[\frac{g-4}{2}\right]$ fibres has dimension

$$\dim \Sigma - 2\left[\frac{g-4}{2}\right] = g - 4 - 2\left(\frac{g-5}{2}\right) = 1$$

therefore there is a pencil of distinct surfaces $S_{\overline{Q}}$. ⋄

The existence of such surface S has been proved, using a different method, also by Schreyer in [**12**], Sect.6.

Notation. From now on, f will denote the general fibre of S, so f is a conic lying on a plane $F = \langle f \rangle$. Moreover, if T is a surface ruled by conics, we will denote by V_T the scroll whose fibres are the planes spanned by these conics. For example, if S is the surface defined in 1.1, the scroll V_S is exactly V.

Remark 1.2. The fibres of the ruled surface S defined in 1.1 cannot be all singular. Otherwise, from 1.2, [**5**], the surface S would be ruled by lines on a hyperelliptic curve, Y say, via $\alpha: S \to Y$ and the ruling $\pi: S \to \mathbb{P}^1$ would factorize through α.
Hence, taking into account that the restriction $X_K \to Y$ of α has degree two, we obtain that X_K is bi-hyperelliptic, contrary to the assumption made before on X.

Remark 1.3. The surface S introduced in 1.1 is then ruled by conics in the sense of the preliminary Section.

2. Birational models of $X_K \subset S$

In this section we shall study a surface S (not necessarily of minimum degree as that one defined in 1.1) such that S is ruled by conics and $X_K \subset S \subset V$, where V denotes as usual the 3–dimensional scroll spanned by the four–gonal divisors on X_K.
Note that, since X_K is linearly normal, then $S \subset \mathbb{P}^{g-1}$ is linearly normal. Moreover the scroll $V = V_S$ is not a cone (see the forthcoming Corollary 7.9), then 0.2 holds, so the classification of the degenerate fibres of the surface S is the one described there.
In Section 0 we have also summarized the results (contained in [**6**]) which allow us to associate to a surface S, ruled by conics and of a certain level L, the set $\mathbf{GRC}_L(S)$ consisting of all the g.r.c. surfaces linked to S via a sequence of L monoidal transformations.
Here we are looking for the inverse procedure: how to recover the surface S (and the curve X_K) starting from a g.r.c. surface $S_0 \in \mathbf{GRC}_L(S)$.

Notation. Since each surface $S_0 \in \mathbf{GRC}_L(S)$ is geometrically ruled by conics, it admits an invariant $\tau_0 := t(S_0)$, in the sense that $S_0 \cong \mathbb{F}_{\tau_0}$. We denote by $X_{\tau_0} \subset \mathbb{F}_{\tau_0} \cong S_0$ the corresponding model of $X_K \subset S$.
Since $X_{\tau_0} \subset \mathbb{F}_{\tau_0}$ is a four–secant curve, then

$$X_{\tau_0} \sim 4C_0 + (\lambda_0 + \tau_0)f \tag{4}$$

where C_0 and f are the generators of $\mathrm{Num}(\mathbb{F}_{\tau_0})$ (see 0.1) and λ_0 is a suitable integer. Moreover, denoting by $p_a(C)$ the arithmetic genus of a curve C, we set

$$\delta_{\tau_0} := p_a(X_{\tau_0}) - g.$$

Note that, if all the singularities of X_{τ_0} are ordinary double points, then $\delta_{\tau_0} = \deg(Sing(X_{\tau_0}))$.

Remark 2.1. Let us recall the Adjunction Formula for the dualizing sheaf ω_{X_R} of a curve X_R on a smooth surface R (see [**7**], Ch.1, (1.5))

$$\omega_{X_R} = \mathcal{K}_R \otimes \mathcal{O}_R(X_R)_{|X_R} \tag{5}$$

where $\mathcal{K}_R = \mathcal{O}_R(K_R)$ denotes the canonical sheaf of R. Taking the degrees we then obtain:

$$2p_a(X_R) - 2 = X_R \cdot (X_R + K_R). \tag{6}$$

In our situation $R = \mathbb{F}_{\tau_0}$ and $X_R = X_{\tau_0}$. Then $\mathcal{K}_{\mathbb{F}_{\tau_0}} = \mathcal{O}_{\mathbb{F}_{\tau_0}}(-2C_0 - (\tau_0 + 2)f)$, so using (4) we obtain

$$\mathcal{K}_{\mathbb{F}_{\tau_0}} \otimes \mathcal{O}_{\mathbb{F}_{\tau_0}}(X_{\tau_0}) = \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f).$$

Hence from (5) we can obtain the dualizing sheaf of the curve X_{τ_0} as:

$$\omega_{X_{\tau_0}} = \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)_{|X_{\tau_0}}.$$

Finally, taking into account that $K_{\mathbb{F}_{\tau_0}} \sim -2C_0 - (\tau_0 + 2)f$, from (6) and (4) we obtain

$$2p_a(X_{\tau_0}) - 2 = 6\lambda_0 - 6\tau_0 - 8.$$

Proposition 2.2. *The following properties hold:*

i) the arithmetic genus of X_{τ_0} is $p_a(X_{\tau_0}) = 3(\lambda_0 - \tau_0 - 1)$;
ii) $\lambda_0 \geq \max\{3\tau_0, \tau_0 + 5\}$;
iii) $\delta_{\tau_0} = 3(\lambda_0 - \tau_0 - 1) - g$.

Proof. *i)* Immediate from the last relation of 2.1.
ii) From [**11**], Ch. V, 2.18, since X_{τ_0} is irreducible, then $\lambda_0 + \tau_0 \geq 4\tau_0$. Therefore $\lambda_0 \geq 3\tau_0$. On the other hand, $p_a(X_{\tau_0}) \geq g \geq 10$ by assumption. Then, using (i), we obtain $\lambda_0 \geq \tau_0 + 5$.
iii) It follows from $\delta_{\tau_0} = p_a(X_{\tau_0}) - g$ and from (i). $\diamond$

We wish to describe how to recover the canonical model X_K starting from the chosen birational model $X_{\tau_0} \subset \mathbb{F}_{\tau_0} \cong S_0 \in \mathbf{GRC}_L(S)$.
Since X_0 is the embedded model of X_{τ_0} obtained via the dualizing sheaf $\omega_{X_{\tau_0}}$ (described before), then, in order to obtain X_0, we have to embed $\mathbb{F}_{\tau_0}$ by the sheaf $\mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)$ (see 2.1). Finally, we will project the obtained curve X_0 from its singular points.

Remark 2.3. Note first that $\lambda_0 - 2 > 2\tau_0$. In fact, if $\tau_0 \leq 2$ then $\lambda_0 > \tau_0 + 4 \geq 2\tau_0 + 2$. If $\tau_0 \geq 3$, then $\lambda_0 \geq 3\tau_0 > 2\tau_0 + 2$ (both arguments follow from 2.2, (ii)).
Therefore (using [**11**], Ch. V, 2.18) the linear system $|2C_0 + (\lambda_0 - 2)f|$ is very ample on $\mathbb{F}_{\tau_0}$. Moreover, from [**4**], Prop.1.8, and from 2.2, (iii) we get that

$$h^0\left(\mathbb{F}_{\tau_0}, \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)\right) = g + \delta_{\tau_0}.$$

Hence there is an isomorphism

$$\varphi : \mathbb{F}_{\tau_0} \xrightarrow{\cong} S_0 \subset \mathbb{P}^{g-1+\delta_{\tau_0}}, \quad \text{where} \quad \varphi = \varphi_{2C_0 + (\lambda_0 - 2)f} \quad \text{and} \quad S_0 := \varphi(\mathbb{F}_{\tau_0}).$$

Clearly S_0 is a projective ruled surface, whose fibers are all smooth conics and $X_0 = \varphi(X_{\tau_0}) \subset S_0$, so we have the commutative diagrams:

$$\begin{tikzcd} \mathbb{F}_{\tau_0} \arrow[r, "\varphi", "\cong"'] & S_0 \arrow[d, shift left, "\pi"] & \subset & \mathbb{P}^{g-1+\delta_{\tau_0}} \arrow[d, "\pi"] \\ & S \arrow[u, shift left, "\rho"] & \subset & \mathbb{P}^{g-1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_{\tau_0} \arrow[r, "\varphi_{|X_{\tau_0}}", "\cong"'] & X_0 \arrow[d, shift left, "\pi"] & \subset & S_0 \arrow[d, "\pi"] \\ & X_K \arrow[u, shift left, "\rho"] & \subset & S \end{tikzcd}$$

where π (which is the inverse of the map ρ) is exactly the desingularization morphism of X_0 or, equivalently, the linear projection centered in $\langle\Sigma\rangle$ is generated by the singular points of X_0 (possibly infinitely near).

Remark 2.4. Since there are at most two singular points on each fibre, then $\langle\Sigma\rangle$ meets S_0 in a zero–dimensional variety of degree δ_{τ_0}. It is then clear that $\delta_{\tau_0} = L$ and $\deg(S) = \deg(S_0) - \delta_{\tau_0}$.

3. Singularities of a birational model X_0

The purpose of this section is to describe all the possible singularities of X_0.
Recall that, from 2.3, the projection $\pi : X_0 \subset S_0 \longrightarrow X_K \subset S$ is centered in the singular points of X_0 and the singular fibres of S correspond to the fibres of S_0 containing the singular points of X_0. Therefore it is enough to examine the singular fibres of S and the four–gonal divisor on each of them.
In order to do this, let us focus on one singular fibre f of S and the corresponding fibre $f_0 \subset S_0$.

Remark 3.1. Note that the curve $X_K \subset S$ intersects each fibre of S in four points (the 4–gonal divisor $\Phi \in g^1_4$). In particular, X_K meets also each singular fibre f in four points. If $f = l \cup m$ and $l \neq m$ then two of them belong to the line l and two are on the other line m (possibly coinciding); where this not the case, X_K would have a trisecant line, hence a trigonal series (from the Geometric Riemann–Roch Theorem). On the other hand, if $l = m$, then the support of $\Phi = X_K \cap f$ consists of two points, possibly coinciding.

Example 3.2. Let $f \subset S$ be an embedded fibre of level 1. Then π is the projection centered at the point $P_0 \in f_0$, where $P_0 \in Sing(X_0)$. Clearly, $f = f_0 + E$, where E is the exceptional divisor and f_0 still denotes the other component of f. Setting $A := f_0 \cdot E$, $P_i \in f_0$ and $Q_i \in E$ (where $P_i \neq A \neq Q_i$ and $P_i \neq Q_i$, for $i = 1, 2$), the possible cases are the following:

$$\begin{array}{ll}
(a) & \Phi = P_1 + P_2 + Q_1 + Q_2 \\
(b) & \Phi = P_1 + P_2 + 2Q_1 \\
(c) & \Phi = 2P_1 + Q_1 + Q_2 \\
(d) & \Phi = 2P_1 + 2Q_1 \\
(e) & \Phi = P_1 + 2A + Q_1 \\
(f) & \Phi = P_1 + 3A \quad (\text{where } X_K \cdot f_0 = P_1 + A \text{ and } X_K \cdot E = 2A) \\
(g) & \Phi = 3A + Q_1 \quad (\text{where } X_K \cdot f_0 = 2A \text{ and } X_K \cdot E = A + Q_1).
\end{array}$$

The picture below illustrates the corresponding singularities of X_0.

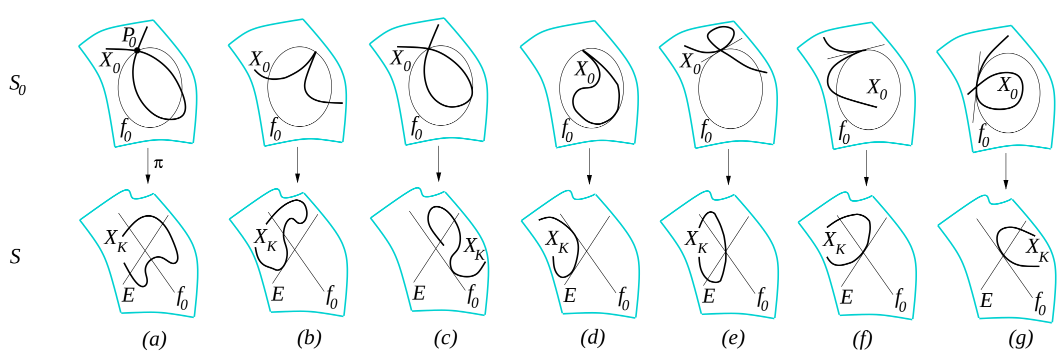

Figure 1

It is clear that, in all the cases above, X_0 has a double point: more precisely, either a node, in cases $(a), (c), (e), (g)$, or an ordinary cusp, in cases $(b), (d), (f)$.

A description of the double points of an algebraic curve can be found, for instance, in [**10**], Lect. 20.
Here let us just recall that a *node of n-th kind* is a double point analitically equivalent to $y^2 - x^{2n} = 0$. In particular, if $n = 1, 2, 3$, it is called *(ordinary) node*, *tacnode*, *oscnode*, respectively.
Moreover, a *cusp of n-th kind* is a double point analitically equivalent to $y^2 - x^{2n+1} = 0$. In particular, if $n = 1, 2$, it is called *(ordinary) cusp* or *ramphoid cusp*, respectively.

Definition. We say for short that a double point P_0 of X_0 is *transversal* if the tangent line to the fibre f_0 at P_0 does not coincide with any of the tangent lines to X_0 at P_0; it is *tangent* otherwise.

Example 3.3. Assume that S is a surface ruled by conics having a fibre f of type $(2A)$, as defined in 0.2. Clearly (see [**6**], Sect. 3) this fibre arises from a fibre $f_0 \subset S_0$ by projecting it from two points. More precisely, the projection $\pi : S_0 \longrightarrow S$ can be factorized by $\pi = \pi_{P_1} \circ \pi_{P_0}$, where $P_0 \in f_0$ and $P_1 \in f_1 := f_0 + E \subset \pi_{P_0}(S_0)$ and $P_1 \neq f_0 \cdot E$. There are two possibilities: either $P_1 \in f_0$ or $P_1 \in E$.
In the first case, $f = E + E^{(1)}$, while in the second one, where P_1 is infinitely near to P_0, we have $f = f_0 + E^{(1)}$ (in both cases $E^{(1)}$ denotes the exceptional divisor of the blowing–up centered at P_1). Moreover, in both configurations, f turns out to be a union of two lines meeting in an ordinary double point for the surface S.
Let us start by scketching the situations corresponding to the configuration (a) (in both cases $f = E + E^{(1)}$ and $f = f_0 + E^{(1)}$) and the configurations (b) and (d) (both in the case $f = f_0 + E^{(1)}$).

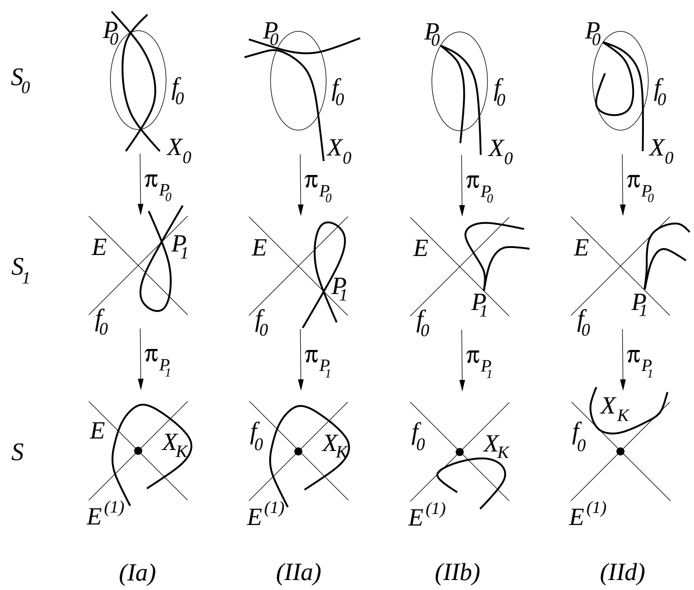

Figure 2

The construction (Ia) gives X_0 to have two nodes on the fibre f_0; in (IIa) the curve X_0 has a tacnode, while in (IIb) and (IId) it has a ramphoid cusp. Finally, one can easily see that the cases related to $(e), (f), (g)$ do not occur.

Remark 3.4. The two examples above lead us to a general pattern. If X_0 has only one singular point $P_0 \in f_0$ and f is of type (nA), then:

- $f = f_0 + E^{(n-1)}$ and π can be factorized by $\pi = \pi_{P_{n-1}} \circ \cdots \circ \pi_{P_1} \circ \pi_{P_0}$, where $P_{i+1} \in E^{(i)}$ for all i;
- the type of the singularity of P_0 depends only on the intersection $X_K \cdot E^{(n-1)}$ on S, so we can always assume that the two points given by $X_K \cdot f_0$ on S are distinct.

We can now complete 3.3: if X_0 has one singular point on f_0, then the significant cases are (IIa) and (IIb), where X_0 has a transversal tacnode or a transversal ramphoid cusp. Note that the difference between these two cases is that X_K is tangent (resp. transversal) to $E^{(1)}$ on S.

Example 3.5. In the same way, we get the possible singularities in the case $F_3(A)$, as this picture shows:

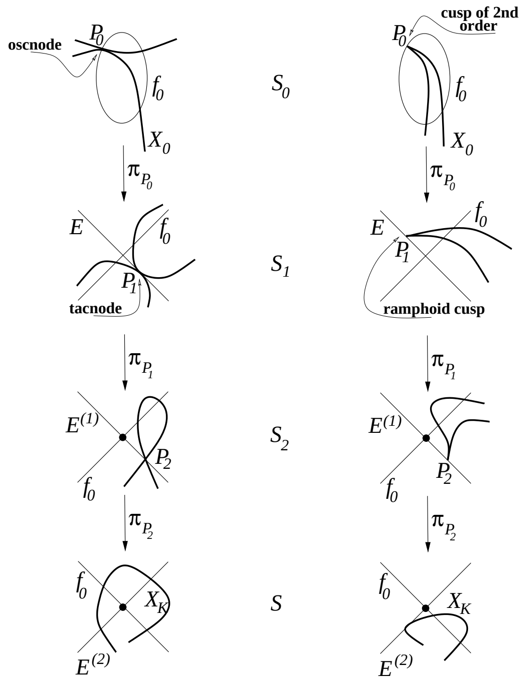

Figure 3

The above study can be easily generalized, obtaining the following result:

Proposition 3.6. *The possible singularities of $X_0 \subset S_0$ arising from a fibre of S of type $F_n(A)$, where $n \geq 2$, are the following points on the same fibre $f_0 \subset S_0$:*

- ($\bullet$) *if $n = 2$ there is either one double point of second kind (either a transversal tacnode or a transversal ramphoid cusp) or two double points of first kind (either node or cusp);*
- ($\bullet$) *if $n \geq 3$ there is either one double point of $n-$th kind (transv.) or two double points of lower kind.* $\diamond$

Note that in the case of two double points on f_0, these two points are of kind h and k, where $h + k = n$.

Example 3.7. Assume now that S is a surface ruled by conics having a fibre f of type $(2D)$. Clearly (see [**6**], Sect. 3) this fibre arises from a fibre $f_0 \subset S_0$ by projecting it from two infinitely near points. More precisely, if $\pi : S_0 \longrightarrow S$ is the considered projection, then $\pi = \pi_{P_1} \circ \pi_{P_0}$, where $P_0 \in f_0$ and, if $f_1 := f_0 + E \subset \pi_{P_0}(S_0)$, then $P_1 := f_0 \cdot E$. As noted in [**6**], the fibre of S corresponding to f_0 is given by $f = 2E^{(2)}$: it is a totally degenerate conic containing two singular points of S, which correspond to the lines f_0 and E. Since f consists of a double line, the four–gonal divisor can be either $2A + 2B$ (where $A, B \in E^{(2)}$ are distinct points non singular for S) or $4A$, as the following picture describes:

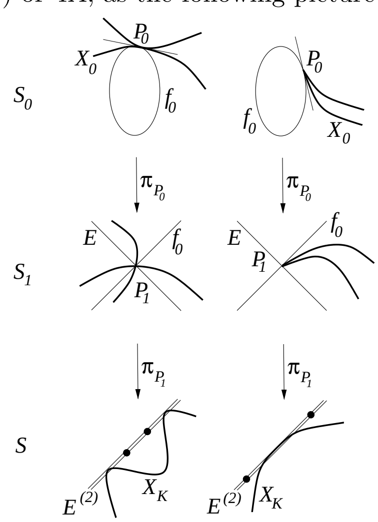

Figure 4

It is clear that the first configuration leads to a tangential tacnode and the second one gives a tangential ramphoid cusp of first order. With the same argument as before, we easily get the following result:

Proposition 3.8. *The possible singularities of $X_0 \subset S_0$ arising from a fibre of S of type $F_n(D)$, where $n \geq 2$, consist of a unique singular point of the corresponding fibre $f_0 \subset S_0$ as follows:*
($\bullet$) *if $n = 2$ then there is either a tangential tacnode or a tangential ramphoid cusp;*
($\bullet$) *if $n \geq 3$ then there is a tangential double point of n-th kind.*

Collecting 3.2, 3.6, 3.8, we obtain the following complete description of the possible singularities of X_0.

