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On the singularities of surfaces ruled by conics

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Abstract

We classify the singularities of a surface ruled by conics: they are rational double points of type A_n or D_n. This is proved by showing that they arise from a precise series of blow–ups of a suitable surface geometrically ruled by conics. We determine also the family of such surfaces which are birational models of a given surface ruled by conics and obtained in a "minimal way" from it.

Keywords: projective surface, conic bundle, ruled surface, rational singularities.

MSC: 14 J 26, 14 E 05, 14 D 06

Introduction

Projective surfaces ruled by conics arise naturally in the study of the moduli space of four–gonal curves $\mathcal{M}_{g,4} \subset \mathcal{M}_g$ inside of the moduli space of curves of genus g. Indeed, it is known that the canonical model of such curves lies on a relative hyperquadric S in a three-dimensional rational normal scroll $V = \mathbb{P}(\mathcal{E})$, that is a divisor of the type $2H - \beta F$, with H tautological divisor and F a fiber.

In analogy with the Maroni invariant in the trigonal case, the splitting type of the vector bundle $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(g-3-a-b)$ gives a first description of the curve. But in [3], using the invariant t of a precise birational smooth model $\mathbb{F}_t$ of S and a further invariant λ (which is substantially the minimum degree of a linear series of the curve, out of the four–gonal one), a more precise description is given, essentially by linking the properties of the surface S with the geometry of the curve (for instance the authors prove that $\deg(S) = g + \lambda - t - 5$). More precisely, with a suitable use of these four invariant, they define and describe a stratification in irreducible locally closed subsets $\mathcal{M}_g^{\lambda,t}(a,b)$ of $\mathcal{M}_{g,4}$, which has been extensively studied in the cited paper.

In this frame, becomes essential a precise description of the surface S, its singular locus and the type of its singularities. This motivates the local and global study of projective surfaces with isolated singularities, fibered over $\mathbb{P}^1$, with general fiber a smooth conic.

We dealt with a very similar subject in [2], but the investigation carried out in the present paper differs from and is more general than the previous article in two important respects. First of all, the previous study considered only rational surfaces, whereas here we treat the general case. Second, it is known that every surface S ruled by conics is birational to a surface which is geometrically ruled by conics and everywhere smooth. This means that S can be obtained from such a surface by means of a finite number of blow-ups and blow-downs (see [6], Ch. 4, Sect. 3). In the previous study, we restricted to the case of surfaces which arise in this way using blow-ups at *distinct* points and the main theorem (1.9) of [2] classifies the singularities of a surface ruled by conics in that situation.

In the present paper, the main result (Theorem 2.4) describes *all* the possible singularities of a surface ruled by conics.

Let us point out that the nature of the classification given in the present paper is of an algorithmic type, and in principle could be used for computational purposes.

Surfaces ruled by conics are roughly regarded as a special case of conic bundles. This subject has been widely developed in the literature, but mainly in the case of the dimension of the base variety is at least two (only recall the well–known papers [10], [7], [8], among the several works spread in decades). In Section 0 we first remind the basic notions and the general results concerning conic bundles contained in the paper of Sarkisov [9] which apply also when the base of the conic bundle has dimension one. Then we prove that a surface ruled by conics can be regarded as an embedded conic bundle, at least if it has isolated singularities.

In Sect.1 we describe in detail how the singularities of a surface ruled by conics can be computed *via* a series of blow-ups and contractions from a surface *geometrically* ruled by conics. In Sect.2 we give a complete classification of *all* surfaces ruled by conics, by showing that the singularities computed in the previous section are the all possible ones.

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We also solve the inverse problem: given a surface ruled by conics, how can one recover, in a minimal way, a birationally–equivalent surface which is geometrically ruled by conics? We determine the (finite) family of such birational models in Sect. 3. This last result will play an important role in the study of the moduli space of four–gonal curves cited before.