Theorem 3.9. *Let S be a surface ruled by conics containing X_K and let $X_0 \subset S_0$ be birational models of X_K and S respectively, where S_0 is a g.r.c. surface. Let $\pi : S_0 \longrightarrow S$ be the usual projection. Assume that f is the unique singular fibre of S and set f_0 the corresponding fibre of S_0.*
Then the singular points of X_0 belong to f_0 and are, as far as f is of type F_1, of one of the following types, $F_n(A)$, $F_n(D)$, for $n \geq 2$:

F_1 - one singular point: either a node or a cusp, both of them either tangential or transversal;

$F_n(A)$ - only transversal singular points and precisely:

(a) one double point of n-th kind;

(b) two double points of orders $h, k < n$, where $h + k = n$;

$F_n(D)$ - only one tangential double point of n-th kind;

In particular, all the singular points of X_0 are double points. ◇

4. "Standard" birational models of $X_K \subset S$

In Section 2 we studied the set $\mathbf{GRC}_L(S)$ consisting of the g.r.c. surfaces S_0 such that S can be obtained from S_0 by a sequence of L monoidal transformations (here L is the level of S). In this section we are going to determine one of such surfaces in a sort of "canonical" way: this will be called "standard" birational model of S.

Proposition 4.1. *Let $X_0 \subset S_0 \in \mathbf{GRC}_L(S)$ be as usual. Then*

$$\mathbf{GRC}_L(S) = \{elm_\Sigma(S_0) \mid \Sigma \subseteq Sing(X_0)\}$$

i.e. each $S'_0 \in \mathbf{GRC}_L(S)$ can be obtained from S_0 by a sequence of elementary transformations centered in singular points of X_0 (or infinitely near to them) and conversely.

<u>Proof</u>. Consider a surface $S'_0 \in \mathbf{GRC}_L(S)$ and the corresponding model of X_K, say $X'_0 \subset S'_0$. As in 2.2, denote by π and π' the projections centered in the singular points (possibly infinitely near) of X_0 and X'_0, respectively. We get then the diagram

$$\begin{array}{ccc} S_0 & \dashrightarrow & S'_0 \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & S & \end{array}$$

where the horizontal arrow denotes a suitable sequence of elementary transformations centered in (some of) the singular points of X_0.
Conversely, note that each elementary transformation of S_0 can be obtained by considering an embedded model of S_0 which is ruled by lines and projecting it from a finite number of points. In this way, we get a birational model S'_0 of S which is a geometrically ruled surface. If $X'_0 \subset S'_0$ is the corresponding curve, it is clear that $\delta(X'_0) = \delta(X_0)$ if and only if the above projection is centered in singular points of X_0 (this is due to the fact that the singular points of X_0 are double points for 3.9). Therefore, if $S'_0 = elm_\Sigma(S_0)$, where $\Sigma \subseteq Sing(X_0)$, using 2.4, the level of S'_0 coincides with $\delta(X'_0) = \delta(X_0) = L$, hence $S'_0 \in \mathbf{GRC}_L(S)$, as requested. ◇

Among the surfaces S_0 geometrically ruled by conics belonging to $\mathbf{GRC}_L(S)$ (and the corresponding curves X_0), we are going to establish a way for choosing one particular model of S (and hence of X_K). In order to do this, we give the following notion.

Definition. Given a surface S ruled by conics, we say that a surface $\overline{S}_0 \in \mathbf{GRC}_L(S)$ is a *standard model* of S if its invariant is

$$t := \min\{\tau_0 = t(S_0) \mid S_0 \in \mathbf{GRC}_L(S)\}.$$

Let us consider now the curve $X_K \subset S$ and the corresponding birational model, say $\overline{X}_0 := \rho(X_K) \subset \overline{S}_0$, where $\overline{S}_0$ is a standard model of S. We say also that $\overline{X}_0$ is a *standard model* of X_K.
Finally, if $\overline{S}_0$ is a standard model of S, we denote the corresponding invariant λ_0 by λ.

Theorem 4.2. *Let S be as before, L be its level, $S_0 \in \mathbf{GRC}_L(S)$ be a birational model of S of invariant τ_0 and X_0 be the model of X_K on S_0. If we assume that $t > 0$, then the following facts hold:*

i) if S_0 is a standard model, then the singular points of X_0 belong to the minimum unisecant C_0 of S_0;
ii) there is exactly one standard model $\overline{S}_0$ of S;
iii) if the singular points of X_0 belong to the minimum unisecant C_0 of S_0, then $S_0 = \overline{S}_0$.

Proof. Consider first the model $X' \subset R_{1,\tau_0+1} \cong S_0$. We know that $X' \sim 4C_0 + (\lambda_0 + \tau_0)f$ and $\delta(X') = 3(\lambda_0 - \tau_0 - 1) - g$ by 2.2. In particular, the level of S is $L = 3(\lambda_0 - \tau_0 - 1) - g$.
Consider a singular point T of X' and the projection π_T from T. From 4.1, $\pi_T(R_{1,\tau_0+1})$ belongs to $\mathbf{GRC}_L(S)$.
(i) If S_0 is a standard model, then $\tau_0 = t$. Assume that the point T does not belong to C_0. Then the invariant of $\pi_T(R_{1,t+1})$ is $t-1$, while t is the minimum invariant of the surfaces belonging to $\mathbf{GRC}_L(S)$.
(ii) Let $\overline{S}_0 \cong R_{1,t+1}$ be a standard model and let S_0' be another surface in $\mathbf{GRC}_L(S)$. From 4.1, we know that $S_0' = elm_\Sigma(\overline{S}_0)$, where $\Sigma \subseteq Sing(\overline{X}_0)$. For simplicity, assume that $\Sigma = \{T\}$, where T is a singular point of $\overline{X}_0$. From (i), we have that $T \in C_0$ and, from 3.9, we know that T is a double point of $\overline{X}_0$, so $T = A_1 + A_2$, where $\Phi := A_1 + A_2 + A_3 + A_4$ is the four–gonal divisor on the fibre $\overline{f}_0$ containing T.
Clearly, $S_0' = \pi_T(R_{1,t+1})$, so the curve X_0' has a double point on the fibre $\overline{f}_0'$ given by $A_3 + A_4$ and such point does not belong to the unisecant curve C_0' of S_0'. Therefore we get from (i) that S_0' is not a standard model of S.
(iii) An analogous argument. ◇

Proposition 4.3. *With the above notation, if $t > 0$ then the singular points of $\overline{X}_0$ belong to distinct fibres.*

Proof. Also in this case consider the model $X' \subset R_{1,t+1} \cong \overline{S}_0$ and assume that there exists a fibre containing two distinct singular points of X', P_1 and P_2, say . Clearly, one of them, P_1 say, does not belong to C_0. So, by projecting $R_{1,t+1}$ from P_1 we get a contraddiction with the argument used in 4.2. ◇

Theorem 4.4. *With the notation above, the surface S has degree*

$$\deg(S) = 4(\lambda - t - 2) - \delta_t = g + \lambda - t - 5.$$

Proof. Since $\overline{S}_0 = \varphi_{2C_0+(\lambda-2)f}(\mathbb{F}_t)$ and $C_0^2 = -t$, then

$$\deg(\overline{S}_0) = (2C_0 + (\lambda-2)f)^2 = 4(\lambda - t - 2).$$

Moreover, from 2.4 we have that $\deg(S) = \deg(\overline{S}_0) - \delta_t$, so the first equality holds. The second equality follows immediately from $\delta_t = 3(\lambda - t - 1) - g$ (see 2.2, (iii)). ◇

5. Bounds on the invariants λ and t

Let us come back to the global description of the four–gonal curve X of genus g whose canonical model is $X_K \subset S \subset V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \subset \mathbb{P}^{g-1}$ and the surface S is (as in 1.1) the surface of minimum degree.
We have chosen $\overline{X}_0 \subset \overline{S}_0 \cong \mathbb{F}_t$ as a pair of standard models of $X_K \subset S$ respectively. Since the model $X_t \subset \mathbb{F}_t$ is again a four–secant curve, it is of the type $X_t \sim 4C_0 + (\lambda + t)f$.

So far we have defined a set of integers, $a, b, c, t, \delta, \lambda$ (here, for simplicity, $\delta := \delta_t$), that are *invariants* of the curve X. All of them will be useful to describe its geometry.

Let us start with the dependence of the first three invariants a, b, c on the others t, δ, λ.

Remark 5.1. Consider the isomorphism

$$\varphi_{2C_0+(\lambda-2)f} : \mathbb{F}_t \longrightarrow \overline{S}_0 \subset \mathbb{P}^{g-1+\delta}$$

and the volume $V_{\overline{S}_0} \subset \mathbb{P}^{g-1+\delta}$ generated by $\overline{S}_0$. From 1.8, [**4**], we have that

$$V_{\overline{S}_0} = \mathbb{P}(\mathcal{O}(\lambda - 2 - 2t) \oplus \mathcal{O}(\lambda - 2 - t) \oplus \mathcal{O}(\lambda - 2)).$$

If we consider the projection $\pi : \ \mathbb{P}^{g-1+\delta} \to \mathbb{P}^{g-1}$ centered at the singular locus of $\overline{X}_0$, it is clear that $\pi(V_{\overline{S}_0}) = V_S$.
Using 4.2 (i), if $t > 0$ then the singular points of $\overline{X}_0$ are contained in the unisecant of minimum degree of $\overline{S}_0$ and hence of $V_{\overline{S}_0}$. Moreover, if these points are all distinct, then V_S has the form:

$$V_S = \mathbb{P}(\mathcal{O}(\lambda - 2 - 2t - \delta) \oplus \mathcal{O}(\lambda - 2 - t) \oplus \mathcal{O}(\lambda - 2)).$$

On the other hand, taking into account that $c = g - 3 - a - b$, the scroll above is:

$$V_S = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(g - 3 - a - b)).$$

Hence, comparing the two expressions of V_S and using the equality $\delta = 3(\lambda - t - 1) - g$ (see 2.2 (iii)), we obtain:

$$a = g + t - 2\lambda + 1 \quad \text{and} \quad b = \lambda - t - 2.$$

Note that, if $t > 0$ but the δ double points of $\overline{X}_0$ are not all distinct, then $a \geq g + t - 2\lambda + 1$.

Proposition 5.2. *With the above notation, if $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, then*

$$a + b \geq \frac{g-5}{2}.$$

Proof. Let us consider the curve $X_K \subset V$ and the ruled surface $R_{a,b} = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \subset V$. In order to apply the Intersection Formula (IF) in Section 0, we observe first that $R_{a,b}$ and X_K meet properly on V, i.e.

$$\dim(R_{a,b} \cap X_K) = \dim(R_{a,b}) + \dim(X_K) - \dim(V) = 0.$$

To see this note that X_K cannot be contained in $R_{a,b}$, otherwise the general 4–gonal divisor on X_K would span a line instead of a plane, against the Geometric Riemann–Roch Theorem.
Hence $\dim(R_{a,b} \cap X_K) = 0$ and we can apply (IF), which gives the (non–negative) degree of the intersection:

$$0 \leq \deg_V(R_{a,b} \cdot X_K) = 4(a + b) + 2g - 2 - 4(g - 3) = 2(a + b) - g + 5$$

and this proves the requested inequality. $\diamond$

The lower bound of λ in terms of t given in the previous section can be improved. Namely, we saw that $\lambda \geq \max\{3t, t + 5\}$ (see 2.2).

Remark 5.3. Assume that $t \geq 1$ and the δ singular points of X_t are distinct. Clearly

$$2\delta \leq \int C_0 \cdot X_t = \int C_0 \cdot (4C_0 + (\lambda + t)f) = \lambda - 3t$$

hence

$$\lambda \geq 2\delta + 3t.$$

Since $\delta = 3(\lambda - t - 1) - g$ (see 2.2 (iii)), we easily obtain:

$$\lambda \leq \frac{2g + 3t + 6}{5}. \tag{7}$$

Proposition 5.4. *The following properties hold :*
(i) *for any* t:

$$\lambda \geq \frac{g}{3} + t + 1;$$

(ii) *if* $t = 0$ *then*

$$\lambda \leq \frac{g+3}{2};$$

(iii) *if* $t \geq 1$ *then*

$$\lambda \leq t + \frac{g+3}{2} \quad \text{and} \quad t \leq \frac{g+3}{4};$$

(iv) *if* $t \geq 1$ *and the double points of* X *are all distinct, then*

$$\lambda \leq \frac{g+3}{2} \quad \text{and} \quad t \leq \frac{g+3}{6}.$$

Proof. (i) It comes from 2.2 (i), since $p_a(\overline{X}_0) = 3(\lambda - t - 1) \geq g$.
(ii) $-$ (iii) Using 1.1 and 4.4 we have

$$g + \lambda - t - 5 = \deg(S) \leq \left\lceil \frac{3g-8}{2} \right\rceil \quad \Rightarrow \quad \lambda - t \leq \left\lceil \frac{3g-8}{2} \right\rceil - g + 5 = \left\lceil \frac{g+2}{2} \right\rceil$$

hence, we obtain the required bounds either if $t = 0$ or if $t \geq 1$. Moreover, from 2.2 we have $\lambda \geq 3t$; so, using the previous bound of λ in (iii), we finally get $t \leq \lambda/3 \leq t/3 + \frac{g+3}{6}$ and this concludes the proof.
(iv) In this case, we can apply 5.3. Using $3(\lambda - t - 1) - g = \delta \geq 0$ followed by (7), we get:

$$t \leq \lambda - \frac{g+3}{3} \leq \frac{2g+3t+6}{5} - \frac{g+3}{3} \quad \Rightarrow \quad t \leq \frac{g+3}{6}.$$

Using this bound and (7) we finally get $\lambda \leq \dfrac{g+3}{2}$. $\diamond$

6. Geometric meaning of the invariant λ

Let us keep the notation of the previous section: S is a surface ruled by conics such that $X_K \subset S \subset V$ and L denotes its level. Take a standard model $\overline{S}_0 \in \mathbf{GRC}_L(S)$ and consider its embedded model $R_{1,t+1} \subset \mathbb{P}^{t+3}$. Let us denote as usual by $X' \subset R_{1,t+1}$ the corresponding model of X_K, where $X' \sim 4C_0 + (\lambda + t)f$.

Remark 6.1. Note that such X' has only double points as singularities (see 3.9).

Remark 6.2. Denote by $H_{X'}$ the hyperplane section of $X' \subset R := R_{1,t+1} \subset \mathbb{P}^{t+3}$.
Since $H_R \sim C_0 + (t+1)f$ then

$$H_{X'} = H_R \cdot X' \sim \Phi + \Delta, \quad \text{where} \quad \Phi \in g^1_4 \quad \text{and} \quad \Delta \in g^{1+t}_{\lambda+t}.$$

In particular

$$\deg(H_{X'}) = \lambda + t + 4$$

and one can easily verify that X' is the embedding of minimum degree of the curve X_K.

Definition. A linear system $|D|$ on a curve X is called *primitive* if, for each point $P \in X$, the linear system $|D + P|$ has P as base point. Equivalently, $\dim |D + P| = \dim |D|$.

It is not difficult to see that the following property of $X' \subset \mathbb{P}^{t+3}$, here stated for a standard model $\overline{S}_0$, holds also for any birational model $S_0 \in \mathbf{GRC}_L(S)$.

Proposition 6.3. *Let* $\overline{S}_0 \cong R_{1,t+1} \subset \mathbb{P}^{t+3}$ *be a standard model of* S. *Let* Φ *and* Δ *be as before and* $X' = X_{\Phi+\Delta} \subset R_{1,t+1}$ *be as usual. If* $g > 13$ *then the following facts hold:*
(i) the divisor $\Phi + \Delta$ *is a special divisor on* X*; in particular* $K - \Phi - \Delta$ *is an effective divisor.*
(ii) The curve $X' \subset \mathbb{P}^{t+3}$ *is linearly normal.*

Proof. (i) It is enough to show that $h^0(\mathcal{O}(K-\Phi-\Delta))>0$ or, equivalently by Riemann–Roch Theorem, that $\lambda<g-1$. If $t=0$, it follows immediately from 5.4 (ii).
If $t\geq 1$, still from 5.4 (iii), we have:

$$\lambda \leq t+\frac{g+3}{2} \quad \text{and} \quad t\leq \frac{g+3}{4} \quad \Rightarrow \quad \lambda\leq \frac{3g+9}{4}<g-1$$

where the last inequality is true since $g>13$ by assumption. Finally, observe that $\Phi+\Delta$ special implies that $K-\Phi-\Delta$ is an effective divisor.
(ii) Let us recall that (as in 5.1) the surface $\overline{S}_0$ is naturally embedded, via the isomorphism $\varphi_{2C_0+(\lambda-2)f}$, in a projective space: namely $\overline{S}_0\subset V_{\overline{S}_0}\subset \mathbb{P}^{g-1+\delta}$, where

$$V_{\overline{S}_0}=\mathbb{P}(\mathcal{O}(\lambda-2-2t)\oplus\mathcal{O}(\lambda-2-t)\oplus\mathcal{O}(\lambda-2))$$

and $t\geq 0$. If $t>0$, denoting by $M:=\langle\varphi_{2C_0+(\lambda-2)f}((\lambda-3-t)\Phi)\rangle$, it is clear that

$$\pi_M: V_{\overline{S}_0}\longrightarrow \mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(t+1))=R_{1,t+1}.$$

This map can be factorized as follows: setting Σ the divisor of the singular points of $\overline{X}_0$ and taking into account that $K-\Phi-\Delta$ is an effective divisor on X from (i), put:

$$L:=\langle\varphi_{2C_0+(\lambda-2)f}(\Sigma)\rangle,\quad N:=\langle\varphi_K(K-\Phi-\Delta)\rangle.$$

Then we have the following diagram:

$$\begin{array}{ccccccccc}
 & & \overline{X}_0 & \subset & \overline{S}_0 & \subset & V_{\overline{S}_0} & \subset & \mathbb{P}^{g-1+\delta}\\
 & \overline{\varphi}\nearrow & \downarrow & & \downarrow & & \downarrow & & \big\downarrow \pi_L\\
\mathbb{F}_t\supset X_t & \xrightarrow{\varphi_K} & X_K & \subset & S & \subset & V & \subset & \mathbb{P}^{g-1}\\
 & \varphi'\searrow & \downarrow & & & \searrow & \downarrow & & \big\downarrow \pi_N\\
 & & X' & & \subset & & R_{1,t+1} & \subset & \mathbb{P}^{t+3}
\end{array} \tag{8}$$

where $\overline{\varphi}:=\varphi_{2C_0+(\lambda-2)f}$, $\varphi'=\varphi_{\Phi+\Delta}$ and

$$\pi_N\circ\pi_L=\pi_M.$$

Note that $\overline{X}_0$ is not linearly normal. Namely, $\overline{X}_0$ is not special; if it was linearly normal, then $\dim\langle\Phi\rangle=3$ in $\mathbb{P}^{g-1+\delta}$, while $\overline{X}_0$ is contained in the scroll $V_{\overline{S}_0}$ which is ruled by planes.
Hence we have to consider its normalization $\widetilde{X}\subset\mathbb{P}^{g-1+2\delta}$, and the corresponding scroll

$$W:=\bigcup_{\Phi\in g^1_4}\langle\Phi\rangle\subset\mathbb{P}^{g-1+2\delta}.$$

It is easy to see that W is ruled by planes. Setting $\widetilde{L}:=\langle\Sigma\rangle\subset\mathbb{P}^{g-1+2\delta}$, the projection $\pi_{\widetilde{L}}$ factorizes through the normalization map, say Π, as follows:

$$\begin{array}{ccccc}
\widetilde{X} & \subset & W & \subset & \mathbb{P}^{g-1+2\delta}\\
\downarrow & & \downarrow & & \big\downarrow \Pi\\
\overline{X}_0 & \subset & V_{\overline{S}_0} & \subset & \mathbb{P}^{g-1+\delta}\\
\downarrow & & \downarrow & & \big\downarrow \pi_L\\
X_K & \subset & V & \subset & \mathbb{P}^{g-1}
\end{array} \tag{9}$$

and

$$\pi_L\circ\Pi=\pi_{\widetilde{L}}.$$

Setting

$$\widetilde{M} := \langle(\lambda - 3 - t)\Phi\rangle \subset \mathbb{P}^{g-1+2\delta}$$

and keeping into account (8) and (9) we finally obtain:

$$\begin{array}{ccccc}
\widetilde{X} & \subset & W & \subset & \mathbb{P}^{g-1+2\delta} \\
\downarrow & & \downarrow & & \Big\downarrow \pi_{\widetilde{L}} \\
X_K & \subset & V & \subset & \mathbb{P}^{g-1} \\
\downarrow & & \downarrow & & \Big\downarrow \pi_N \\
X' & \subset & R_{1,t+1} & \subset & \mathbb{P}^{t+3}
\end{array}$$

where

$$\pi_N \circ \pi_{\widetilde{L}} = \pi_{\widetilde{M}}.$$

Since $\pi_{\widetilde{M}} : \widetilde{X} \longrightarrow X'$ and $\widetilde{X}$ is linearly normal, than also X' is linearly normal.
If $t = 0$, the proof runs in a similar way. $\diamond$

Proposition 6.4. *Let $\overline{S}_0 \cong R_{1,t+1} \subset \mathbb{P}^{t+3}$, Φ, Δ and $X' = X_{\Phi+\Delta}$ be as usual. If $g > 13$ then the following facts hold:*

i) The linear system $|\Delta|$ defined before is primitive;

ii) if $B \subset \Delta$ is a divisor on X' such that $B \in g^1_\beta \neq g^1_4$, then $B \sim \Delta - A_1 - \cdots - A_t$, for suitable $A_i \in X' \setminus C_0$ for all i. In particular, $\beta = \lambda$.