0. Preliminary notions and known results

All varieties in this paper are assumed to be algebraic over a fixed algebraically closed field of characteristic 0. By *projective surface* we mean an irreducible and reduced surface in $\mathbb{P}^N$.

Notation.

(a) If X is an algebraic surface and $P \in X$ is any point, then σ_P will denote the *blow–up morphism* at the point P and the obtained surface, the *blow–up* of X at P, will be denoted by $Bl_P(X)$ (or $\widetilde{X}$ if the centre of the monoidal transformation is clear). Briefly:

$$\sigma_P: \ Bl_P(X) = \widetilde{X} \longrightarrow X.$$

(b) If $\widetilde{X}$ is is an algebraic surface and $E \subset \widetilde{X}$ is a contractible curve, we will set:

$$con(E): \ \widetilde{X} \longrightarrow X$$

the *blow–down morphism* giving the contraction of E to a point of X.

(c) If X is as before and D is a divisor on X, then

$$\Phi_D: \quad X \longrightarrow X' \subset \mathbb{P}(H^0(\mathcal{O}_X(D)))$$

denotes the morphism associated to D.

Definition 0.1. Let C be a smooth irreducible curve of genus g and let $\mathcal{F}$ be a rank 2 vector bundle on C. If D is a very ample k–secant divisor on $\mathbb{P}(\mathcal{F})$, then the surface $S_0 := \Phi_D(\mathbb{P}(\mathcal{F}))$ is said to be *geometrically k–ruled* over C. Equivalently, a projective surface $S_0 \subset \mathbb{P}^N$ is geometrically k–ruled over C if there exists a surjective morphism $\pi: \ S_0 \longrightarrow C$ such that the fibre $\pi^{-1}(y)$ is a smooth rational curve of degree k for every point $y \in C$.

In particular, a geometrically k–ruled surface S_0 is smooth hence $\pi: \ S_0 \longrightarrow C$ is flat, being C a smooth curve (see [1], pg. 91).

Definition 0.2. Let C be as before. We say that a projective surface $S \subset \mathbb{P}^N$ is *k–ruled* over C if there exists a surjective morphism $\pi: \ S \longrightarrow C$ and a non–empty open subset $U \subseteq C$ such that:

- π is flat with fibres $\pi^{-1}(y)$ of degree k and arithmetic genus 0 for every point $y \in C$;
- the fibre $\pi^{-1}(y)$ is smooth for every point $y \in U$.

In this paper we will treat 2–ruled surfaces, hence the very ample divisor D on $\mathbb{P}(\mathcal{F})$ is bisecant. In this case the surface $S_0 = \Phi_D(\mathbb{P}(\mathcal{F}))$ will be called *geometrically ruled by conics* while the surface S will be said *ruled by conics*.
In Sections 1 and 2 we show that a surface S ruled by conics is birational to a surface S_0 geometrically ruled by conics and in Section 3 we analyze these birational models of S.

It is clear that a surface S ruled by conics corresponds to a curve in $\mathbb{P}^5$, the space parametrizing the conics of the plane. Since this curve can be locally approximated to a line, then the surface S can be locally expressed as a pencil of conics. In this way one can see that if P is a singular point of S then necessarily three facts occur: P is a base–point of the above pencil of conics, the fibre F_P of S containing P is singular at P and, finally, F_P contains at most one more singular point of S.
This leads to the following basic fact:

Proposition 0.3. *Let $\pi : S \to C$ be a surface ruled by conics on a smooth curve C. Then S has only isolated singularities.*

Proof. Let Σ be a one–dimensional component of the singular locus S_{sing}. If Σ intersect all fibres of S, this implies (from the above observation) that each fibre is a singular conic. But this is impossible.
Hence Σ contains a fibre; so such fibre is contained in the singular locus of S. But, again from the previous remark, each fibre contains at most two singular points of S. So also this case cannot occur.
Therefore S_{sing} is a zero–dimensional subset of S. ⋄

There is an extensive literature on conic bundles; here we mention some notions and results collecting them mainly from the work of Sarkisov (see [9]).