Proof. *i)* Assume that there exists $P \in X'$ such that $\Delta + P \in g^{2+t}_{\lambda+t+1}$ and consider the model of X_K given by $X_{\Delta+P} \subset \mathbb{P}^{t+2}$. Keeping into account 6.3, we have that $X' = X_{\Phi+\Delta}$ is linearly normal in $\mathbb{P}^{t+3}$. Hence we can consider the following diagram:

$$\begin{array}{ccccc}
 & & X_{\Phi+\Delta} & \subset R_{1,t+1} \subset & \mathbb{P}^{t+3} \\
 & \nearrow & \Big\downarrow & & \Big\downarrow \pi_{\langle\Phi-P\rangle} \\
X & \longrightarrow & X_{\Delta+P} & \subset & \mathbb{P}^{t+2} \\
 & \searrow & \Big\downarrow & & \Big\downarrow \pi_P \\
 & & X_\Delta & \subset & \mathbb{P}^{t+1}
\end{array}$$

therefore $\Phi - P$ is a triple point of $X' = X_{\Phi+\Delta}$, in contrast with 6.1.
ii) The result is obvious for $t = 0$, so we can assume that $t > 0$.
Since $\langle\Phi\rangle$ is a fibre of $R_{1,t+1}$, then the projection centered in the line $\langle\Phi\rangle$ maps $R_{1,t+1}$ onto a cone:

$$\begin{array}{rcl}
\pi_{\langle\Phi\rangle} : \mathbb{P}^{t+3} & \longrightarrow & \mathbb{P}^{t+1} \\
R_{1,t+1} & \mapsto & R_{0,t}.
\end{array}$$

Moreover, recalling that $H_{X'} \sim \Phi + \Delta$, we have $\pi_{\langle\Phi\rangle}(X') = X_\Delta = \varphi_\Delta(X) \subset R_{0,t}$. Since all the singularities of X' belong to C_0 (see 4.2), then necessarily X_Δ has only one singular point in $C := \pi_{\langle\Phi\rangle}(C_0)$, which is the vertex of the cone $R_{0,t}$.
In order to obtain a linear series of dimension 1 on $X_\Delta \subset \mathbb{P}^{t+1}$, it is necessary to project it from t points, say $A_1, \ldots, A_t$, of X_Δ. If each of these points if different from C, then we get the required $B \in g^1_\beta$, where $\beta = \deg(\Delta) - t = \lambda$. If, for some i, it occurs that $A_i = C$, then $\pi_C(R_{0,t}) = \mathcal{C} \subset \mathbb{P}^t$, where $\mathcal{C}$ is a rational normal curve of degree t: in this case $B \in g^1_4$, in contrast with the assumption $g^1_\beta \neq g^1_4$. $\diamond$

Definition. A linear system $|\Delta|$ on the curve X is called *minimal* if it satisfies the conditions *i)* and *ii)* of 6.4.

Remark 6.5. Note that, if we perform the previous construction with respect to a birational model $S_0 \in \mathbf{GRC}_L(S)$ which is not a standard model, then the corresponding series $|\Delta|$ is primitive but not minimal.

Remark 6.6. If $t = 0$, i.e. $|\Delta| = g^1_\lambda$, then $|\Delta|$ is minimal if and only if is primitive.

We have seen in 6.4 that, if $R_{1,t+1}$ is isomorphic to a standard model, then the associated series $|\Delta|$ on X' is minimal. The converse is also true, as the following result shows.

Proposition 6.7. *Let X be as usual and consider two divisors $\Phi \in g^1_4$ and $\Delta \in g^{1+t}_{\lambda+t}$. If the linear series $|\Delta|$ is minimal on X, then $X_{\Phi+\Delta} \subset R_{1,t+1}$ is isomorphic to a standard model of $X_K \subset S$.*

Proof. We have to consider two cases: either $\dim\langle\varphi_{\Phi+\Delta}(\Phi)\rangle = 1$ or $\dim\langle\varphi_{\Phi+\Delta}(\Phi)\rangle = 2$.
(1) In this case, since $\deg(\Phi) = 4$, then $X_{\Phi+\Delta}$ is contained in a geometrically ruled surface as a four–secant curve. Moreover, since $\dim|\Delta| = t+1$, then the invariant of such ruled surface is t. Therefore $X_{\Phi+\Delta} \subset R_{h,t+h}$ for a suitable $h \geq 1$.
Assume first that $h \geq 2$. With a construction as in the proof of 6.4 (ii), consider the projection

$$\pi_{\langle\Phi\rangle} : \ R_{h,t+h} \longrightarrow R_{h-1,t+h-1}$$

where $\pi_{\langle\Phi\rangle}(X_{\Phi+\Delta}) = X_\Delta$.
Note that $H_R \sim U + hf$, where U is a unisecant curve of degree $t+h$. Therefore, as noted in 6.2,

$$\Phi + \Delta = H_R \cdot X_{\Phi+\Delta} \sim h\Phi + U \cdot X_{\Phi+\Delta}.$$

Since $h \geq 2$, it follows that $\Delta \sim (h-1)\Phi + U \cdot X_{\Phi+\Delta}$, so $\Phi \subset \Delta$. Hence $\Delta - \Phi \in g^{t-1}_{\lambda+t-4}$. Therefore there exist $t-2$ points, say $A_1, \ldots, A_{t-2}$, such that $\Delta - \Phi - A_1 - \cdots - A_{t-2} \in g^1_{\lambda-2}$. But this is impossible since $|\Delta|$ is minimal, hence it satisfies (ii) of 6.4. This proves that $h = 1$, so $X_{\Phi+\Delta} \subset R_{1,t+1}$.
If $X_{\Phi+\Delta}$ has a multiple point P not belonging to C_0, then we can project it from P and $t-1$ general points of the curve, obtaining a divisor $B \subset \Delta$ such that $B \in g^1_{\overline{\lambda}}$ and $\overline{\lambda} < \lambda$. Therefore all the singular points of $X_{\Phi+\Delta} \subset R_{1,t+1}$ belong to C_0 and this implies (from 4.2) that $R_{1,t+1}$ is a standard model.
(2) In this case the curve is contained in the scroll V, ruled by planes, whose fibers are $\langle\varphi_{\Phi+\Delta}(\Phi)\rangle$, $\Phi \in g^1_4$. So we set, for suitable $a \leq b \leq c$:

$$X_{\Phi+\Delta} \subset V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)).$$

Clearly, among the unisecant curves U^b of degree b such that $U^b \subset R_{a,b} \subset V$, we can choose one of them, say U, which does not meet $X_{\Phi+\Delta}$ (otherwise $X_{\Phi+\Delta}$ would be contained in the ruled surface $R_{a,b} \subset V$, against the assumption). Therefore, if we consider the projection

$$\pi_{\langle U\rangle} : \quad V \longrightarrow R_{a,c}$$

it is clear that $\pi_{\langle U\rangle}(X_{\Phi+\Delta})$ is again a curve, say $\overline{X}_{\Phi+\Delta}$, whose hyperplane divisor is still $\Phi + \Delta$, but $\overline{X}_{\Phi+\Delta} \subset R_{a,c}$, contrary to the assumption as well. $\diamond$

The remaining part of this Section is devoted to the case $t = 0$. Here the linear series $|\Delta|$ will be denoted by $|\Lambda|$, since its degree is λ, as noted in 6.6.
We will show that this linear series is, in general, not unique. In order to determine all such series g^1_λ, let us describe the situation and notation.

Let $X_K \subset S \subset V$ be as usual and assume that $t(S) = 0$. Let $\Phi \in g^1_4$, $\Lambda' \in g^1_{\lambda'}$ (where $\lambda' > 4$) and $X_{\Phi+\Lambda'} := \varphi_{\Phi+\Lambda'}(X) \subset R_{1,1}$. Denote by $|l|$ and $|l'|$ the two rulings of $R_{1,1}$.

Notation. If $P \in R_{1,1}$, denote by l_P and l'_P the lines of the two rulings passing through P. Moreover, if A is a double point of $X_{\Phi+\Lambda'}$, denote by A_1 and A_2 the corresponding points on the canonical model of the curve, i.e. $A_1, A_2 \in X_K$ are such that $\varphi_{\Phi+\Lambda'}(A_1) = \varphi_{\Phi+\Lambda'}(A_2) = A$.

Proposition 6.8. *In the above situation, each pair of double points, A and B say, of $X_{\Phi+\Lambda'}$ such that $l_A \neq l_B$ and $l'_A \neq l'_B$, determines a linear series $|\overline{\Lambda}'| \neq |\Lambda'|$ of degree λ'.*

Proof. Consider the four–gonal divisors and the λ'-gonal divisors of $|\Lambda'|$ containing, respectively, the two double points, i.e.

$$\begin{array}{ll} A_1 + A_2 + A'_1 + A'_2 \in g^1_4, & A_1 + A_2 + P_1 + \cdots + P_{\lambda'-2} \in |\Lambda'| \\ B_1 + B_2 + B'_1 + B'_2 \in g^1_4, & B_1 + B_2 + Q_1 + \cdots + Q_{\lambda'-2} \in |\Lambda'| \end{array}.$$

Consider the divisor $\overline{\Lambda}' = \Phi + \Lambda' - (A_1 + A_2 + B_1 + B_2)$; it is clear that $|\overline{\Lambda}'|$ is a linear series of degree λ' which is distinct from $|\Lambda'|$. $\diamond$

Remark 6.9. Let $X_K \subset S$ be as usual and assume that $t = 0$ and λ are the invariants of S. Let $\Phi \in g^1_4$, $\Lambda \in g^1_\lambda$ be two divisors on X. In the general case, the δ double points of $X' = X_{\Phi+\Lambda} \subset R_{1,1}$ belong to different lines of the two rulings $|l|$ and $|l'|$. Therefore from the above result it is clear that there are $\binom{\delta}{2}$ linear series $|\Lambda|$ of degree λ; to each of them we can associate a model of X lying on $R_{1,1}$. In particular, if $|\overline{\Lambda}|$ is one of these series, the corresponding model $X_{\Phi+\overline{\Lambda}}$ still has δ double points since the pair (A, B) has been replaced by (A', B'), where $A' := \varphi_{\Phi+\overline{\Lambda}}(A'_1) = \varphi_{\Phi+\overline{\Lambda}}(A'_2)$ and $B' := \varphi_{\Phi+\overline{\Lambda}}(B'_1) = \varphi_{\Phi+\overline{\Lambda}}(B'_2)$, following the notation in 6.8.

Theorem 6.10. *Let $X_K \subset S \subset V$ and let S be a surface ruled by conics of minimum degree. Let t and λ be the invariants of S defined before. If $t = 0$ then the invariant λ is the minimum degree of a linear series distinct from the g^1_4, i.e.*

$$\lambda = \min\{r \mid X \textit{ has a complete and base–point–free linear series } g^1_r \textit{ and } r > 4\}.$$

Moreover, assume that $|\Lambda|$ and $|\Lambda'|$ are two distinct linear series of degree λ and let S and S' be the associated surfaces. Then the following facts hold:

(i) if $\lambda \neq \frac{g+3}{2}$, then $S = S'$;

(ii) if $\lambda = \frac{g+3}{2}$, then S and S' are not necessarely coincident but belong to a pencil of surfaces, ruled by conics, each of them associated to a linear series of degree λ and has degree $\frac{3g-7}{2}$.

Proof. Recall that λ is defined at the beginning of this Section as the invariant of X such that a standard model of X is a divisor of type $(4, \lambda)$ on $R_{1,1}$. Consider a linear series $g^1_{\lambda'} \neq g^1_\lambda$; we need to show that $\lambda' \geq \lambda$. Suppose that $\lambda' < \lambda$.

If $g^1_{\lambda'}$ is minimal, consider $\Lambda' \in g^1_{\lambda'}$. Clearly, $X_{\Phi+\Lambda'} \subset R_{1,1}$ is a standard model.

If $g^1_{\lambda'}$ is not minimal, then it is not primitive (from 6.6); so there exist t' points, say $A_1, \ldots, A_{t'}$ such that $\Delta := \Lambda' + A_1 + \cdots + A_{t'}$ is both primitive and minimal. Therefore $X_{\Phi+\Delta} \subset R_{1,t'+1}$ is a standard model. Hence the corresponding surface S' ruled by conics is such that $X_K \subset S' \subset V$ and $\deg(S') = g + \lambda' - t' - 5$. Assume that $S' \neq S$; since $X_K \subseteq S \cap S'$, by (IF) we have:

$$\deg(X_K) \leq \int_V S \cdot S' = 2\ \deg(S) + 2\ \deg(S') - 4\ \deg(V)$$

hence

$$2g - 2 \leq 2(2g + \lambda + \lambda' - t - t' - 10) - 4(g - 3) \quad \Rightarrow \quad \lambda + \lambda' \geq t + t' + g + 3.$$

Since $\lambda' < \lambda$ then the above relation gives:

$$\lambda > \frac{g+3}{2} + \frac{t+t'}{2} = \frac{g+3}{2} + \frac{t'}{2}$$

where the last equality comes from the assumption $t = 0$.

On the other hand, $\lambda \leq \frac{g+3}{2}$ from 5.4. Hence $t' < 0$ and this is impossible.

Therefore we have proved that, if $S' \neq S$ then $\lambda' \geq \lambda$.

Assume now that $S' = S$. Clearly, $t' = t = 0$ and $\deg(S) = \deg(S')$. Hence, from 4.4, it follows that $\lambda = \lambda'$. In this way, we have proved the first part of the statement.

(i) Assume now that $\lambda \neq \frac{g+3}{2}$ and $S \neq S'$. Then we can use the (IF) as before and, from the assumption $\lambda = \lambda'$, we obtain

$$\lambda \geq \frac{g+3}{2} + \frac{t'}{2}.$$

Again we apply 5.4 to S, so:

$$\lambda \leq \frac{g+3}{2}.$$

Comparing these inequalities, we obtain:

$$t' = 0 \quad \text{hence} \quad \lambda = \frac{g+3}{2}$$

contrary to the assumption.

(ii) Suppose now that $\lambda = (g+3)/2$. In this case, from 4.4,

$$\deg(S) = g + \lambda - 5 = \frac{3g-7}{2}.$$

Therefore

$$\deg(S') = g - \lambda - t' - 5 \leq \deg(S)$$

and this implies $t' = 0$ and

$$\deg(S') = \deg(S) = \frac{3g-7}{2}.$$

So, by 1.1, the result follows. $\diamond$

7. Bounds for the invariants a and b

In this section we determine the range of the invariants a and b of the four–gonal curve X.
Let us keep the notation of Section 5, where $\overline{X}_0 \subset \overline{S}_0 \subset \overline{V}$ are standard models of $X_K \subset S \subset V$ and $\pi : \mathbb{P}^{g-1+\delta} \longrightarrow \mathbb{P}^{g-1}$ is the projection centered on the singular locus of $\overline{X}_0$.
Recall also that $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ and $\overline{V} = V_{\overline{S}} = \mathbb{P}(\mathcal{O}(\lambda-2-2t) \oplus \mathcal{O}(\lambda-2-t) \oplus \mathcal{O}(\lambda-2))$.
Moreover, from 2.2 (iii), we have $\delta = 3(\lambda - t - 1) - g$ and, from 5.4, we obtain the following range of the invariant λ:

$$\frac{g+3}{3} \leq \lambda - t \leq \frac{g+3}{2}. \tag{10}$$

Remark 7.1. Note that, from the above expression of $\overline{V}$, it follows that $a \leq \lambda - 2 - 2t$, $b \leq \lambda - 2 - t$, $c \leq \lambda - 2$. Moreover, since $a + b + c = g - 3$, there are only two independent invariants, a and b say.

Notation. Clearly, if $a < b$, there exists a unique directrix on V having degree a. In this case, let us denote by A such directrix of V, by $\overline{A} \subset \overline{V}$ the preimage of A via π, by δ_A the number of the double points (possibly infinitely near) of $\overline{X}_0$ lying on $\overline{A}$ and by $\overline{a}$ the degree of $\overline{A}$. Then

$$a = \overline{a} - \delta_A. \tag{11}$$

Proposition 7.2. *Let $t > 0$ and U be a directrix on $\overline{S}_0$. If $\deg(U) < \lambda - 2$, then $U = C_0$.*

Proof. It is enough to consider the isomorphism

$$\varphi_{2C_0+(\lambda-2)f} : \quad \mathbb{F}_t \longrightarrow \overline{S}_0$$

and the unisecant irreducible curves C_0 and $U = C_0 + \alpha f$ on $\mathbb{F}_t$.
If $U \neq C_0$, then $\alpha \geq t$ from 0.1. So

$$\deg_{\overline{S}_0}(U) = \int_{\overline{S}_0} (C_0 + \alpha f) \cdot (2C_0 + (\lambda-2)f) = \lambda - 2 + 2\alpha - 2t \geq \lambda - 2$$

and the result follows. $\diamond$

Proposition 7.3. *Let $t \geq 0$. Then the directrix $\overline{A}$ of $\overline{V}$ is contained in $\overline{S}_0$*

<u>Proof</u>. Assume that $\overline{A} \not\subset \overline{S}_0$. Then, taking into account that $\deg(\overline{S}_0) = 4(\lambda - t - 2)$ as computed in 4.4 and $\deg(\overline{V}) = 3(\lambda - t - 2)$, using the Intersection Formula we have:

$$\int_{\overline{V}} \overline{X}_0 \cdot \overline{A} \leq \int_{\overline{V}} \overline{S}_0 \cdot \overline{A} = \deg(\overline{S}_0) + 2\deg(\overline{A}) - 2\deg(\overline{V}) = 2\overline{a} - 2\lambda + 2t + 4.$$

Therefore, if the δ_A singular points are distinct, it follows that:

$$\delta_A \ \leq \ \frac{1}{2}\int_{\overline{V}} \overline{X}_0 \cdot \overline{A} = \overline{a} - \lambda + t + 2.$$

In the case of infinitely near points, it is not so difficult to show that the same relation holds.
In this way, from (11), we have the following bound of a:

$$a = \overline{a} - \delta_A \geq \lambda - t - 2,$$

which is the minimum degree of a directrix of V.
Consider the directrix $\pi(C_0) \subset V$. Since $\deg_{\overline{V}}(C_0) = \lambda - 2t - 2$ and the center of π contains at least one point of C_0, then $\deg_V(\pi(C_0)) \leq \lambda - 2t - 3 < \lambda - t - 2$; this concludes the proof. ⋄

Next we determine bounds for the invariant a.

Remark 7.4. Consider the unisecant $\overline{A} \subset \overline{S}_0 \cong \mathbb{F}_t$. Clearly, from 0.1, we have:

$$\overline{A} \sim C_0 + \alpha f, \quad \text{for some } \alpha \geq t \text{ or } \alpha = 0.$$

Therefore, as computed in the proof of 7.2, we have:

$$\overline{a} = \deg_{\overline{S}_0}(\overline{A}) = \lambda - 2t + 2\alpha - 2 \tag{12}$$

$$\overline{A} \cdot \overline{X}_0 = \int_{\overline{S}_0} (C_0 + \alpha f)(4C_0 + (\lambda + t)f) = \lambda - 3t + 4\alpha$$

$$\delta_A \leq \frac{\overline{A} \cdot \overline{X}_0}{2} = \frac{\lambda - 3t + 4\alpha}{2}. \tag{13}$$

It is immediate to see that, from (11), (12) and (13):

$$a = \overline{a} - \delta_A \geq \frac{\lambda - t - 4}{2}. \tag{14}$$

Note that this bound of a does not depend on α.

Remark 7.5. Note that, since $\delta_A \leq \delta$, from (11) we have:

$$a = \overline{a} - \delta_A \geq \overline{a} - \delta$$

so, taking into account that $\delta = 3(\lambda - t - 1) - g$, from (12) we immediately obtain

$$a \geq \lambda - 2t + 2\alpha - 2 - 3(\lambda - 1 - t) + g = g - 2\lambda + t + 2\alpha + 1 \geq g - 2\lambda + t + 1. \tag{15}$$

Remark 7.6. In order to compare the two bounds of a given by (14) and (15), just note that

$$\frac{\lambda - t - 4}{2} < g - 2\lambda + t + 1 \quad \Leftrightarrow \quad \lambda < \frac{2g + 3t + 6}{5}.$$

This leads us to consider the best lower bound of a in each of the two ranges of λ.