Definition 0.4. Let S and C be irreducible algebraic varieties and C be non singular. A triple (S, C, π), where $\pi : \ S \longrightarrow C$ is a rational map whose generic fibre is an irreducible rational curve, is called a *conic bundle over the base C*.

Definition 0.5. A conic bundle (S, C, π) is called *regular* if π is a flat morphism of nonsingular varieties.

Let (S, C, π) be a conic bundle. If $y \in C$, denote by S_y the fibre $\pi^{-1}(y)$ and, if $\mathcal{F}$ is a sheaf on S, denote by $\mathcal{F}_y$ the restriction of $\mathcal{F}$ to the fibre S_y. Finally, set ω_S and K_S, respectively, the canonical sheaf and the canonical divisor of S.

Remark 0.6. Let (S, C, π) be a regular conic bundle. Then $\mathcal{O}_S(-K_S)$ is flat over C, since for all $x \in S$, $(\mathcal{O}_S(-K_S))_x \cong \mathcal{O}_{x,S}$, which is flat over $\mathcal{O}_{\pi(x),C}$ by assumption.

Remark 0.7. Let (S, C, π) be a regular conic bundle. Then $H^i(S_y, \mathcal{O}_S(-K_S)_y) = 0$ for $i \geq 1$, for each $y \in C$. To show this, note that the cohomology groups vanish for all $i \geq 2$ since $\dim(S_y) = 1$.
Let $\mathcal{N}_{y/C}$ denote the normal bundle at the point $y \in C$; then, since C is smooth, $\mathcal{N}_{y/C}$ is free free of rank $r := \dim(C)$. On the other hand, $\pi^*(\mathcal{N}_{y/C}) = \mathcal{N}_{S_y/S}$, so also the latter is (locally) free of rank r. Therefore

$$\bigwedge^r \mathcal{N}_{S_y/S} = \mathcal{O}_{S_y}.$$

Since $S_y \subset S$ is a nonsingular subvariety of codimension r, it holds (by [6], Ch. II, 8.20 and the above equality) that

$$\omega_{S_y} \cong \omega_S \otimes \bigwedge^r \mathcal{N}_{S_y/S} = \omega_S \otimes \mathcal{O}_{S_y}.$$

Finally, dualizing the above equality, we obtain that the restriction to the fibre of the anticanonical sheaf is exactly the anticanonical sheaf of the fibre:

$$\mathcal{O}_S(-K_S) \otimes \mathcal{O}_{S_y} \cong \mathcal{O}_{S_y}(-K_{S_y}).$$

But each fibre of π is isomorphic to $\mathbb{P}^1$, hence

$$H^1(S_y, \mathcal{O}_S(-K_S)_y) = H^1(S_y, \mathcal{O}_{S_y}(-K_{S_y})) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 0.$$

Proposition 0.8. *If (S, C, π) is a regular conic bundle, then we have:*
- $R^i\pi_*(\mathcal{O}_S(-K_S)) = 0$ *for all* $i \geq 1$;
- $R^0\pi_*(\mathcal{O}_S(-K_S))$ *is a locally free sheaf of rank 3.*

Proof. Using 0.6 and 0.7, we can apply [6], Ex 11.8, Ch. III , obtaining that $R^i\pi_*(\mathcal{O}_S(-K_S)) = 0$ for all $i \geq 1$ (in a neighborhood of each $y \in C$).
With the same argument, it is easy to see that $\dim H^0(S_y, \mathcal{O}_S(-K_S)_y)$ is constant on C. So, using Grauert Theorem ([6], 12.9, Ch. III), we obtain that $R^0\pi_*(\mathcal{O}_S(-K_S))$ is locally free on S and

$$(R^0\pi_*(\mathcal{O}_S(-K_S)))_y \cong H^0(S_y, \mathcal{O}_S(-K_S)_y) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$$

which is a vector space of dimension 3. ⋄

Setting $\mathcal{E} := R^0\pi_*(\mathcal{O}_S(-K_S))$ the above locally free sheaf of rank 3 on C and taking into account the above isomorphism $\mathcal{E}_y \cong H^0(S_y, \mathcal{O}_S(-K_S)_y)$, it is clear that the morphism associated to the divisor $-K_S$ of S is an embedding $S \hookrightarrow \mathbb{P}(\mathcal{E})$ under which the image of each fibre S_y of π is a conic in the corresponding fibre $\mathbb{P}(\mathcal{E})_y \cong \mathbb{P}^2$. For this reason, S is said *conic* bundle.
The above construction, performed in the regular conic bundle case, leads to the described embedding $S \hookrightarrow \mathbb{P}(\mathcal{E})$. But one can have an analogous embedding also in a more general case:

Definition 0.9. Let $\mathcal{E}$ be a locally free sheaf of rank 3 on C and $\tau : \mathbb{P}(\mathcal{E}) \longrightarrow C$ be the natural projection. An irreducible reduced divisor $S \subset \mathbb{P}(\mathcal{E})$ such that the triple $(S, C, \tau_{|S})$ is a conic bundle is called an *embedded conic bundle.*

The case of 1–dimensional base is not particularly focused in [9], but it turns out that in this frame can be settled also the surfaces ruled by conics (in the sense of 0.1 and 0.2), as the following result shows:

Proposition 0.10. *Let $S \subset \mathbb{P}^N$ be a surface ruled by conics over the curve C via $\pi : S \to C$. Then:*
i) each singular point of S is a Gorenstein singularity;
ii) S is normal;
iii) (S, C, π) is an embedded conic bundle.

Proof. *i)* For each $P \in S$, let $T_P(S)$ be the tangent space to S at P, F_P the fibre of S containing P and U_P a unisecant on S passing through P. Then

$$\dim(T_P(S)) \leq \dim(T_P(F_P)) + \dim(T_P(U_P)).$$

Since C is a smooth curve, then $\dim(T_P(U_P)) = 1$, while $\dim(T_P(F_P))$ is 1 or 2 accordingly if P is a smooth or singular point of the fibre F_P. This is due to the fact that F_P has degree 2 and F_P has no embedded points since π is flat. Therefore $\dim(T_P(S)) \leq 3$; in particular, if S is not smooth at P then P is a hypersurface singularity of S, hence a Gorenstein singularity.
ii) It is immediate from a well–known theorem (see for instance [11]): if P is a two–dimensional singularities, then P is normal if and only if it is Cohen–Macaulay and an isolated singularity.
iii) It is a consequence of [4], Proposition 2.1: if $\pi : X \longrightarrow Z$ is a morphism, X is a Gorenstein scheme, Z is smooth and the fibres of π are all (possibly degenerate) conics in $\mathbb{P}^2$, then X is a flat conic bundle. ⋄

The last notion we want to recall about conic bundles is that of *degeneration divisor*. In [9] it is defined as the vanishing locus of a homomorphism of certain cohomology groups (see [9], 1.6). Even if the author gives a complete characterization of such divisor in the case of regular conic bundle (in particular when S is smooth), let us recall the properties which hold also in the general case: the degeneration divisor Δ of a conic bundle (S, C, π) is such that, for all points $y \in C \setminus \Delta$, the conic $\pi^{-1}(y)$ has rank 3, i.e. it is smooth. If $y \in \Delta$ then the corresponding conic has rank ≤ 2. Finally, if π is flat, the rank of each conic is non–zero.

In a forthcoming paper of the moduli space of 4–gonal curves, we need a very detailed description of all the possible singularities of S and the relations between the invariant of its non–singular models and the invariants of $\mathbb{P}(\mathcal{E})$. As far as we know, there are not such results in the literature. Hence, in what follows, we determine a procedure (a sort of constructive algorithm) in order to detect and classify the singularities of a projective surface ruled by conics.