Keeping into account the previous remarks, we have immediately:

Proposition 7.7. *The invariant a has the following lower bound:*

$$a_{\min} := a_{\min}(g,\lambda,t) = \begin{cases} \left\lceil \dfrac{\lambda - t - 4}{2} \right\rceil & \text{if} \quad \lambda \geq \frac{2g+3t+6}{5} \\ g - 2\lambda + t + 1 & \text{if} \quad \lambda \leq \frac{2g+3t+6}{5} \end{cases}$$

and these bounds are attained if and only if $\overline{A} = C_0$. $\diamond$

Remark 7.8. We can also obtain an "absolute" lower bound of a, just observing that $a_{\min}$ can be realized when $\delta_A = \delta$ hence when $\frac{\lambda-t-4}{2} = g - 2\lambda + t + 1$ or, equivalently (from 7.6) when $\lambda = \frac{2g+3t+6}{5}$.
It is immediate to see that, on this line of the plane (t,λ) the two functions giving $a_{\min}(g,\lambda,t)$ coincide and are equal to

$$a_{\min}(g,t) = \frac{g-t-7}{5}. \tag{16}$$

Clearly, the minimum value of a is obtained for the maximum value of t (if $t > 0$). Therefore, keeping into account that $\lambda \geq 3t$ (by 2.2), it is clear that the minimum value of a corresponds to the common point of the lines $\lambda = \frac{2g+3t+6}{5}$ and $\lambda = 3t$. We finish the argument by observing that

$$\frac{2g+3t+6}{5} = 3t \quad \Leftrightarrow \quad t = \frac{g+3}{6}$$

and substituting this value in (16) we obtain:

$$a_{\min}(g) = \frac{g-9}{6}.$$

Note that, in this case, $\lambda = 3t = \frac{g+3}{2}$. Summing up we have proved that:

$$\text{if} \quad t > 0 \quad \text{then} \quad a_{\min}(g) = \frac{g-9}{6}, \quad \text{for} \quad t = \frac{g+3}{6} \quad \text{and} \quad \lambda = \frac{g+3}{2}.$$

Note also that, if $t = 0$, the value of $a_{\min}$ of (16) can be realized for $\lambda = \frac{2g+6}{5}$ and we immediately have:

$$\text{if} \quad t = 0 \quad \text{then} \quad a_{\min}(g) = \frac{g-7}{5}, \quad \text{for} \quad \lambda = \frac{2g+6}{5}.$$

Therefore, from 7.8, we obtain:

Corollary 7.9. *With the notation above we have:*

$$\textit{for all } t \geq 0, \quad a \geq \frac{g-9}{6} \qquad \textit{while, if } t = 0, \quad a \geq \frac{g-7}{5}.$$

In particular, V_S is not a cone for $t \geq 0$ and $g \geq 10$ or $t = 0$ and $g \geq 8$.

$\diamond$

Proposition 7.10. *Keeping the notation above, the invariants a and b can vary in the following two ranges:*

$$a_{\min} \leq a \leq \frac{g-3}{3} \tag{R_2}$$

$$g - \lambda - 1 \leq a + b \leq \frac{2(g-3)}{3}. \tag{R_3}$$

Proof. The two inequalities on the right in (R_2) and (R_3) follow from $a \leq b \leq c$ and $a+b+c = g-3$. For the left inequality of (R_3), note that $c \leq \lambda - 2$ by 7.1, hence $a + b = g - 3 - c \geq g - 3 - (\lambda - 2)$, as requested. $\diamond$

Remark 7.11. If $a < \frac{g-\lambda-1}{2}$ then $a < b$, hence A is unique.

8. Existence of curves of given invariants λ, a, b when $t = 0$.

Remark 8.1. Let us examine the situation corresponding to $t = 0$. Here a standard model $\overline{S}_0$ of S is isomorphic to the quadric $\mathbb{F}_0$ via

$$\varphi_{2l+(\lambda-2)l'}: \quad \mathbb{F}_0 \longrightarrow \overline{S}_0 \subset \mathbb{P}^{3\lambda-4}$$

and $\overline{X}_0 \sim 4l + \lambda l'$ on $\overline{S}_0$. Moreover, the projection from $\overline{V}$ to V is $\pi : \mathbb{P}^{3\lambda-4} \longrightarrow \mathbb{P}^{g-1}$, $\overline{V} = \mathbb{P}(\mathcal{O}(\lambda-2)^{\oplus 3})$ and the previous 2.2 (iii), (10), (R_2), (R_3) become, respectively:

$$\delta = 3(\lambda - 1) - g \tag{17}$$

$$\frac{g+3}{3} \leq \lambda \leq \frac{g+3}{2}. \tag{R_1}$$

$$a_{\min} \leq a \leq \frac{g-3}{3} \tag{R_2}$$

$$g - \lambda - 1 \leq a + b \leq \frac{2(g-3)}{3} \tag{R_3}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda-4}{2} \right\rceil & \text{if} \quad \lambda \geq \frac{2g+6}{5} \\ g - 2\lambda + 1 & \text{if} \quad \lambda \leq \frac{2g+6}{5} \end{cases}.$$

Note that $\frac{2g+6}{5}$ belongs to the range of λ given in (R_1). Moreover, $\lambda = \frac{2g+6}{5}$ if and only if $\delta = \frac{\lambda}{2}$.

At this point, beside the map $\varphi := \varphi_{2l+(\lambda-2)l'}$ defined before, it is useful to introduce a further model of S given by the following isomorphism

$$\psi := \varphi_{4l+\lambda l'} : \mathbb{F}_0 \longrightarrow S' \subset \mathbb{P}^{5\lambda+4}.$$

Notation. From now on, we denote a geometrically ruled surface $\varphi_{nl+ml'}(\mathbb{F}_0) \subset \mathbb{P}^{(n+1)(m+1)-1}$ by $S_{n,m}$.

In this way, $S' = S_{4,\lambda}$ and we set $f : S' \longrightarrow \overline{S}_0$ the isomorphism being given by $\varphi = f \circ \psi$.

Remark 8.2. A hyperplane section $H \cdot S'$ of $S' \subset \mathbb{P}^{5\lambda+4}$ corresponds, via the morphism ψ, to a curve $X_H \subset \mathbb{F}_0$ of type $(4, \lambda)$. It is not difficult to show, using 3.9, that $P \in \mathbb{F}_0$ is a double point of X_H if and only if H contains the tangent plane $T_P(S')$ (here P means $\psi(P) \in S'$).

Remark 8.3. Let $S := S_{n,m} \subset \mathbb{P}^{(n+1)(m+1)-1}$ and $Y \subset S$ be a divisor whose decomposition into irreducible and reduced components is $Y = Y_1 \cup \ldots \cup Y_s$. Let $P_1, \ldots, P_\delta$ be points of Y and denote by δ_i the number of these points belonging to the component Y_i. Let

$$L := \left\langle T_{P_1}(S), \ldots, T_{P_\delta}(S) \right\rangle$$

be the linear space spanned by the δ tangent planes. Clearly, if H is any hyperplane containing L, then H intersects Y_i in at least $2\delta_i$ points. Therefore, if $2\delta_i > \deg(Y_i)$, then H contains Y_i.

The above observation leads to the following:

Definition. We say that $P_1, \ldots, P_\delta$ *trivially degenerate* the component Y_i if $2\delta_i > \deg(Y_i)$. Moreover, we say that $P_1, \ldots, P_\delta$ *trivially degenerate* the curve Y if this occurs for at least one component of Y.

Remark 8.4. Let $S' = S_{4,\lambda}$ be as before. Assume that $a \leq b \leq c$ fulfil the relations $(R_1), (R_2), (R_3)$.

(a) Let $M \sim l$ be a divisor of S'. Clearly $\deg(M) = H \cdot M = \lambda$. Let us consider $\lambda - 2 - a$ distinct points of M, say $P_1, \ldots, P_{\lambda-2-a}$. Clearly $P_1, \ldots, P_{\lambda-2-a}$ do not trivially degenerate M if and only if

$$2(\lambda - 2 - a) \leq \deg(M) = \lambda \quad \Leftrightarrow \quad a \geq \frac{\lambda - 4}{2}$$

and this is true by (R_2).

(b) In the same way, if $N \sim l$ is a divisor of S' and $P_1, \ldots, P_{\lambda-2-b}$ are distinct points of N, then

$$2(\lambda - 2 - b) \leq 2(\lambda - 2 - a) \leq \deg(N) = \lambda$$

again by (R_2). So $P_1, \ldots, P_{\lambda-2-b}$ do not trivially degenerate N.

(c) Consider now a divisor $Q \sim (\lambda - 2 - c)l'$ consisting of $\lambda - 2 - c$ distinct components and a set of distinct points $P_1, \ldots, P_{\lambda-2-c}$, one on each component of Q. Obviously $P_1, \ldots, P_{\lambda-2-c}$ do not trivially degenerate Q.

Theorem 8.5. *Let g, a, b, λ be positive integers, with $g \geq 10$, and consider the following inequalities:*

$$\frac{g+3}{3} \leq \lambda \leq \frac{g+3}{2} \tag{R_1}$$

$$a_{\min} \leq a \leq \frac{g-3}{3} \tag{R_2}$$

$$g - \lambda - 1 \leq a + b \leq \frac{2(g-3)}{3} \tag{R_3}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda - 4}{2} \right\rceil & \text{if} \quad \lambda \geq \frac{2g+6}{5} \\ g - 2\lambda + 1 & \text{if} \quad \lambda < \frac{2g+6}{5} \end{cases}.$$

Then there exists a 4–gonal curve of genus g and invariants a, b, λ if and only if (R_1), (R_2), (R_3) are verified.

Proof. If there exists a 4–gonal curve of genus g and invariants a, b, λ then (R_1), (R_2), (R_3) come from 8.1.
Conversely, let us choose g, λ, a, b satisfying the inequalities (R_1), (R_2), (R_3). Using 8.2, it is enough to show that there exists an irreducibile hyperplane section $H \cdot S'$ of $S' = S_{4,\lambda}$, i.e. a curve $X_H \sim 4l + \lambda l'$ on $\mathbb{F}_0$, of genus g and invariants a, b.
Take the following three divisors of S': M, N, Q, where $M \sim l \sim N$ ($M \neq N$) and $Q \sim (\lambda - 2 - c)l'$ consists of distinct lines; moreover consider $\lambda - 2 - a$ distinct points of M, $\lambda - 2 - b$ distinct points of N and $\lambda - 2 - c$ distinct points of Q, one on each line and none belonging to M or N.
Note that $M + N + Q \in |2l + (\lambda - 2 - c)l'|$ and the equality $(\lambda - 2 - a) + (\lambda - 2 - b) + (\lambda - 2 - c) = \delta$ holds from (17).
Therefore, taking into account also 8.4, it is immediate to see that the hypotesis of the forthcoming lemma 9.4 are verified; then we can deduce that the linear space L spanned by the tangent planes to S' at the above δ points does not contain any further point of S'. In particular, a general hyperplane $H \supset L$ corresponds to an irreducible curve $X_H \sim 4l + \lambda l'$ having exactly δ nodes; so its genus is $g(X_H) = 3(\lambda - 1) - \delta = g$.
Consider the isomorphism $f : S' \longrightarrow \overline{S}_0$ defined before and set $\overline{A} := f(M)$, $\overline{B} := f(N)$. Clearly

$$\deg(\overline{A}) = \deg(\overline{B}) = \lambda - 2.$$

Set $\overline{X}_0 := \varphi(X_H) \subset \overline{S}_0$ and denote by δ_A and δ_B the number of the double points of $\overline{X}_0$ lying on $\overline{A}$ and on $\overline{B}$, respectively. From the construction, it is clear that:

$$\delta_A = \lambda - 2 - a \quad \text{and} \quad \delta_B = \lambda - 2 - b.$$

Setting $A, B \subset S \subset V$ the projections of $\overline{A}$ and $\overline{B}$, respectively, via $\pi_{\langle \overline{\Delta} \rangle} : \overline{S}_0 \to S$, from (11) we have that $\deg(A) = \deg(\overline{A}) - \delta_A = \lambda - 2 - \delta_A = a$ and $\deg(B) = \deg(\overline{B}) - \delta_B = \lambda - 2 - \delta_B = b$.
In this way one can easily deduce that $V = V_S = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, so a and b are the other two invariants of X. $\diamond$

In order to complete the proof of the Theorem above, we need to prove the "Key–lemma" stated in 9.4. Next section will be devoted to this purpose.

9. Proof of the Key–lemma

In order to prove the Key–lemma 9.4, we need some preliminary technical results.

Lemma 9.1. *Let $S := S_{n,m}$ and $D \sim hl + kl' \subset S$ be a divisor, where $h \leq n+1$ and $k \leq m+1$. Then the following facts hold:*

i)

$$\dim\langle D\rangle = h(m+1) + k(n+1) - hk - 1.$$

Moreover, if D is irreducible:

ii) D is a non–special curve;

iii) D is a linearly normal curve in $\langle D\rangle$.

Proof.

i) Assume first that $h \leq n$ and $k \leq m$. It is clear that, setting $S' := S_{n-h,m-k}$, we have $\dim\langle D\rangle = h^0(\mathcal{O}_S(1)) - h^0(\mathcal{O}_{S'}(1)) - 1$ and this proves the above relation.
The remaining cases are: $h = n+1$ and $k \leq m+1$ or $h \leq n+1$ and $k = m+1$. In both of them, $D \sim hl + kl'$ cannot be contained in any hyperplane section $H \cdot S \sim nl + ml'$ of S. Hence $\langle D\rangle = \langle S\rangle$, so $\dim\langle D\rangle = \dim\langle S\rangle = (n+1)(m+1) - 1$ and this gives the formula in the statement when $h = n+1$ or $k = m+1$.

ii) It is enough to show that $\deg(D) > 2p_a(D) - 2$. Taking into account that $\deg(D) = hm + kn$ and $p_a(D) = hk - h - k + 1$, and using the assumption $n \geq h-1$ and $m \geq k-1$, we obtain:

$$\deg(D) = hm + kn \geq h(k-1) + (h-1)k > 2hk - 2h - 2k = 2p_a(D) - 2.$$

iii) It is enough to prove that $h^0(D, \mathcal{O}_D(1)) = \dim\langle D\rangle + 1$.
Since D is non–special, as proved before, applying the Riemann–Roch Theorem, we obtain

$$h^0(\mathcal{O}_D(1)) = \deg(D) - p_a(D) + 1$$

and this coincides with $\dim\langle D\rangle + 1$, as one can easily verify. Hence D is linearly normal in $\langle D\rangle$. $\diamond$

Lemma 9.2. *Let $S := S_{2,k}$, where $k \geq 2$, and consider d distinct points: $P_1, \dots, P_d \in S$, where $d \leq 2k+1$. Setting $J := \langle P_1, \dots, P_d\rangle$, if $\dim(J) < d-1$, then there exists a unisecant curve U on S such that $\#(U \cap \{P_1, \dots, P_d\}) \geq \deg(U) + 1$. In particular, $U \subset S \cap J$.*

Proof. Assume for simplicity that the considered points belong to distinct fibres of S'.
Since $\dim|l + kl'| = 2k+1 \geq d$, there exists a unisecant curve linearly equivalent to $l + kl'$ containing $P_1, \dots, P_d$. Therefore we can find a unisecant, U' say, of minimum degree containing $P_1, \dots, P_d$. Clearly, $U' \sim l + \epsilon l'$, where $\epsilon \leq k$; moreover $U' = U + l'_1 + \dots + l'_\alpha$, where U is irreducible, $P_1, \dots, P_{d-\alpha} \in U$ and $P_{d-\alpha+i} \in l'_i \setminus U$, for $i = 1, \dots, \alpha$. Let us show that U is the required unisecant curve. Were this not the case, setting

$$\beta := \deg(U) + 1 - (d - \alpha)$$

it follows that $\beta > 0$. Consider the linear space $T := \langle J, A_1, \dots, A_\beta\rangle$, where $A_j \in U$. Clearly $U \subset T$, hence T meets each fiber l'_i in two points: $P_{d-\alpha+i}$ and $U \cap l'_i$. Since the fibers are conics then, choosing $B_i \in l'_i$, the linear space

$$\Sigma := \langle J, A_1, \dots, A_\beta, B_1, \dots, B_\alpha\rangle$$

contains $\langle U'\rangle$. Therefore $\dim\langle U'\rangle \leq \dim(\Sigma) \leq \dim(J) + \alpha + \beta = \dim(J) + \deg(U) + 1 - d + 2\alpha$. On the other hand, using 9.1, $\dim\langle U'\rangle = \deg(U') = \deg(U) + 2\alpha$, so $\dim(J) \geq d - 1$, against the assumption.
It is not difficult to generalize this proof to the case where at most two of the d points belong to the same fibre. $\diamond$

Lemma 9.3. *Let $S := S_{4,\lambda}$, where $\lambda \geq 4$, and $\widetilde{D} \in |2l + \epsilon l'|$ be a bisecant curve on S such that $\widetilde{D}$ does not contain any fiber of S. Consider $d+1$ points $P, P_1, \dots, P_d$ as follows: $P \in S$, $P_1, \dots, P_d \in \widetilde{D}$ such that they do not trivially degenerate $\widetilde{D}$ and at most two of them belong to the same fibre. Assume that $P_1, \dots, P_m$ are double points of $\widetilde{D}$ (for $0 \leq m \leq d$) and $P_{m+1}, \dots, P_d$ are simple points of $\widetilde{D}$. Let*

$$T := \langle P, T_{P_1}(S), \dots, T_{P_m}(S), t_{P_{m+1}}(\widetilde{D}), \dots, t_{P_d}(\widetilde{D})\rangle$$

where $T_{P_i}(S)$ and $t_{P_i}(\widetilde{D})$ denote the tangent plane to S and the tangent line to $\widetilde{D}$, respectively, at P_i.
If $\epsilon \leq \lambda$ and $d \leq \lambda$, then $\dim(T) = 2d + m$.

<u>Proof</u>. For simplicity, assume that $P \in \widetilde{D}$ and $P_1, \ldots, P_d$ belong to distinct fibres of S. In this situation, $T \subseteq \langle \widetilde{D} \rangle$ and $m \leq d \leq \epsilon$.
Claim: T is a proper subspace of $\langle \widetilde{D} \rangle$.
In order to prove this, observe that, by 9.1 and the assumption $d \leq \lambda$, we have

$$\dim\langle \widetilde{D} \rangle = 2\lambda + 3\epsilon + 1 \geq 2d + 3\epsilon + 1.$$

As noted at the beginning, $m \leq \epsilon$ hence $\dim\langle \widetilde{D} \rangle \geq 2d + 3m + 1 > 2d + m \geq \dim(T)$ and this proves the claim.
Let $N := \dim\langle \widetilde{D} \rangle$ and consider the projection $\pi_T : \mathbb{P}^N \to \mathbb{P}^n$ with center T, for a suitable n. Clearly, by the claim above, $n > 0$.
Let $R := R(\widetilde{D})$ be the ruled surface generated by $\widetilde{D}$ via the ruling on S. Since T is a multisecant space of this ruled surface and $P_1, \ldots, P_d$ belong to distinct fibers, then $T \cap R$ contains a unisecant curve (see [**4**], 1.5), Y say. Therefore $\pi_T(R) = \pi_T(\widetilde{D})$ is a rational normal curve of degree n in $\mathbb{P}^n$. In particular:

$$N - n = \dim\langle \widetilde{D} \rangle - \dim\langle \pi_T(\widetilde{D}) \rangle = \dim(T) + 1. \tag{18}$$

In order to prove the statement, observe that it holds that $\dim(T) \leq 2d + m$.
First case: $\widetilde{D}$ is irreducible.
Since $\pi_{T|\widetilde{D}}$ is a map of degree two, then

$$n = \deg(\pi_T(\widetilde{D})) = \frac{\deg(\widetilde{D}) - \int T \cdot \widetilde{D}}{2}. \tag{19}$$

Moreover, from 9.1 (iii) we have that:

$$N = \dim\langle \widetilde{D} \rangle = h^0(\mathcal{O}_{\widetilde{D}}(1)) - 1 = \deg(\widetilde{D}) - p_a(\widetilde{D})$$

so, using (18) we finally obtain:

$$\dim(T) = N - n - 1 = \deg(\widetilde{D}) - p_a(\widetilde{D}) - \frac{\deg(\widetilde{D}) - \int T \cdot \widetilde{D}}{2} - 1 = \frac{\deg(\widetilde{D}) + \int T \cdot \widetilde{D}}{2} - p_a(\widetilde{D}) - 1.$$

Note that $\deg(\widetilde{D}) = 4\epsilon + 2\lambda$ and $p_a(\widetilde{D}) = \epsilon - 1$; moreover, by the definition of T, $\int T \cdot \widetilde{D} \geq 2d + 2m + 1$. Hence we obtain

$$\dim(T) \geq \epsilon + \lambda + d + m + 1/2.$$

Thus, if we assume $\dim(T) < 2d + m$, we get

$$\epsilon + \lambda + d + m + 1/2 < 2d + m \quad \Rightarrow \quad d > \lambda + \epsilon + 1/2$$

contrary to the assumption $d \leq \lambda$.
Second case: $\widetilde{D}$ is reducible.
Let $\widetilde{D} = U_1 + U_2$, where U_i are irreducible unisecant curves. Let d_i be the number of points among $P_1, \ldots, P_d$ belonging to U_i. Clearly, $P_1, \ldots, P_m$ belong to $U_1 \cap U_2$, so $d = d_1 + d_2 - m$. Moreover, we have

$$\dim\langle \widetilde{D} \rangle = \dim\langle U_1 \rangle + \dim\langle U_2 \rangle - \int U_1 \cdot U_2 + 1. \tag{20}$$

Since T is a proper subspace of $\langle \widetilde{D} \rangle$ as proved in the previous claim, then $\widetilde{D} \not\subset T$; therefore only two cases can occur: either $U_i \not\subset T$ for $i = 1, 2$ or (for instance) $U_1 \subset T$ and $U_2 \not\subset T$.
If $U_i \not\subset T$ for $i = 1, 2$, then $\pi_T(\widetilde{D}) = \pi_T(U_1) = \pi_T(U_2)$ so

$$n = \dim\langle \pi_T(\widetilde{D}) \rangle = \dim\langle \pi_T(U_i) \rangle = \deg(\pi_T(U_i)) = \deg(U_i) - \int T \cdot U_i \quad \text{for} \quad i = 1, 2. \tag{21}$$

Adding the previous relations (21) for $i = 1$ and $i = 2$, we obtain that $2n = \deg(U_1 + U_2) - \int T \cdot (U_1 + U_2)$, so this equality coincides with (19) and we conclude the proof as in the first case.
We are left to study the case $U_1 \subset T$, i.e. $U_1 = Y$. Since T contains the tangent lines to U_2 at all the d_2 points defined before and since $U_1 \subset T$ and the m double points of $\widetilde{D}$ belong to $U_1 \cap U_2$, then

$$\int T \cdot U_2 = 2d_2 + \int U_1 \cdot U_2 - m.$$

In this case (21) holds only for U_2, so it becomes:

$$\dim\langle \pi_T(\widetilde{D})\rangle = \deg(U_2) - \left(2d_2 + \int U_1 \cdot U_2 - m\right).$$

Therefore, using the relation above and (20), and taking into account that $\dim\langle U_i\rangle = \deg(U_i)$, we obtain:

$$\dim\langle \widetilde{D}\rangle - \dim\langle \pi_T(\widetilde{D})\rangle = \deg(U_1) + 2d_2 - m + 1.$$

Now we substitute $d_2 = d + m - d_1$ and use (18), obtaining

$$\dim(T) + 1 = \deg(U_1) + 2d + 2m - 2d_1 - m + 1.$$

Finally recall that the P_i's do not trivially degenerate $\widetilde{D}$, hence $2d_1 \leq \deg(U_1)$; so we obtain

$$\dim(T) + 1 \geq 2d + m + 1$$

as required. In the general case, the proof runs in a similar way. $\diamond$

Notation. Since we will consider, in the following result, both $S' := S_{4,\lambda}$and $S_{2,c+2}$, we denote the divisors on these surfaces by: $D_4, \widetilde{D}_4, \ldots$ and $D_2, \widetilde{D}_2, \ldots$, respectively.

Key–Lemma 9.4. *Let g, a, b, c, λ be positive integers satisfying* (2), $(R_1), (R_2), (R_3)$.
Let $S' := S_{4,\lambda} \subset \mathbb{P}^{5\lambda+4}$ and $D_4 \in |2l + (\lambda - 2 - c)l'|$ be a curve on S' of type

$$D_4 = \widetilde{D}_4 + \sum_{i=1}^{\alpha} l'_i$$

where α is an integer such that $0 \leq \alpha \leq \lambda - 2 - c$ and $\widetilde{D}_4$ is a suitable bisecant divisor not containing any irreducible component linearly equivalent to l'.
Let us take $\delta = 3(\lambda - 1) - g$ distinct points on D_4 which do not trivially degenerate D_4 and set

$$P_1, \ldots, P_{\delta-\alpha} \in \widetilde{D}_4 \quad \text{and} \quad P'_1, \ldots, P'_\alpha \in \sum_{i=1}^{\alpha} l'_i$$

such that $P'_i \in l'_i \setminus \widetilde{D}_4$ for $i = 1, \ldots, \alpha$. Consider the linear space

$$L := \big\langle T_{P_1}(S'), \ldots, T_{P_{\delta-\alpha}}(S'), T_{P'_1}(S'), \ldots, T_{P'_\alpha}(S')\big\rangle$$

spanned by the tangent planes to S' at these δ points.
If $P \in S'$ is any further point such that $P \notin L$ and $L' := \langle P, L\rangle$, then:

$$\dim(L') = 3\delta.$$

In particular, $\dim(L) = 3\delta - 1$, *i.e. L is of maximum dimension and the intersection of L and S' consists only of the points $P_1, \ldots, P_{\delta-\alpha}, P'_1, \ldots, P'_\alpha$.*

Proof. Note first that $\dim(L') \leq 3\delta$ and $\dim(L) \leq 3\delta - 1$. So it is enough to show that $\dim(L') \geq 3\delta$.
Assume first that $P \notin \widetilde{D}_4$.

Step 1. Computation of the dimension of $\Sigma := \langle L', D_4 \rangle$.
Among the choosen points $P_1, \ldots, P_{\delta-\alpha} \in \widetilde{D}_4$, consider those which are singular points of $\widetilde{D}_4$, say $P_1, \ldots, P_m$, for some $0 \le m \le \delta - \alpha$.

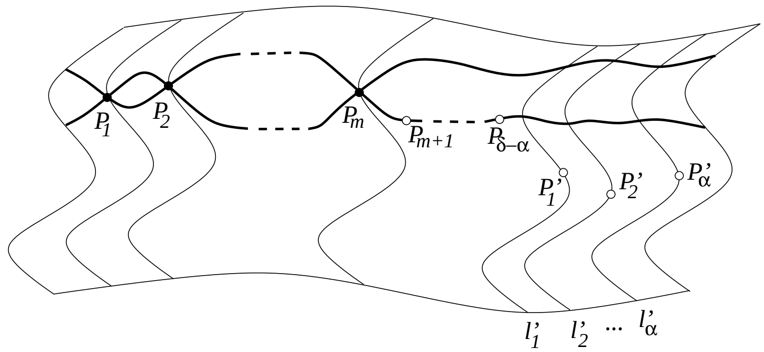

Figure 5

Clearly, since they are double points of $\widetilde{D}_4$, the tangent plane at each of them is contained in $\langle \widetilde{D}_4 \rangle$. On the other hand, the tangent plane at the remaining $\delta - m$ points intersects $\langle D_4 \rangle$ in a line (either tangent to $\widetilde{D}_4$ for $P_{m+1}, \ldots, P_{\delta-\alpha}$, or tangent to l'_i for the points of type P'_i). Briefly:

$$\begin{array}{rll} T_{P_i}(S') \subset \langle \widetilde{D}_4 \rangle, & & \text{for } i = 1, \ldots, m \\ T_{P_i}(S') \cap \langle D_4 \rangle = t_{P_i}(D_4) = t_{P_i}(\widetilde{D}_4), & & \text{for } i = m+1, \ldots, \delta - \alpha \\ T_{P'_j}(S') \cap \langle D_4 \rangle = t_{P'_j}(D_4) = t_{P'_j}(l'_j), & & \text{for } j = 1, \ldots, \alpha. \end{array} \tag{22}$$

Consider now the projection

$$\pi := \pi_{\langle D_4 \rangle} : \ S' = S_{4,\lambda} \longrightarrow S_{2,c+2}$$

and set

$$J := \pi(\Sigma) = \langle \overline{P}, \overline{P}_{m+1}, \ldots, \overline{P}_{\delta-\alpha}, \overline{P}'_1, \ldots, \overline{P}'_\alpha \rangle$$

where

$$\overline{P} := \pi(P), \ \overline{P}_i := \pi(T_{P_i}(S')), \text{ for } i = m+1, \ldots, \delta - \alpha, \text{ and } \quad \overline{P}'_j := \pi(T_{P'_j}(S')), \text{ for } j = 1, \ldots, \alpha.$$

By the definition of J, we clearly have:

$$\dim(\Sigma) = \dim(J) + \dim\langle D_4 \rangle + 1. \tag{23}$$

Step 2. Computation of the dimension of J.
Observe that the isomorphisms $\varphi_{4l+\lambda l'}$ and $\varphi_{2l+(c+2)l'}$ induce a canonical isomorphism, say χ, as follows

$$\begin{tikzcd} & \mathbb{F}_0 \arrow[dl, "\varphi_{4l+\lambda l'}"'] \arrow[dr, "\varphi_{2l+(c+2)l'}"] & \\ S_{4,\lambda} \arrow[rr, "\chi"] & & S_{2,c+2} \end{tikzcd}$$

and χ coincides with π on $S_{4,\lambda} \setminus D_4$.
Therefore, setting $D_2 := \chi(D_4) \subset S_{2,c+2}$, the points $\overline{P}_{m+1}, \ldots, \overline{P}_{\delta-\alpha}, \overline{P}'_1, \ldots, \overline{P}'_\alpha$ belong to D_2.
Clearly, $\dim(J) \le \delta - m$. We want to show that $\dim(J) = \delta - m$.
Assume that $\dim(J) < \delta - m$. In order to apply 9.2, we need to compare the number of points spanning J with the integer c.
On one hand, from (17) and (R_1) we have:

$$\delta = 3(\lambda - 1) - g \le \frac{g+3}{2}.$$

On the other hand, from (R_3), we get $c \geq \frac{g-3}{3}$, i.e. $g \leq 3c+3$. Therefore we obtain:

$$\delta - m \leq \delta \leq \frac{g+3}{2} \leq \frac{3c+6}{2} < 2c+5 \quad \Rightarrow \delta - m + 1 \leq 2(c+2)+1.$$

So, we can apply Lemma 9.2 to J (which is spanned by $\delta - m + 1$ points and has dimension smaller than $\delta - m$) and $S_{2,c+2}$. In this way we obtain that there exists a unisecant curve $\overline{U} \subset J \cap S_{2,c+2}$ such that, setting r the number of the points among $\overline{P}, \overline{P}_{m+1}, \ldots, \overline{P}_{\delta-\alpha}, \overline{P}'_1, \ldots, \overline{P}'_\alpha$ belonging to $\overline{U}$, then

$$\deg(\overline{U}) \leq r - 1.$$

Let $\overline{U} \sim l + \epsilon l'$; then $\deg(\overline{U}) = c + 2 + 2\epsilon$.
Claim. The unisecant $\overline{U}$ is not contained in D_2.
If not, let $U := \chi^{-1}(\overline{U})$ and h be the number of the points among P, the P_i's and the P'_j's belonging to U. On one hand, since these points do not trivially degenerate D_4 (by assumption) and $U \subset D_4$ (since $\overline{U} \subset D_2$ by the assumption of the Claim), then $2h \leq \deg(U)$.
On the other hand, $h \geq r$ by the definitions of h and r and from $\chi(U) = \overline{U}$. From all these observations, it follows

$$\deg(U) \geq 2h \geq 2r \geq 2(\deg(\overline{U}) + 1) = 2(c + 3 + 2\epsilon).$$

Since $\deg(U) = \lambda + 4\epsilon$, we obtain $2c + 6 \leq \lambda$. Using the bound $c \geq (g-3)/3$, we finally get $\lambda \geq (2/3)g + 4$, against (R_1). In this way the claim is proved.

Since $\overline{U}$ is not contained in D_2, we can consider their intersection, which surely contains the r points introduced before. So

$$r \leq \int_{S_{2,c+2}} \overline{U} \cdot D_2 = (l + \epsilon l') \cdot (2l + (\lambda - 2 - c)l') = \lambda - 2 - c + 2\epsilon.$$

The above relation and $\deg(\overline{U}) \leq r - 1$ give:

$$c + 2 + 2\epsilon = \deg(\overline{U}) \leq r - 1 \leq \lambda - 3 - c + 2\epsilon$$

so $\lambda \geq 2c + 5$ and this leads to a contraddiction, as in the proof of the claim above.
Hence such unisecant curve $\overline{U}$ does not exist and this implies

$$\dim(J) = \delta - m. \tag{24}$$

Step 3. Computation of the dimension of L'.
Putting together (23) and (24) we finally obtain:

$$\dim(\Sigma) = \dim\langle D_4 \rangle + \delta - m + 1. \tag{25}$$

Now let us compare $\dim(\Sigma)$ with $\dim(L')$. Consider the linear space

$$T := \langle P, T_{P_1}(S'), \ldots, T_{P_m}(S'), t_{P_{m+1}}(\widetilde{D}_4), \ldots, t_{P_{\delta-\alpha}}(\widetilde{D}_4) \rangle \subseteq L'.$$

Note that, from (R_1), we have $g \geq 2\lambda - 3$; hence

$$\delta - \alpha \leq \delta = 3(\lambda - 1) - g \leq \lambda.$$

Therefore the assumption in 9.3 are satisfied by $S_{4,\lambda}$, $\widetilde{D}_4$ and T with respect to the points $P, P_1, \ldots, P_{\delta-\alpha}$: we then obtain

$$\dim(T) = 2(\delta - \alpha) + m. \tag{26}$$

Since $T \subseteq \langle \widetilde{D}_4, P\rangle$ by (22), there exist β points, say $R_1, \dots, R_\beta \in \widetilde{D}_4$ such that $\langle T, R_1, \dots, R_\beta\rangle$ coincides with $\langle \widetilde{D}_4, P\rangle$, where

$$\beta = \dim\langle \widetilde{D}_4, P\rangle - \dim(T) \leq \dim\langle \widetilde{D}_4\rangle - \dim(T) + 1. \tag{27}$$

Therefore the linear space $\langle L', R_1, \dots, R_\beta\rangle$ contains $\langle \widetilde{D}_4, P\rangle$, so it meets each fibre $l'_{P'_j}$ (for $j = 1, \dots, \alpha$) in four points: two of them are $l'_{P'_j} \cap \widetilde{D}_4$ and the remaining ones are $l'_{P'_j} \cap T_{P'_j}(S')$. Hence, if we add to this space a further point, say A_j, on each fiber, the obtained linear space contains also the quartic curves $l'_{P'_1}, \dots, l'_{P'_\alpha}$, hence the whole divisor D_4. In this way we have proved that

$$\langle L', R_1, \dots, R_\beta, A_1, \dots, A_\alpha\rangle \supset \langle L', D_4\rangle = \Sigma$$

so

$$\dim(\Sigma) \leq \dim(L') + \alpha + \beta. \tag{28}$$

Using (25) and (28) we obtain:

$$\dim\langle D_4\rangle + \delta - m + 1 = \dim(\Sigma) \leq \dim(L') + \alpha + \beta$$

and from this, using (27) we get:

$$\dim\langle D_4\rangle + \delta - m + 1 \leq \dim(L') + \alpha + \dim\langle \widetilde{D}_4\rangle - \dim(T) + 1.$$

Finally, using (26) we obtain:

$$\begin{aligned}\dim(L') &\geq \delta - m + \dim\langle D_4\rangle - \dim\langle \widetilde{D}_4\rangle - \alpha + 2(\delta - \alpha) + m = \\ &= 3\delta - 3\alpha + \dim\langle D_4\rangle - \dim\langle \widetilde{D}_4\rangle = \\ &= 3\delta\end{aligned}$$

where the last equality easily comes from 9.1.
Note that the statement has been proved in the case $P \notin \widetilde{D}_4$, but the case $P \in \widetilde{D}_4$ runs in a similar way, with some cautions. Namely, in Step 1, the main difference concernes the linear space $J := \pi(\Sigma) = \langle \overline{P}_{m+1}, \dots, \overline{P}_{\delta-\alpha}, \overline{P}'_1, \dots, \overline{P}'_\alpha\rangle$ obtained from Σ by projecting from $\langle D_4\rangle$ and the relation (23) still holds. In Step 2, since $\delta - m + 1 \leq 2(c+2)+1$ then, *a fortiori*, it holds $\delta - m \leq 2(c+2)+1$. So also in this case Lemma 9.2 can be applied to J, which is spanned by $\delta - m$ points and it is assumed to have dimension smaller then $\delta - m - 1$. With the same argument can be proved the analogous of (24) i.e. $\dim(J) = \delta - m - 1$. Finally, in Step 3 we obtain the analogous of (25) and precisely $\dim(\Sigma) = \dim\langle D_4\rangle + \delta - m$. In the following argument the result 9.3 is used; since it holds for any P, also in this case (26) is verified. Now it is immediate to see that (27) becomes $\beta = \dim\langle \widetilde{D}_4\rangle - \dim(T)$ and we obtain again that

$$\dim\langle D_4\rangle + \delta - m = \dim(\Sigma) \leq \dim(L') + \alpha + \beta.$$

Using the new form of (27) we finally obtain:

$$\dim\langle D_4\rangle + \delta - m \leq \dim(L') + \alpha + \dim\langle \widetilde{D}_4\rangle - \dim(T)$$

which leads to the end of the proof as in the general case. $\diamond$

Remark 9.5. The result stated in 9.4 holds also if at most two of the points $P_1, \dots, P_d$ belong to the same fibre.

The following immediately follows from 9.4:

Corollary 9.6. *For every curve $\overline{D} \sim 2l + (\lambda - 2 - c)l' \subset \overline{S}_0 \cong \mathbb{F}_0$ and for every choice of $P_1, \dots, P_\delta \in \overline{D}$ which do not trivially degenerate $\overline{D}$, there exists a curve $\overline{X}_0 \subset \overline{S}_0$ whose double points are exactly $P_1, \dots, P_\delta$ and whose characters are a, b, λ, where $a + b = g - 3 - c$.* $\diamond$

We conclude this section with some remark about the construction of the bisecant curves D_4 and $\widetilde{D}_4$.

Let us consider a geometrically ruled surface contained in V and having minimum degree; each of such surfaces corresponds to a quotient of type

$$\mathcal{F} := \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \longrightarrow \mathcal{O}(a) \oplus \mathcal{O}(b) \longrightarrow 0 \tag{29}$$

i.e. it is of the type $R := R_{a,b} = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$.

Remark 9.7. Since the above quotients correspond to the sections of $\mathcal{F}(-c)$, tensorizing (29) by $\mathcal{O}(-c)$ we obtain:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(a-c) \oplus \mathcal{O}(b-c) \oplus \mathcal{O} \longrightarrow \mathcal{O}(a-c) \oplus \mathcal{O}(b-c) \longrightarrow 0$$

so

$$h^0(\mathcal{F}(-c)) = \begin{cases} 3 & \text{if} \quad a=b=c \\ 2 & \text{if} \quad a<b=c \\ 1 & \text{if} \quad \quad b<c \end{cases} \quad \text{or, equivalently:} \quad \dim|R_{a,b}| = \begin{cases} 2 & \text{if} \quad a=b=c \\ 1 & \text{if} \quad a<b=c \\ 0 & \text{if} \quad \quad b<c \end{cases}.$$

Remark 9.8. Set $\overline{V} := V_{\overline{S}_0}$ and let as usual Σ be the set of the double points of $\overline{X}_0$. We have the diagram

$$\begin{array}{ccccc} \overline{S}_0 & \subset & \overline{V} & \supset & \overline{R} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_\Sigma} & & \big\downarrow \\ S & \subset & V & \supset & R \end{array}$$

where $\overline{R} := \pi_\Sigma^{-1}(R)$. Setting $\delta_R := \sharp(\Sigma \cap \overline{R})$, i.e. the number of the double points (possibly infinitely near) of $\overline{X}_0$ lying on $\overline{R}$, it is clear that $\deg(\overline{R}) = \deg(R) + \delta_R = a + b + \delta_R$.

Lemma 9.9. *Let $R \in |R_{a,b}|$ be a fixed ruled surface on $V = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$ and $\overline{S}_0 = S_{2,\lambda-2}$ be as usual. Then*

$$\widetilde{D} := \overline{R} \cdot \overline{S}_0 \sim 2l + (\lambda - 2 - c - \delta + \delta_R)l'$$

and there exists a unique bisecant curve $\overline{D} \sim 2l + (\lambda-2-c)l' \subset \overline{S}_0$ such that $\Sigma \subset \overline{D}$ and $\overline{D} \supseteq \widetilde{D}$. Moreover, as soon as R varies in $|R_{a,b}|$, $\overline{D}$ varies in a linear system of dimension $0, 1, 2$ if $b < c$, $a < b = c$, $a = b = c$, respectively.

Proof. Let $H_{\overline{V}}$ be a hyperplane section of $\overline{V}$ containing $\overline{R}$. Since each hyperplane section cannot contain any other unisecant component out of $\overline{R}$, then $H_{\overline{V}} \sim \overline{R} + \tau F_{\overline{V}}$, where $F_{\overline{V}}$ is the generic fibre of $\overline{V}$ and τ is a non negative integer.
Clearly, since $\deg(H_{\overline{V}}) = \deg(\overline{V}) = \deg(V) + \delta = a + b + c + \delta$ and $\deg(\overline{R}) = a + b + \delta_R$, we obtain that

$$\overline{R} \sim H_{\overline{V}} - (c + \delta - \delta_R)F_{\overline{V}}.$$

Taking into account that $H_{\overline{V}} \cdot \overline{S}_0 = 2l + (\lambda-2)l'$ and $F_{\overline{V}} \cdot \overline{S}_0 = l'$, we obtain:

$$\overline{R} \cdot \overline{S}_0 \sim 2l + (\lambda-2)l' - (c + \delta - \delta_R)l' = 2l + (\lambda - 2 - c - \delta + \delta_R)l'$$

as required. Note that only δ_R points of Σ lie on $\widetilde{D}$ and the remaining $\delta - \delta_R$ lie on $\delta - \delta_R$ fibres (possibly coincident) of $\overline{S}_0$, say $l'_1, \ldots, l'_{\delta-\delta_R}$. Hence

$$\Sigma \subset \ \widetilde{D} \cup l'_1 \cup \ldots \cup l'_{\delta-\delta_R} \sim 2l + (\lambda - 2 - c)l'$$

so, setting $\overline{D} := \widetilde{D} \cup l'_1 \cup \ldots \cup l'_{\delta-\delta_R}$, we obtain that $\overline{D}$ is linearly equivalent to $2l + (\lambda-2-c)l'$ and contains both Σ and $\widetilde{D}$, as required. Finally, from the above construction, the divisor $\overline{D}$ is unique, for each $\overline{R}$. The last statement follows from 9.7. $\diamond$

Keeping the notation above, one can immediately compute the degree of $\overline{D}$:

$$\deg(\overline{D}) = \int (2l + (\lambda-2-c)l') \cdot (2l + (\lambda-2)l') = 4(\lambda-2) - 2c. \tag{30}$$

Observe that $\overline{R}$ is the ruled surface generated by the ruling of $\overline{V}$ on $\widetilde{D}$, i.e.

$$\overline{R} = \bigcup\nolimits_{P,Q \in \widetilde{D} \cap F_{\overline{V}}} l_{P,Q}$$

where $l_{P,Q}$ denotes the line passing through the points P and Q. In particular, $\overline{R}$ is determined by $\widetilde{D}$; to stress this fact, we will write $\overline{R} = \overline{R}(\widetilde{D})$.

10. Moduli spaces of 4–gonal curves with $t = 0$

In this section we study the moduli spaces of 4–gonal curves with given invariants; in particular we determine whether they are irreducible and find their dimension. Moreover we give a stratification of these spaces using the invariants introduced in the previous sections.

Let X be a 4–gonal curve of genus g and consider its canonical model $X_K \subset S \subset V \subset \mathbb{P}^{g-1}$, where (from 1.1) S is a surface ruled by conics, of minimum degree and unique, unless g is odd and $\deg(S) = \frac{3g-7}{2}$. In this case, there is a pencil of such surfaces.

Assume that S has invariant $t = 0$, i.e. its (embedded) standard model is the quadric surface $R_{1,1} \subset \mathbb{P}^3$, on which X can be realized as a curve $X' \sim 4l + \lambda l'$ having only double points as singularities: we will write $X = X(g, \lambda)$. Moreover, if $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, then a and b are further invariants of X and we will write $X = X(g, \lambda, a, b)$.

Remark 10.1. If X is as before, then by 6.9 it is clear that it has a finite number of models X', at most $\binom{\delta}{2}$, on $R_{1,1}$ unless g is odd and $\deg(S) = \frac{3g-7}{2}$. In this case, there is a one–dimensional family of such models of X. More precisely, one model comes from another via an elementary transformation of type $elm_{A,B}$, where A and B are two double points of X' as in 6.9. In this way, denoting by X'' another model of X on $R_{1,1}$ and by ξ an elementary transformations as before, the set

$$\Xi_{X'} := \{\xi : X' \longrightarrow X''\}$$

consists of at most $\binom{\delta}{2}$ elements if $\deg(S) \leq \lceil \frac{3g-8}{2} \rceil$, while $\dim(\Xi_{X'}) = 1$ if $\deg(S) = \frac{3g-7}{2}$.
Note that $\Xi_{X'}$ has exactly $\binom{\delta}{2}$ elements in the general case.

Let us denote by $\mathcal{A}_\lambda$ the open subset of the linear system $|4l + \lambda l'|$ on $R_{1,1}$ parametrizing the irreducible curves of such linear system and set

$$\mathcal{W}_g^\lambda := \{X' \in \mathcal{A}_\lambda \mid X = X(g, \lambda) \text{ and } X' \text{ has } \delta \text{ double points on distinct fibres}\}$$
$$\mathcal{W}_g^\lambda(a, b) := \{X' \in \mathcal{W}_g^\lambda \mid X = X(g, \lambda, a, b)\}.$$

Let us denote by $\mathcal{M}_{g,4}$ the moduli space of 4–gonal curves of genus g and let

$$\theta : \mathcal{W}_g^\lambda \longrightarrow \mathcal{M}_{g,4}$$

be the usual projection defined by $\theta(X') = [X]$, where $[X]$ is the isomorphism class of the four–gonal curve X in $\mathcal{M}_{g,4}$. Finally set

$$\mathcal{M}_g^\lambda := \theta(\mathcal{W}_g^\lambda), \quad \mathcal{M}_g^\lambda(a, b) := \theta(\mathcal{W}_g^\lambda(a, b)).$$

It is clear that, in order to compute the dimension of these moduli spaces, we need to find both the dimensions of $\mathcal{W}_g^\lambda$ (resp. $\mathcal{W}_g^\lambda(a, b)$) and of the general fibre of θ.

Remark 10.2. From 8.5, the locally closed subsets $\mathcal{W}_g^\lambda(a, b)$ and hence $\mathcal{W}_g^\lambda$ are not empty, as soon as a, b, λ fulfil $(R_1), (R_2), (R_3)$.

Lemma 10.3. *Let $X', Y' \in \mathcal{W}_g^\lambda$ be two curves on $R_{1,1}$. If $[X] = [Y]$ in $\mathcal{M}_g^\lambda$, there exists an automorphism β of the quadric surface $R_{1,1}$ and a morphism $\xi \in \Xi_{Y'}$ such that*

$$Y' = \xi(\beta(X')).$$

Therefore the dimension of the general fibre of θ is:

$$\dim(\theta^{-1}([X])) = \begin{cases} 7 & \text{if } g \text{ is odd and } \lambda = \lceil \frac{g+2}{2} \rceil \\ 6 & \text{otherwise} \end{cases}.$$

<u>Proof</u>. Since $X \cong Y$, then $X_K \cong Y_K$ and there exists a linear automorphism, α say, of $\mathbb{P}^{g-1}$ such that $\alpha(X_K) = Y_K$.

Let S_X and S_Y be the surfaces, ruled by conics and of minimum degree such that $X_K \subset S_X \subset \mathbb{P}^{g-1}$ and $Y_K \subset S_Y \subset \mathbb{P}^{g-1}$. Assume that these surfaces are unique: therefore $\alpha(S_X) = S_Y$.
Let us consider the diagram (8) for both X and Y: defining with obvious notation $N_X := \langle \varphi_X(K_X - \Phi_X - \Lambda_X)\rangle$ and N_Y analogously, we have

$$\begin{array}{ccccccccc} \mathbb{P}^{g-1} & \supset & S_X & \supset & X_K & \xrightarrow{\pi_{N_X}} & X_{\Phi_X+\Lambda_X} = X' & \subset & R_{1,1}(X) \\ \alpha\big\downarrow & & \alpha\big\downarrow & & \alpha\big\downarrow & & & & \beta\big\downarrow \\ \mathbb{P}^{g-1} & \supset & S_Y & \supset & Y_K & \xrightarrow{\pi_{N_Y}} & Y_{\Phi_Y+\Lambda_Y} = Y' & \subset & R_{1,1}(Y) \end{array}$$

where β is the isomorphism between the quadrics $R_{1,1}(X)$ and $R_{1,1}(Y)$ induced by α. Up to a linear change of coordinates in $\mathbb{P}^3$, we can assume that $R_{1,1}(X) = R_{1,1}(Y)$ so $\beta \in \mathrm{Aut}(R_{1,1})$.
Consider then the curves Y' and $\beta(X')$ lying on $R_{1,1}$: from the construction above, we obtain that they are both models of Y on a quadric. Therefore, applying 10.1, we get that there exists $\xi \in \Xi_{Y'}$ such that $Y' = \xi(\beta(X'))$, as requested.
When S_X and S_Y are not unique they vary in a pencil (see 1.1) and the proof runs in a similar way.
The second part of the statement follows from the first part; namely, it is clear that

$$\dim(\theta^{-1}([X])) = \dim(\mathrm{Aut}(R_{1,1})) + \dim(\Xi_X).$$

On one hand, observe that $\mathrm{Aut}(R_{1,1}) \cong \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \cong PGL(2) \times PGL(2)$ has dimension 6.
On the other hand, by 10.1,

$$\dim(\Xi_X) = \begin{cases} 1 & \text{if } g \text{ is odd and } \deg(S) = \frac{3g-7}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Finally note that (using 4.4):

$$g + \lambda - 5 = \deg(S) = \frac{3g-7}{2}$$

or, equivalently

$$\lambda = \frac{g+3}{2} = \left\lceil \frac{g+2}{2} \right\rceil$$

where the last equality holds since g is odd. ◇

Let us recall (see Section 8) that, if $X' \in \mathcal{W}_g^\lambda$ then $X' \subset R_{1,1} \cong \mathbb{F}_0$ and $\varphi_{4l+\lambda l'} : \mathbb{F}_0 \longrightarrow S' \subset \mathbb{P}^{5\lambda+4}$; in particular, we can associate to X' a hyperplane H_X of $\mathbb{P}^{5\lambda+4}$. By 8.2 we have that X' has $P_1, \ldots, P_\delta$ as double points if and only if H_X contains the linear space

$$L_{P_1,\ldots,P_\delta} := \langle T_{P_1}(S'), \ldots, T_{P_\delta}(S')\rangle.$$

In this way we can identify $\mathcal{W}_g^\lambda$ with its image via the injective morphism

$$\begin{array}{rl} i: & \mathcal{W}_g^\lambda \longrightarrow \check{\mathbb{P}}^{5\lambda+4} \\ & X' \mapsto H_X \end{array}.$$

In order to compute the dimension of $\mathcal{W}_g^\lambda$ and of $\mathcal{W}_g^\lambda(a,b)$ and to prove their irreducibility, we need further preliminary observations.

Remark 10.4. The Key–Lemma 9.4 has been proved under the assumption that $(P_1, \ldots, P_\delta)$ are distinct points. For instance, if $\delta = 2$, this result says that

$$\dim L_{P_1,P_2} = \dim\langle T_{P_1}(S'), T_{P_2}(S')\rangle = 5.$$

If P_2 is infinitely near to P_1, given a local system of coordinates of S' in a neighbourhood of P_1, the tangent plane to S' at P_1 is generated by P_1 and the first derived vectors both along the bisecant $\widetilde{D}$ and along the fibre l'_1. Hence it is easy to see that the linear space L_{P_1,P_2} is generated by the above generators of $T_{P_1}(S')$ and by two further second derived vectors and a third derived vector. One can show that all of them are linearly independent so, also in this case, $\dim L_{P_1,P_2} = 5$.
It is not difficult to prove that, if k is any integer $(1 \le k \le \delta - 1)$ and the considered points are $P_1, P_2, \ldots, P_{k+1}, \ldots, P_\delta$ where $P_2, \ldots, P_{k+1}$ are infinitely near to P_1, then

$$\dim L_{P_1,\ldots P_\delta} \ge 3\delta - k.$$

Lemma 10.5. *Let us consider the morphism*

$$\begin{aligned}\Psi:\ \mathcal{W}_g^\lambda &\longrightarrow \mathrm{Sym}^\delta(R_{1,1})\\ X' &\mapsto (P_1,\dots,P_\delta)\end{aligned}$$

where $\Sigma = P_1+\cdots+P_\delta$ is the singular locus of $X'\subset R_{1,1}$. Then the general fibre of Ψ has dimension

i) $\dim(\Psi^{-1}(P_1,\dots,P_\delta)) = 5\lambda+4-3\delta$ *if $P_1,\dots,P_\delta$ are distinct points;*

ii) $\dim(\Psi^{-1}(P_1,\dots,P_\delta)) \leq 5\lambda+3-3\delta+k$ *if $P_2,\dots,P_{k+1}$ are infinitely near to P_1, for some $k\geq 1$.*

Proof. By definition, $\mathcal{W}_g^\lambda$ consists of the irreducible curves of type $(4,\lambda)$ on $R_{1,1}$ having δ double points on distinct fibres. So, taking into account the above injective morphism $i:\ \mathcal{W}_g^\lambda \longrightarrow \check{\mathbb{P}}^{5\lambda+4}$ and the fact that $X'\in\mathcal{W}_g^\lambda$ has $P_1,\dots,P_\delta$ as double points if and only if the hyperplane $H_X := i(X')$ contains the linear space $L_{P_1,\dots,P_\delta}$, it is clear that the general fibre $\Psi^{-1}(P_1,\dots,P_\delta)$ is isomorphic to an open subset of $\{H\in\check{\mathbb{P}}^{5\lambda+4}\mid H\supset L_{P_1,\dots,P_\delta}\}$, since the general hyperplane containing $L_{P_1,\dots,P_\delta}$ contains the tangent planes to S' only at the choosen points. This means exactly that

$$\dim(\Psi^{-1}(P_1,\dots,P_\delta)) = 5\lambda+4-(\dim L_{P_1,\dots P_\delta}+1).$$

i) If $P_1,\dots,P_\delta$ are distinct, then in the Key–Lemma 9.4 we have shown that the dimension of $L_{P_1,\dots,P_\delta}$ is $3\delta-1$ independently on the position of the considered points. So, in this case, $\Psi^{-1}(P_1,\dots,P_\delta)$ is irreducible of dimension $5\lambda+4-3\delta$.

ii) If $P_1,\dots,P_\delta$ are not distinct – as in the assumption – then the fibre of Ψ could have bigger dimension. Nevertheless, we can get an upper bound on this dimension by taking into account 10.4, obtaining that $\dim(\Psi^{-1}(P_1,\dots,P_\delta))$ is at most $5\lambda+4-(3\delta-k+1)$ and this proves the second part of the statement. ⋄

Proposition 10.6. *For each λ satisfying*

$$\frac{g+3}{3}\leq\lambda\leq\left\lceil\frac{g+2}{2}\right\rceil \tag{R_1}$$

the locally closed subset $\mathcal{W}_g^\lambda$ is irreducible of dimension $g+2\lambda+7$.

Proof. Setting $\mathrm{Sym} := \mathrm{Sym}^\delta(R_{1,1})$, consider the map $\Psi:\ \mathcal{W}_g^\lambda\to\mathrm{Sym}$ defined in 10.5. Note that Ψ is dominant and $\dim(\mathrm{Sym}) = 2\delta$.
Recall also that the δ singular points of the general curve $X'\in\mathcal{W}_g^\lambda$ are in general position on $R_{1,1}$ by 9.4.
If $P_1,\dots,P_\delta$ are distinct points, by 10.5 we get that $\dim(\Psi^{-1}(P_1,\dots,P_\delta)) = 5\lambda+4-3\delta$. Therefore

$$\begin{aligned}\dim(\mathcal{W}_g^\lambda) &= \dim(\Psi^{-1}(P_1,\dots,P_\delta))+\dim(\mathrm{Sym}) =\\ &= 5\lambda+4-\delta =\\ &= g+2\lambda+7\end{aligned}$$

where the last equality follows from $\delta = 3(\lambda-1)-g$.
Assume now that $P_2,\dots,P_{k+1}$ are infinitely near to P_1 for some $k\geq 1$. Then the fibre of Ψ at the point $(P_1,\dots,P_\delta)\in\mathrm{Sym}$ has dimension at most $5\lambda+3-3\delta+k$ by 10.5. The difference between such integer and $5\lambda+4-3\delta$ is at most

$$k-1<2k=\mathrm{codim}_{\,\mathrm{Sym}}(\Delta)$$

where $\Delta := \{(Q_1,\dots,Q_\delta)\in\mathrm{Sym}\mid Q_1=\cdots=Q_{k+1}\}$. Clearly Δ is a closed subset of Sym and contains the considered element $(P_1,\dots,P_\delta)$. Therefore, the variety consisting of the fibres on the points of Δ is a proper closed subset of $\mathcal{W}_g^\lambda$. ⋄

Remark 10.7. Recall that $\mathcal{M}_{g,4}$ is a closed irreducible subset of the moduli space $\mathcal{M}_g$ and has dimension $2g+3$. Let us set the maximum value of λ (see (R_1)):

$$\lambda_{\max} := \left\lceil\frac{g+2}{2}\right\rceil.$$

Then, from 10.6
$$\dim(\mathcal{W}_g^{\lambda_{\max}}) = g + 2\lambda_{\max} + 7.$$
Let us recall that the fibre of $\theta : \mathcal{W}_g^{\lambda_{\max}} \to \mathcal{M}_g^{\lambda_{\max}}$ has dimension either 6 or 7, accordingly to wheter g is even or odd, respectively (from 10.3). Hence
$$\dim(\mathcal{M}_g^{\lambda_{\max}}) = \begin{cases} g + 2\frac{g+2}{2} + 1 = 2g+3, & \text{if } g \text{ is even ;} \\ g + 2\frac{g+3}{2} = 2g+3, & \text{if } g \text{ is odd.} \end{cases}$$
Therefore, in both cases, we have that $\dim(\mathcal{M}_g^{\lambda_{\max}}) = \dim(\mathcal{M}_{g,4})$; in other words, the general 4–gonal curve has invariant $\lambda_{\max}$.

Remark 10.8. We know that, if $t > 0$, then X admits a standard model $X' \subset R_{1,t+1}$. Nevertheless, also in this case, it is possible to define another model of X, X'' say, on a quadric surface $R_{1,1}$. Clearly, in this situation, X'' will have not only double points as singularities, but also triple points.
Namely, let $Q_1, \ldots, Q_t$ be simple points of X', belonging to t distinct fibres of $R_{1,t+1}$ and consider the projection from these points:
$$\begin{tikzcd} X' \arrow[d] & \subset & R_{1,t+1} \arrow[d, "\pi_{Q_1,\ldots,Q_t}"] \\ X'' & \subset & R_{1,1} \end{tikzcd}$$
Since X' meets each fibre of $R_{1,t+1}$ in the four points of the gonal divisor, the singularities of X'' are the δ double points of X' and, in addition, t triple points, all of them belonging to the same line l.
It is clear that the closure $\overline{\mathcal{W}_g^\lambda}$ of $\mathcal{W}_g^\lambda$ in $\mathcal{A}_\lambda$ contains also the curves of invariants g, λ and $t > 0$ and it is not difficult to see that the closed subset consisting of such curves has dimension smaller then $\dim(\mathcal{W}_g^\lambda)$.

Using 10.2, 10.3, 10.6, 10.7 and 10.8, we immediately obtain the following result, which is the first part of the Main Theorem stated in the Introduction (here $\overline{\mathcal{M}}_g^\lambda$ denotes the closure of $\mathcal{M}_g^\lambda$ in the moduli space $\mathcal{M}_{g,4}$ of 4–gonal curves):

Theorem 10.9. *There exists a stratification of the moduli space $\mathcal{M}_{g,4}$ of 4–gonal curves given by:*
$$\mathcal{M}_{g,4} = \overline{\mathcal{M}}_g^{\lceil \frac{g+2}{2} \rceil} \supset \overline{\mathcal{M}}_g^{\lceil \frac{g}{2} \rceil} \supset \cdots \supset \overline{\mathcal{M}}_g^{\lambda} \supset \cdots \supset \overline{\mathcal{M}}_g^{\lceil \frac{g+3}{3} \rceil}$$
and $\overline{\mathcal{M}}_g^\lambda$ are irreducible locally closed subsets of dimension $g+2\lambda+1$, for each λ satisfying $\frac{g+3}{3} \leq \lambda < \lceil \frac{g+2}{2} \rceil$.
$\diamond$

In order to show the second part of the Main Theorem, let us start with some preliminary fact.

We keep the notation of 9.9, where $\widetilde{D}$ denotes a divisor of $\overline{S}_0 = S_{2,\lambda-2} \subset \mathbb{P}^{g-1+\delta}$ linearly equivalent to $2l + (\lambda - 2 - c - \delta + \delta_R)l'$ and containing δ_R points among $P_1, \ldots, P_\delta$.
Recall also that, referring to Section 7, the unisecant $\overline{A} \subset \overline{V}$ is the preimage, via π, of the (unique if $a < b$) unisecant of degree a of V. Moreover, $\overline{R} := \pi^{-1}(R)$, where $R := R_{a,b}$, so $\overline{A} \subset \overline{R} = \overline{R}(\widetilde{D})$ as described in 9.9.
In the forthcoming computations we will use a few times the following relations (coming from $a+b+c = g-3$ and from (17)):
$$c = g - 3 - a - b, \qquad 3\lambda = \delta + g + 3. \tag{31}$$

Lemma 10.10. *Let $\widetilde{D} \subset \overline{S}_0$ and $\overline{R} := \overline{R}(\widetilde{D})$ be as before. Let $\overline{A} \in Un^{a+\delta_R}(\overline{R})$ and $\Gamma := \widetilde{D} \cdot \overline{A}$. Assume that $a \geq (g - \lambda - 1)/2$. Then:*
i) $\deg(\Gamma) = 4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R)$;
ii) $h^0(\mathcal{O}_{\overline{R}}(\overline{A})) = h^0(\mathcal{O}_{\widetilde{D}}(\Gamma))$;
iii) assume also that $\delta_R = \delta$ and either $a > (g - \lambda - 1)/2$ or $a = (g - \lambda - 1)/2$ and $a < b$; then:
$$H^0(\mathcal{O}_{\overline{R}}(\overline{A})) \cong H^0(\mathcal{O}_{\widetilde{D}}(\Gamma)).$$

Proof. $i)$ Recall that, keeping the notation in 9.9, $\overline{D} = \widetilde{D} + (\delta - \delta_R)l'$. So $\deg(\widetilde{D}) = \deg(\overline{D}) - 2(\delta - \delta_R)$ since $\overline{S}_0$ is ruled by conics. Hence, using (30), we obtain that $\deg(\widetilde{D}) = 4(\lambda - 2) - 2(c + \delta - \delta_R)$. Therefore, applying (IF) and 9.8, we have that

$$\begin{aligned}\deg(\Gamma) &= 2\deg(\overline{A}) + \deg(\widetilde{D}) - 2\deg(\overline{R}) = \\ &= 2(a + \delta_R) + 4(\lambda - 2) - 2(c + \delta - \delta_R) - 2(a + b + \delta_R) = \\ &= 4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R).\end{aligned}$$

$ii)$ Let us show first that Γ is a non special divisor on $\widetilde{D}$. Since $\widetilde{D}$ is of type $(2, \lambda - 2 - c - (\delta - \delta_R))$ on the quadric, then $p_a(\widetilde{D}) = \lambda - 3 - c - (\delta - \delta_R)$. A sufficient condition in order to have Γ non special is $\deg(\Gamma) > 2p_a(\widetilde{D}) - 2$, or, equivalently:

$$4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R) > 2(\lambda - 3 - c - (\delta - \delta_R)) - 2 \quad \iff \quad \lambda - b > 0$$

and this is true since $b \leq \lambda - 2$. Therefore $h^1(\mathcal{O}_{\widetilde{D}}(\Gamma)) = 0$ and, by Riemann–Roch theorem, using also (31), we obtain that

$$h^0(\mathcal{O}_{\widetilde{D}}(\Gamma)) - 1 = \deg(\Gamma) - p_a(\widetilde{D}) = a - b + \delta_R + 1.$$

Moreover $h^0(\mathcal{O}_{\overline{R}}(\overline{A})) - 1 = \dim_{\overline{R}}(|\overline{A}|) = \dim(\mathrm{Un}^{a+\delta_R}(\overline{R})) = a - b + \delta_R + 1$ by (UF). Hence we obtain that

$$h^0(\mathcal{O}_{\widetilde{D}}(\Gamma)) = a - b + \delta_R + 2 = h^0(\mathcal{O}_{\overline{R}}(\overline{A})).$$

$iii)$ In order to prove the claim, consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{\widetilde{D}/\overline{R}}(\overline{A}) \longrightarrow \mathcal{O}_{\overline{R}}(\overline{A}) \longrightarrow \mathcal{O}_{\widetilde{D}}(\Gamma) \longrightarrow 0. \tag{32}$$

By $ii)$, it suffices to show that the map $f:\ H^0(\mathcal{O}_{\overline{R}}(\overline{A})) \to H^0(\mathcal{O}_{\widetilde{D}}(\Gamma))$ induced by (32) is injective.
Clearly this holds if and only if there exists a unique $\overline{A} \in Un^{a+\delta_R}(\overline{R})$ passing through Γ and this holds if $\int \overline{A}^{\,2} < \deg(\Gamma)$. From (IF) and 9.8 we obtain that

$$\int \overline{A}^{\,2} = 2\deg(\overline{A}) - \deg(\overline{R}) = 2(a + \delta_R) - (a + b + \delta_R) = a - b + \delta_R.$$

Therefore the condition $\int \overline{A}^{\,2} < \deg(\Gamma)$ becomes

$$a - b + \delta_R < 4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R).$$

Using again (31), the above inequality is equivalent to:

$$\lambda - g + a + b + 1 - (\delta - \delta_R) > 0.$$

By assumption $\delta - \delta_R = 0$, so

$$a + b > g - \lambda - 1$$

and using the further assumptions on a and b, the claim is proved. $\diamond$

Before stating the second part of the Main Theorem, let us set

$$\epsilon := \begin{cases} 0, & \text{if } b < c \\ 1, & \text{if } a < b = c \\ 2, & \text{if } a = b = c \end{cases} \quad , \quad \tau := \begin{cases} 0, & \text{if } a < b \\ 1, & \text{if } a = b \end{cases} \quad \text{and} \quad \xi := \begin{cases} 1, & \text{if } \lambda = \frac{g+3}{2} \\ 0, & \text{otherwise} \end{cases} .$$

Theorem 10.11. *Let g, λ, a, b be positive integers satisfying (R_1), (R_2), (R_3) and $c = g-3-a-b$. If $a \geq (g-\lambda-1)/2$ then $\mathcal{M}_g^\lambda(a,b)$ is an irreducible variety of dimension $2(2a+b+\lambda)+10-g-\epsilon-\tau-\xi$.*

Proof. From 10.2 and 10.3, it is enough to show that $\mathcal{W}_g^\lambda(a,b)$ is irreducible of the right dimension.
Keeping the notation in 10.5, set $Y_g^\lambda(a,b) := \Psi(\mathcal{W}_g^\lambda(a,b))$.
Claim: $\Psi^{-1}(Y_g^\lambda(a,b)) \subset \mathcal{W}_g^\lambda(a,b)$.
This is equivalent to the following property: let $X'' \in \mathcal{W}_g^\lambda$ be such that $\Psi(X'') = (P_1, \ldots, P_\delta) = \Psi(X')$, where $X' \in \mathcal{W}_g^\lambda(a,b)$; then $X'' \in \mathcal{W}_g^\lambda(a,b)$. This is true, since $\pi_{\langle P_1,\ldots,P_\delta\rangle}(\overline{V})$ is the scroll $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ associated both to X' and to X'' and this proves the claim.
Step 1. Irreducibility and dimension of $\mathcal{W}_g^\lambda(a,b)$.
From the claim above we can consider the restriction of Ψ

$$\psi : \mathcal{W}_g^\lambda(a,b) \longrightarrow Y_g^\lambda(a,b).$$

From 10.5, $\dim(\psi^{-1}(P_1,\ldots,P_\delta)) = 5\lambda+4-3\delta$ if $P_1,\ldots,P_\delta$ are distinct points.
With the same argument as the one in the proof of 10.6, in the case of infinitely near points one easily shows that the variety consisting of the fibres on the points of Δ is a proper closed subset of $\mathcal{W}_g^\lambda(a,b)$. For this reason, $\mathcal{W}_g^\lambda(a,b)$ is irreducible if $Y_g^\lambda(a,b)$ is irreducible and

$$\dim(\mathcal{W}_g^\lambda(a,b)) = \dim(Y_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta. \tag{33}$$

Step 2. Irreducibility and dimension of $Y_g^\lambda(a,b)$.
Recall that the singular locus $\Sigma = P_1 + \cdots + P_\delta$ of $X' \subset R_{1,1}$ is contained in a suitable bisecant curve $\overline{D} \sim 2l + (\lambda-2-c)l' \subset R_{1,1}$ by 9.9 (there the result concerns $\overline{S}_0$, here $R_{1,1}$).
It is not hard to show that there exists an open subset, Y^0 say, of $Y_g^\lambda(a,b)$ whose elements $(P_1,\ldots,P_\delta)$ fulfil the following property: there exists $\overline{D} \in |2l+(\lambda-2-c)l'|$ not containing fibres and such that $P_1,\ldots,P_\delta \in \overline{D} \cap \overline{A}$, for a suitable $\overline{A} \in \mathrm{Un}^{a+\delta}(\overline{R})$, where $\overline{R} := \overline{R}(\overline{D})$. In particular, on this subset $\delta_R = \delta$.
Let us check that the above condition is compatible with the degrees of the involved divisors i.e., setting $\Gamma := \overline{D} \cap \overline{A}$, we must have that $\delta \leq \deg(\Gamma)$. From 10.10 (ii), taking into account that here $\delta = \delta_R$ and using (31) as usual, it is easy to see that $\deg(\Gamma) = 2a+\lambda-g+1+\delta \geq \delta$, since $2a+\lambda-g+1 \geq 0$: namely this is equivalent to $a \geq (g-\lambda-1)/2$, which holds by assumption.
Consider then the following correspondence:

$$Z_{a,b}^\lambda \quad \subset \quad |2l+(\lambda-2-c)l'| \times \mathrm{Sym}^\delta(R_{1,1})$$

defined by:

$$Z_{a,b}^\lambda := \left\{ (\overline{D}, P_1, \ldots, P_\delta) \mid \text{there exists } \overline{A} \in \mathrm{Un}^{a+\delta}(\overline{R}(\overline{D})) \text{ such that } P_1,\ldots,P_\delta \in \overline{D} \cap \overline{A} \right\}.$$

Consider now the two canonical projections, where Ω is the open subset of $|2l+(\lambda-2-c)l'|$ consisting of curves not containing fibres:

$$\begin{array}{ccccc} & & Z_{a,b}^\lambda & & \\ & {}^{p}\swarrow & & \searrow_{q} & \\ |2l+(\lambda-2-c)l'| \supset \Omega & & & & Y^0 \subset Y_g^\lambda(a,b) \subset \mathrm{Sym}^\delta(R_{1,1}) \end{array}.$$

By 9.9, every element $(P_1,\ldots,P_\delta)$ of Y^0 determines either a unique $\overline{D} \sim 2l+(\lambda-2-c)l'$ (if $b<c$) or a pencil (if $a<b=c$) or a two–dimensional linear system (if $a=b=c$) of such curves. This implies that the general fibre of q is irreducible of dimension ϵ, where $\epsilon = 0,1,2$ as soon as $b<c$, $a<b=c$, $a=b=c$, respectively. Furthermore p is surjective by 9.6.
Denoting by $Z_{\overline{D}} := p^{-1}(\overline{D})$ any fibre of p, we have that: if $Z_{\overline{D}}$ is irreducible, then $Y_g^\lambda(a,b)$ is irreducible and

$$\begin{aligned} \dim(Y_g^\lambda(a,b)) &= \dim(Z_{a,b}^\lambda) - \epsilon = \dim(Z_{\overline{D}}) + \dim(|\overline{D}|) - \epsilon \\ &= \dim(Z_{\overline{D}}) + 3(\lambda-1-c) - 1 - \epsilon. \end{aligned} \tag{34}$$

Step 3. Irreducibility and dimension of $Z_{\overline{D}}$
It is clear that

$$Z_{\overline{D}} \cong \{(P_1, \dots, P_\delta) \in \mathrm{Sym}^\delta(\overline{D}) \mid \text{there exists } \overline{A} \in Un^{a+\delta}(\overline{R}) \text{ such that } P_1, \dots, P_\delta \in \overline{D} \cap \overline{A}\}.$$

In order to compute the dimension and to prove the irreducibility of $Z_{\overline{D}}$, consider the following correspondence (where $\Gamma = \overline{D} \cap \overline{A}$ is as before):

$$T_{\overline{D}} := \{(P'_1, \dots, P'_\delta, \overline{A}) \mid P'_1, \dots, P'_\delta \in \Gamma\} \subset \mathrm{Sym}^\delta(\overline{D}) \times Un^{a+\delta}(\overline{R})$$

and the two projections:

$$\begin{array}{ccc} & T_{\overline{D}} & \\ & {\scriptstyle\pi_1} \swarrow \quad \searrow {\scriptstyle\pi_2} & \\ \mathrm{Sym}^\delta(\overline{D}) & & Un^{a+\delta}(\overline{R}) \end{array}$$

Obviously, $Im(\pi_1) = Z_{\overline{D}}$ and π_2 is a finite surjective morphism; hence, denoting by τ the dimension of the fibres of π_1, we obtain:

$$\dim(Z_{\overline{D}}) = \dim(T_{\overline{D}}) - \tau = \dim(Un^{a+\delta}(\overline{R})) - \tau = a - b + \delta + 1 - \tau. \tag{35}$$

Let us find the possible values of τ.
In the proof of 10.10 (iii) we show that $\int \overline{A}^{\,2} = a - b + \delta$; with the same argument used there to prove the uniqueness of the unisecant $\overline{A}$ passing through a certain divisor, it is immediate to see that

$$\tau = 0 \quad \Leftrightarrow \quad \int \overline{A}^{\,2} < \delta \quad \Leftrightarrow \quad a - b + \delta < \delta \quad \Leftrightarrow \quad a < b.$$

With the same argument we obtain:

$$\tau \geq 1 \quad \Leftrightarrow \quad \int \overline{A}^{\,2} \geq \delta \quad \Leftrightarrow \quad a - b + \delta \geq \delta \quad \Leftrightarrow \quad a = b \text{ and } \int \overline{A}^{\,2} = \delta.$$

Hence, necessarily, $\tau = 1$ and $a = b$.
We are left to show that $Z_{\overline{D}}$ is irreducible. Since $Z_{\overline{D}} = \pi_1(T_{\overline{D}})$, it is enough to show that $T_{\overline{D}}$ itself is irreducible.
Assume first that

$$a > \frac{g - \lambda - 1}{2} \quad \text{or} \quad a = \frac{g - \lambda - 1}{2} < b.$$

It follows from 10.10 (iii) that $H^0(\mathcal{O}_{\overline{R}}(\overline{A})) \cong H^0(\mathcal{O}_{\overline{D}}(\Gamma))$, hence

$$T_{\overline{D}} \cong \{(P'_1, \dots, P'_\delta, \Gamma') \mid P'_1, \dots, P'_\delta \in \Gamma'\} \subset \mathrm{Sym}^\delta(\overline{D}) \times |\Gamma|.$$

Consider the morphism associated to $|\Gamma|$:

$$\varphi_\Gamma : \quad \overline{D} \longrightarrow \mathbb{P}^r$$

where $r = \dim|\Gamma| = a - b + \delta + 1$ (as computed in the proof of 10.10 (ii)); if $\overline{D}'$ denotes the image of $\overline{D}$ in $\mathbb{P}^r$, it is clear that

$$T_{\overline{D}} \cong \{(P'_1, \dots, P'_\delta, H) \mid P'_1, \dots, P'_\delta \in H \cap \overline{D}'\} \subset \mathrm{Sym}^\delta(\overline{D}') \times \check{\mathbb{P}}^r.$$

The irreducibility of $T_{\overline{D}}$ is a consequence of the forthcoming lemma 10.12.
Finally, we have to consider the last case:

$$a = \frac{g - \lambda - 1}{2} = b.$$

Since $c = g - 3 - (a + b) = \lambda - 2$, from 10.10 (i) we have

$$\deg(\Gamma) = 4(\lambda - 2) - 2b - 2c = 3\lambda - 3 - g = \delta.$$

Therefore $\pi_2 : T_{\overline{D}} \to Un^{a+\delta}(\overline{R})$ is an isomorphism, hence $T_{\overline{D}}$ is irreducible of dimension $\delta + 1$ (since $a = b$).
Finally observe that, if $\overline{\overline{D}} \notin \Omega$ in Step 2, then one can easily prove that $\dim(Z_{\overline{\overline{D}}}) = a - b + \delta_R + 1 - \tau$. In particular, $\dim(Z_{\overline{\overline{D}}}) < \dim(Z_{\overline{D}})$ hence $p^{-1}\left(|2l + (\lambda - 2 - c)l'| \setminus \Omega\right)$ is a Zariski locally closed subset of $Z^\lambda_{a,b}$.

Step 4. Final computation

We can now compute the dimension of the moduli space using (33), (34), (35) and (31):

$$\begin{aligned}\dim(\mathcal{W}_g^\lambda(a,b)) &= \dim(Y_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta = \\ &= \dim(Z_{\overline{D}}) + 3(\lambda - 1 - c) + 3 - \epsilon + 5\lambda - 3\delta = \\ &= 2(2a + b + \lambda) + 16 - g - \epsilon - \tau\end{aligned}$$

hence, from 10.3, we obtain

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \dim(\mathcal{W}_g^\lambda(a,b)) - 6 - \xi = 2(2a + b + \lambda) + 10 - g - \epsilon - \tau - \xi$$

and this proves the claim. ◇

We are left to show the following fact:

Lemma 10.12. *Let $X \subset \mathbb{P}^r$ be a (smooth) irreducible curve, k an integer such that $k \leq \deg(X)$ and let*

$$V_X := \{(P_1, \ldots, P_k; H) \mid P_1, \ldots, P_k \in H \cap X\} \subset \mathrm{Sym}^k(X) \times \check{\mathbb{P}}^r.$$

Then the variety V_X is irreducible.

Proof. It is a straightforward generalization of the argument used in the proof of the Uniform Position Lemma, [**9**]. ◇

Now we are going to prove the last part of the Main Theorem. We need first some preliminary results; let us recall that, if $a < (g - \lambda - 1)/2$, then $\overline{A} \subset \overline{S}_0 \subset \overline{V}$ (from 7.3).

Lemma 10.13. *Let $a < (g - \lambda - 1)/2$ and $[X] \in \mathcal{M}_g^\lambda(a,b)$. Then in $\theta^{-1}([X])$ there exists a curve $X' \subset R_{1,1}$ such that $\overline{A} \sim l$. In particular, $\deg(\overline{A}) = \lambda - 2$ and $\delta_A = \lambda - 2 - a$.*

Proof. Let $\overline{A} \sim l + \alpha l' \subset \overline{S}_0 = \varphi_{2l+(\lambda-2)l'}(\mathbb{F}_0) \subset \mathbb{P}^{3\lambda-4}$ and assume $\alpha \geq 1$. Since

$$\deg_{\overline{S}_0}(\overline{A}) = \int (l + \alpha l') \cdot (2l + (\lambda - 2)l') = \lambda - 2 + 2\alpha \tag{36}$$

and $\deg(A) = a \leq \lambda - 2$ (from 7.1), the number of double points of $\overline{X}_0$ lying on $\overline{A}$ is, from (11), $\delta_A = \deg(\overline{A}) - \deg(A) = \lambda - 2 + 2\alpha - a \geq 2\alpha$. Therefore, since $\overline{A}$ meets each line of the ruling l of $\overline{S}_0$ in α points, there are at least two double points of $\overline{X}_0$, N_1 and N_2 say, belonging to $\overline{A}$ and not belonging to a same line l.

Consider now the isomorphism

$$\varphi_{l+2l'}: \quad R_{1,1} \cong \overline{S}_0 \longrightarrow \tilde{S} \cong R_{2,2}$$

and set $\tilde{A} := \varphi(\overline{A}) \sim \tilde{l} + \alpha\tilde{l}'$; for simplicity, we still denote by N_1 and N_2 the images of these points in $\widetilde{S}$. Clearly $\deg(\tilde{A}) = \alpha + 2$ and the projection

$$\pi_{\langle N_1, N_2\rangle}: \quad \tilde{S} \longrightarrow R_{1,1}$$

maps $\tilde{A}$ to a unisecant curve $\overline{A}^*$ of degree α (since $N_1, N_2 \in \tilde{A}$) lying on $R_{1,1}$; hence $\overline{A}^* \sim l + (\alpha - 1)l'$; in particular, from (36), $\deg_{\overline{S}_0}(\overline{A}^*) = \lambda - 2 + 2(\alpha - 1)$.

Set $X' := \left(\pi_{\langle N_1,N_2\rangle} \circ \varphi_{l+2l'}\right)(X) \subset R_{1,1}$ and $A^* \subset S$ be the curve corresponding to $\overline{A}^* \subset R_{1,1}$. Since the number of the double points of X' lying on $\overline{A}^*$ is $\delta_A - 2$, we get that

$$\deg(A^*) = \deg_{\overline{S}_0}(\overline{A}^*) - (\delta_A - 2) = \lambda - 2 + 2\alpha - \delta_A = a = \deg(A)$$

and this implies that $A^* = A$. Iterating this procedure we obtain a model of X such that $\alpha = 0$, hence $\overline{A} \sim l$ and the other requirements are fulfilled. ◇

Corollary 10.14. *Let $a < (g-\lambda-1)/2$ and let $\widetilde{\mathcal{W}}_g^\lambda(a,b) \subset \mathcal{W}_g^\lambda(a,b)$ be the following set:*

$$\widetilde{\mathcal{W}}_g^\lambda(a,b) := \{X' \in \mathcal{W}_g^\lambda(a,b) \mid X' \subset R_{1,1}, \overline{A} \sim l\}.$$

Then the restriction

$$\theta :\ \widetilde{\mathcal{W}}_g^\lambda(a,b) \longrightarrow \mathcal{M}_g^\lambda(a,b)$$

is surjective and the fibres have dimension 6 unless g is odd and $\lambda = (g+3)/2$: in this case they have dimension 7.

Proof. The surjectivity is immediate by 10.13 and the dimension of the fibres can be computed with the same argument of 10.3. $\diamond$

Let us set

$$\epsilon := \begin{cases} 0, & \text{if } b < c \\ 1, & \text{if } a < b = c \end{cases} \quad \text{and} \quad \xi := \begin{cases} 1, & \text{if } \lambda = \frac{g+3}{2} \\ 0, & \text{otherwise} \end{cases}.$$

Note that the case $a = b = c$ (which corresponds to $\epsilon = 2$ in 10.11) here does not occur. Namely we now consider the range $a < (g-\lambda-1)/2$: the relation $a = b = c$ would contradict (R_1).

Theorem 10.15. *Let g, λ, a, b be positive integers satisfying (R_1), (R_2), (R_3) and $c = g-3-a-b$. If $a < (g-\lambda-1)/2$ then $\mathcal{M}_g^\lambda(a,b)$ is an irreducible variety of dimension $2(a+b)+\lambda+8-\epsilon-\xi$.*

Proof. Using 10.14, we can slightly modify the construction in 10.11; essentially we use $\widetilde{\mathcal{W}}_g^\lambda(a,b)$ instead of $\mathcal{W}_g^\lambda(a,b)$. In particular, we consider models $X' \subset R_{1,1}$ of X such that $\overline{A} \sim l$ and $\overline{A} \subset \overline{D} \sim 2l + (\lambda-2-c)l'$. Namely, if $\overline{A} \not\subset \overline{D}$, then $\delta_A \leq \overline{A} \cdot \overline{D}$; but $\delta_A = \lambda - 2 - a$ (from 10.13) while $\overline{A} \cdot \overline{D} = \lambda - 2 - c$ and this is impossible since $a < c$.
Setting $\widetilde{Y}_g^\lambda(a,b)$ the image of $\widetilde{\mathcal{W}}_g^\lambda(a,b)$ via the map $\Psi : \mathcal{W}_g^\lambda \to \mathrm{Sym}^\delta(R_{1,1})$ we have

$\widetilde{Y}_g^\lambda(a,b) = \{(P_1,\ldots,P_\delta) \mid \text{there exist } \overline{A} \in |l|, \overline{B} \in |l+(\lambda-2-c)l'| : P_1,\ldots,P_{\lambda-2-a} \in \overline{A}, P_{\lambda-1-a},\ldots,P_\delta \in \overline{B}\}$

and the analogous of (33) holds:

$$\dim(\widetilde{\mathcal{W}}_g^\lambda(a,b)) = \dim(\widetilde{Y}_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta. \tag{37}$$

Consider the following correspondence

$$Z_{a,b}^\lambda \quad \subset \quad |l| \times |l + (\lambda-2-c)l'| \times \mathrm{Sym}^\delta(R_{1,1})$$

defined by:

$$Z_{a,b}^\lambda := \left\{(\overline{A}, \overline{B}, (P_1,\ldots,P_\delta)) \mid P_1,\ldots,P_{\lambda-2-a} \in \overline{A}, P_{\lambda-1-a},\ldots,P_\delta \in \overline{B}\right\}.$$

Note that b is determined from a and c. Consider now the two canonical projections:

$$\begin{array}{ccc} & Z_{a,b}^\lambda & \\ & {}^{p}\swarrow \quad \searrow_{q} & \\ |l| \times |l+(\lambda-2-c)l'| & & \widetilde{Y}_g^\lambda(a,b) \subset \mathrm{Sym}^\delta(R_{1,1}) \end{array}.$$

With the same argument as in 10.11, one can see that the fibres of q are irreducible of dimension ϵ. Note that, in this case, ϵ can assume only the values 0 and 1, since the assumption $a < (g-\lambda-1)/2$ implies $a < b$, otherwise $a + b < g - \lambda - 1$, against (R_3) (see 8.5).
Note that p is surjective from 9.6. Moreover the general fibre $p^{-1}(\overline{A}, \overline{B})$ of p is isomorphic to $\mathrm{Sym}^{\lambda-2-a}(\overline{A}) \times \mathrm{Sym}^{\delta-\lambda+2+a}(\overline{B})$, hence it is irreducible of dimension δ. Therefore we can conclude that $Z_{a,b}^\lambda$ and hence $\widetilde{Y}_g^\lambda(a,b)$ are irreducible and

$$\begin{aligned}\dim(\widetilde{Y}_g^\lambda(a,b)) &= \dim(Z_{a,b}^\lambda) - \epsilon = \dim|l| + \dim|l+(\lambda-2-c)l'| + \delta - \epsilon = \\ &= 2(\lambda-1-c) + \delta - \epsilon\end{aligned}$$

so, using (37) we obtain

$$\dim(\widetilde{\mathcal{W}}_g^\lambda(a,b)) = 2(\lambda-1-c) + \delta - \epsilon + 5\lambda + 4 - 3\delta = 2(3\lambda+1-c-\delta) + \lambda - \epsilon.$$

Using (31), we get $3\lambda + 1 - c - \delta = 3\lambda + 1 - (g-3-a-b) - 3(\lambda-1) + g = a + b + 7$, so

$$\dim(\widetilde{\mathcal{W}}_g^\lambda(a,b)) = 2(a+b) + 14 + \lambda - \epsilon.$$

Appliying 10.14, we obtain that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \dim(\widetilde{\mathcal{W}}_g^\lambda(a,b)) - 6 - \xi = 2(a+b) + 8 + \lambda - \epsilon - \xi$$

as required. $\diamond$

Remark 10.16. If $a < (g-\lambda-1)/2$ then $\delta = 3(\lambda-1) - g > 0$; in particular, $\lambda > (g+3)/3$. To show this, just remark that $g \leq 3\lambda - 3$ by (R_1); hence $a < \frac{g-\lambda-1}{2} \leq \frac{3\lambda-3-\lambda-1}{2} = \lambda - 2$ so, from 10.13: $\delta \geq \delta_A = \lambda - 2 - a > 0$.

Corollary 10.17. *Set, as usual, $a \leq b \leq c$ and $a+b+c = g-3$. The following facts hold:*

1) *The general curve $X(g,\lambda,a,b)$ of $\mathcal{M}_g^\lambda$ satisfies $a+b \geq (2g-8)/3$.*
2) *For the general curve $X(g,\lambda,a,b)$ of $\mathcal{M}_g^\lambda$, the values of $a,b,c = g-3-(a+b)$ are determined by the class of g (mod 3); in particular:*

$$\begin{array}{lllll} (i) & \text{if} & g = 3p & \text{then} & (a,b,c) = (p-1,p-1,p-1); \\ (ii) & \text{if} & g = 3p+1 & \text{then} & (a,b,c) = (p-1,p-1,p); \\ (iii) & \text{if} & g = 3p+2 & \text{then} & (a,b,c) = (p-1,p,p). \end{array}$$

3) *Conversely, for the above values of a and b we obtain a stratum of maximal dimension, i.e.*

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \dim(\mathcal{M}_g^\lambda).$$

Consequentely,

$$\text{a curve} \quad X(g,\lambda,a,b) \in \mathcal{M}_g^\lambda \quad \text{is general} \quad \iff \quad a,b,c \in \left\{ \left\lceil \frac{g-3}{3} \right\rceil, \left\lceil \frac{g-1}{3} \right\rceil \right\}.$$

<u>Proof</u>. 1) We have to show that, if $a+b < (2g-8)/3$, then $\dim(\mathcal{M}_g^\lambda(a,b)) < \dim(\mathcal{M}_g^\lambda)$.
Let us rewrite the above condition correspondingly to the possible values of g (mod 3):

- $g = 3p \quad : \ a+b \leq 2p-3 \ \Rightarrow \ a \leq p-2 \ \Rightarrow \ 2a+b \leq 3p-5;$
- $g = 3p+1 \ : \ a+b \leq 2p-3 \ \Rightarrow \ a \leq p-2 \ \Rightarrow \ 2a+b \leq 3p-5;$
- $g = 3p+2 \ : \ a+b \leq 2p-2 \ \Rightarrow \ a \leq p-1 \ \Rightarrow \ 2a+b \leq 3p-3.$

Clearly, in all these cases

$$a+b \leq \frac{2g-9}{3} \qquad \text{and} \qquad 2a+b \leq g-5. \tag{38}$$

From 10.11 (resp. 10.15) and using (38) we immediately obtain:

$$a \geq \frac{g-\lambda-1}{2} \ \Rightarrow \ \dim(\mathcal{M}_g^\lambda(a,b)) \leq 2(2a+b+\lambda) + 10 - g - \xi \leq 2(g-5+\lambda) + 10 - g - \xi = g + 2\lambda - \xi$$

$$a < \frac{g-\lambda-1}{2} \ \Rightarrow \ \dim(\mathcal{M}_g^\lambda(a,b)) \leq 2(a+b) + \lambda + 8 - \xi \leq \frac{4g-18}{3} + \lambda + 8 - \xi = g + \lambda + 1 + \frac{g+3}{3} - \xi$$

where, in both cases, $\xi := \begin{cases} 1, & \text{if } \lambda = \frac{g+3}{2} \\ 0, & \text{otherwise} \end{cases}$.

Note that, in the second case, from 10.16 we have that $(g+3)/3 < \lambda$. Therefore, for every value of a it holds

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda + 1 - \xi. \tag{39}$$

Finally recall that, from 10.9, $\dim(\mathcal{M}_g^\lambda) = g + 2\lambda + 1$ for all $(g+3)/3 < \lambda < \lambda_{\max}$, where $\lambda_{\max} = \lceil \frac{g+2}{2} \rceil$.
On the other hand, from 10.3, 10.6 and 10.7 it turns out that

$$\dim(\mathcal{M}_g^{\lambda_{\max}}) = \begin{cases} g + 2\lambda_{\max}, & \text{if } g \text{ odd} \\ g + 2\lambda_{\max} + 1, & \text{if } g \text{ even} \end{cases}.$$

Therefore, if $\lambda < \lambda_{\max}$ or g even, then $\xi = 0$ so (39) gives

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda + 1 = \dim(\mathcal{M}_g^\lambda).$$

Otherwise, $\lambda = \lambda_{\max}$ and g odd; then $\xi = 1$ so (39) gives

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda = \dim(\mathcal{M}_g^\lambda).$$

and this proves the first part of the statement.

2) Let us consider a general curve $X(g,\lambda,a,b) \in \mathcal{M}_g^\lambda$. We have just proved that $a+b \geq (2g-8)/3$. From the condition (R_3) we get:

$$\frac{2g-8}{3} \leq a+b \leq \frac{2g-6}{3} \quad \Rightarrow \quad a+b = \left\lceil \frac{2g-6}{3} \right\rceil$$

hence $a+b$ is uniquely determined. Therefore, since $c = g-3-(a+b)$ and $a \leq b \leq c$, we obtain:

- $g = 3p \quad : \quad a+b = 2p-2 \;\Rightarrow\; c = p-1 \;\Rightarrow\; (a,b,c) = (p-1,p-1,p-1)$
- $g = 3p+1 \quad : \quad a+b = 2p-2 \;\Rightarrow\; c = p \;\Rightarrow\; (a,b,c) = \begin{cases} (p-1,p-1,p) \\ (p-2,p,p) \end{cases}$
- $g = 3p+2 \quad : \quad a+b = 2p-1 \;\Rightarrow\; c = p \;\Rightarrow\; (a,b,c) = (p-1,p,p)$

Note that the case $g = 3p+1$ and $(a,b,c) = (p-2,p,p)$ does not correspond to a general curve since, in this case, $X(g,\lambda,a,b)$ belongs to a proper closed subset of $\mathcal{M}_g^\lambda$.
To show this, let us consider the two ranges of a and the corresponding dimensions of the moduli spaces found in 10.11 and 10.15, respectively.

(I) $a \geq \frac{g-\lambda-1}{2}$.

$$\dim(\mathcal{M}_g^\lambda(a,b)) \leq 2(2a+b+\lambda)+10-g = 2(3p-4+\lambda)+10-(3p+1) = 3p+2\lambda+1$$

while $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1 = 3p+2\lambda+2$.

(II) $a < \frac{g-\lambda-1}{2}$.

Substituting $g = 3p+1$ in (R_1) and in the bound of a in the assumption, we obtain respectively:

$$\lambda \geq \frac{g+3}{3} = p + \frac{4}{3} \quad \Rightarrow \quad \lambda \geq p+2$$
$$p-2 = a < \frac{g-\lambda-1}{2} \quad \Rightarrow \quad \lambda \leq p+3.$$

Using 10.15, under the assumption $(a,b,c) = (p-2,p,p)$ we obtain that $\epsilon = 1$ and $\xi = 0$, hence

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(a+b)+\lambda+8-\epsilon-\xi = 2(2p-2)+\lambda+8-1 = 4p+\lambda+3.$$

On the other hand

$$\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1 = 3p+2\lambda+2.$$

Examining the two possible cases of λ, we immediately get:

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \begin{cases} 5p+5, & \text{if } \lambda = p+2 \\ 5p+6, & \text{if } \lambda = p+3 \end{cases} \quad \text{while} \quad \dim(\mathcal{M}_g^\lambda) = \begin{cases} 5p+6, & \text{if } \lambda = p+2 \\ 5p+8, & \text{if } \lambda = p+3 \end{cases}$$

and this proves the second part.

3) We are left to show that the strata corresponding to the values $(i),(ii),(iii)$ of (a,b,c) are maximal.
First note that the inequalities $a < \frac{g-\lambda-1}{2}$ and $\lambda \geq \frac{g+3}{3}$ (the latter coming from (R_1)) become, respectively:

$$(i) \qquad p-1 < \frac{3p-\lambda-1}{2} \quad \text{and} \quad \lambda \geq \frac{3p+3}{3}$$
$$(ii) \qquad p-1 < \frac{3p-\lambda}{2} \quad \text{and} \quad \lambda \geq \frac{3p+4}{3}$$
$$(iii) \qquad p-1 < \frac{3p-\lambda+1}{2} \quad \text{and} \quad \lambda \geq \frac{3p+5}{3}$$

and in cases (i) and (ii) we get a contraddiction, while in (iii) we get $\lambda = p+2$. So in cases (i) and (ii) necessarily $a \geq \frac{g-\lambda-1}{2}$.
Secondly, observe that if $a \geq \frac{g-\lambda-1}{2}$ then 10.11 can be applied and we have

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(2a+b+\lambda)+10-g-\epsilon-\tau-\xi \tag{$*$}$$

where $\xi = 1$ if and only if $\lambda = \frac{g+3}{2}$. This happens if g is odd, so $\lambda = \frac{g+3}{2} = \lceil \frac{g+2}{2} \rceil$. Keeping the notation and the result in 10.7, where $\lambda_{\max} := \lceil \frac{g+2}{2} \rceil$, we have that $\dim(\mathcal{M}_g^{\lambda_{\max}}) = 2g+3 = \dim(\mathcal{M}_{g,4})$. Otherwise, $\xi = 0$ and $\lambda < \lceil \frac{g+2}{2} \rceil$; in this case, from 10.9, $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1$.
Consider now each possibility.

Case (i): $g = 3p$, $(a, b, c) = (p-1, p-1, p-1)$.
Since $\epsilon = 2$ and $\tau = 1$, from $(*)$ we obtain:

$$\begin{aligned}\dim(\mathcal{M}_g^\lambda(a,b)) &= 2(3p-3+\lambda)+10-3p-2-1-\xi = 3p+2\lambda+1-\xi = \\ &= g+2\lambda+1-\xi.\end{aligned}$$

Therefore

$$\lambda = \left\lceil \frac{g+2}{2} \right\rceil \Rightarrow \qquad \xi = 1 \quad \text{and} \quad \dim(\mathcal{M}_g^\lambda(a,b)) = g+2\lambda = g+2\,\frac{g+3}{2} = 2g+3 = \dim(\mathcal{M}_g^\lambda);$$
$$\lambda < \left\lceil \frac{g+2}{2} \right\rceil \Rightarrow \qquad \xi = 0 \quad \text{and} \quad \dim(\mathcal{M}_g^\lambda(a,b)) = g+2\lambda+1 = \dim(\mathcal{M}_g^\lambda).$$

Case (ii): $g = 3p+1$, $(a, b, c) = (p-1, p-1, p)$.
Since $\epsilon = 0$ and $\tau = 1$, from $(*)$ we again obtain:

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 3p+2\lambda+2-\xi = g+2\lambda+1-\xi.$$

With the same argument as before we prove the claim.
Case (iii): $g = 3p+2$, $(a, b, c) = (p-1, p, p)$.
I) If $a \geq \frac{g-\lambda-1}{2}$, the proof runs as above, using $(*)$ where $\epsilon = 1$ and $\tau = 0$.
II) If $a < \frac{g-\lambda-1}{2}$, the dimension of the strata is computed in 10.15 where one can find that

$$\dim\left(\mathcal{M}_g^\lambda(a,b)\right) = 2(a+b)+\lambda+8-\epsilon-\xi. \qquad (**)$$

In our situation, $\epsilon = 1$ and $\xi = 0$, since $\lambda \neq \frac{g+3}{2}$ being $g = 3p+2$ and $\lambda = p+2$, as remarked before. So $(**)$ gives $\dim\left(\mathcal{M}_g^\lambda(a,b)\right) = 5p+7$. On the other hand, $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1 = 5p+7$.

The final claim comes from (2) and (3), together with a straightforward computation on the values in (i), (ii), (iii), taking into account that $a \leq b \leq c$. $\diamond$

11. Moduli spaces of 4–gonal curves with $t \geq 1$

Let us recall that if $t \geq 1$ and the double points of the standard model $\overline{X}_0$ are distinct, then the bounds of the invariants λ and t are described in 5.4 $(i)-(iv)$ while the invariants a and b are determined by λ and t (see 5.1). More precisely,

$$\frac{g+3}{3}+t \leq \lambda \leq \frac{g+3}{2}\,, \quad 1 \leq t \leq \frac{g+3}{6}$$

$$a = g-2\lambda+t+1, \quad b = \lambda-t-2, \quad c = \lambda-2.$$

As a consequence, the subvariety of $\mathcal{W}_g^\lambda$ parametrizing the curves of invariants g, λ, t, a, b can be simply denoted by $\mathcal{W}_g^\lambda(t)$.

In order to describe such variety, we perform a construction similar to that in 10.11.

Let us denote by $\mathcal{A}_\lambda^t$ the open subset of the linear system $|4C_0+(\lambda+t)f|$ on $R_{1,t+1}$ parametrizing the irreducible curves of such linear system and set

$$\mathcal{W}_g^\lambda(t) := \{X' \in \mathcal{A}_\lambda^t \mid X = X(g,\lambda,t) \text{ and it has } \delta \text{ distinct double points on } C_0\}.$$

If we consider the morphism

$$\varphi := \varphi_{4C_0+(\lambda+t)f}: \quad R_{1,t+1} \longrightarrow S' \subset \mathbb{P}^N$$

it is clear that $N = h^0(R_{1,t+1}, \mathcal{O}_{R_{1,t+1}}(4C_0+(\lambda+t)f))-1 = 5(\lambda-t)+4$ (from [**4**], 1.8) and we can identify $\mathcal{W}_g^\lambda(t)$ with the following subset of $\check{\mathbb{P}}^N$:

$$\mathcal{W}_g^\lambda(t) \cong \{H \in \check{\mathbb{P}}^N \mid H \supset \langle T_{P_1}(S'), \dots T_{P_\delta}(S')\rangle, P_i \in C_0\}.$$

Therefore, consider the following correspondence

$$\widetilde{W} = \{(H; P_1, \ldots, P_\delta) \mid H \supset \langle T_{P_1}(S'), \ldots T_{P_\delta}(S')\rangle\} \subset \check{\mathbb{P}}^N \times \mathrm{Sym}^\delta(\mathbb{P}^1)$$

and the projections

$$\begin{array}{ccc} & \widetilde{W} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \check{\mathbb{P}}^N & & \mathrm{Sym}^\delta(\mathbb{P}^1) \end{array}$$

Obviously, $\pi_1(\widetilde{W}) = \overline{\mathcal{W}_g^\lambda(t)}$ and π_1 is an isomorphism on an open subset of $\mathcal{W}_g^\lambda(t)$. Moreover, π_2 is surjective and the fibres have dimension $N - \dim\langle T_{P_1}(S'), \ldots T_{P_\delta}(S')\rangle$.
One can show (as in 9.4) that also in the case $t \geq 1$ it holds that the space $\langle T_{P_1}(S'), \ldots T_{P_\delta}(S')\rangle$ has maximum dimension, i.e. $3\delta - 1$. Hence $\dim(\mathcal{W}_g^\lambda(t)) = \dim \widetilde{W} = N - (3\delta - 1) + \delta = 5(\lambda - t + 1) - 2\delta$, so using 2.2 (*iii*), we obtain:

$$\dim(\mathcal{W}_g^\lambda(t)) = 2g + t - \lambda + 11.$$

As well as in the case $t = 0$, one can show that these varieties are not empty. Furthermore, let us recall that the automorphism group of a rational ruled surface $R_{1,t+1} \subset \mathbb{P}^{t+2}$ has dimension $t + 5$, if $t \geq 1$, and 6, if $t = 0$ (as we already noted in 10.3). These two facts, together with the previous computation of $\dim(\mathcal{W}_g^\lambda(t))$, immediately give the following result:

Theorem 11.1. *Let g, λ, t be positive integers satisfying: $g \geq 10$,*

$$\frac{g+3}{3} + t \leq \lambda \leq \frac{g+3}{2}, \quad 1 \leq t \leq \frac{g+3}{6}.$$

Then $\mathcal{M}_g^\lambda(t)$ is an irreducible variety of dimension $2g - \lambda + 6$. ⋄

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