1. Singularities arising from elementary transformations of "main type"

Since, as observed in the previous Section, any 2–ruled surface S has a finite number of singular points, then it can be obtained by a suitable geometrically 2–ruled surface $S_0 = \Phi_D(\mathbb{P}(\mathcal{F}))$ by a finite number of monoidal transformations. In other words, denoting by $\widetilde{S}_0$ the surface obtained by a sequence σ of blow–ups of S_0, setting $\widetilde{D}$ to be the strict transform of D via σ and B the base locus of $\widetilde{D}$, then S can be obtained in the following way:

$$\begin{array}{ccc} \widetilde{S}_0 & \xrightarrow{\ \sigma\ } & S_0 \\ {\scriptstyle \Phi_{\widetilde{D}-B}} \big\downarrow & & \\ S & & \end{array} \tag{1}$$

where $\Phi_{\widetilde{D}-B}$ is a birational morphism.

Note that, for each rank 2 vector bundle $\mathcal{F}$, the scroll $\mathbb{P}(\mathcal{F})$ can be locally expressed as $U \times \mathbb{P}^1$, where U is an open affine subset of the base curve C. Moreover, also the choice of the very ample 2–secant divisor D on $\mathbb{P}(\mathcal{F})$ does not affect S_0 since Φ_D is an isomorphism.

Let us explicitly describe σ as composition of a chain of blow–ups centered in suitable points: we assume that the centre consists of points either belonging to the same fibre or infinitely near to it.
Consider a point $P_1 \in S_0$ and let $f_0 := \pi^{-1}(y)$ be the fibre of S_0 containing P_1. Let us consider the blow–up of S_0 at P_1 and the corresponding projection on C, say π_1:

$$\begin{array}{ccc} Bl_{P_1}(S_0) := S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \supset \pi^{-1}(y) = f_0 \ni P_1 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & C \ni y & \end{array}$$

Denote also by $f_1 := \pi_1^{-1}(y)$ the total transform of f_0 via σ_{P_1}.
Take now $P_2 \in f_1$ and consider the corresponding blow–up $\sigma_{P_2} : \ S_2 \longrightarrow S_1$. With obvious notations, we can repeat this construction and obtain a sequence of blow–ups:

$$\begin{array}{ccccccccc} \widetilde{S}_0 := S_n & \xrightarrow{\sigma_{P_n}} & S_{n-1} & \longrightarrow \cdots \longrightarrow & S_2 & \xrightarrow{\sigma_{P_2}} & S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \\ \cup & & & & \cup & & \cup & & \cup \\ \widetilde{f}_0 := f_n & & & & f_2 & & P_2 \in f_1 & & P_1 \in f_0 \end{array} \tag{2}$$

where, for all i, we set $P_i \in f_{i-1}$, $f_i := \pi_i^{-1}(y)$ and $\pi_i : S_i := Bl_{P_i}(S_{i-1}) \longrightarrow C$ the natural projection.

Remark 1.1. The surface $\widetilde{S}_0$ is smooth at each point of f_n. Moreover, since $\sigma := \sigma_{P_n} \circ \cdots \circ \sigma_{P_1}$ is an isomorphism from $\widetilde{S}_0 \setminus f_n$ to $S_0 \setminus f_0$, then $\widetilde{S}_0$ is smooth everywhere.

Definition 1.2. Let $\widetilde{S}_0 \supset f_n$ be as in (2). Then we say that f_n is a fibre of *level* n *over* f_0.

Let us consider the single fibres of the surfaces involved in diagram (1):

$$\begin{array}{ccc} \widetilde{S}_0 & \xrightarrow{\sigma} & S_0 \\ \Big\downarrow {\scriptstyle \Phi_{\widetilde{D}-B}} & & \\ S & & \end{array} \qquad\qquad \begin{array}{ccc} f_n & \xrightarrow{\sigma} & f_0 \\ \Big\downarrow {\scriptstyle \Phi_{\widetilde{D}-B}} & & \\ F & & \end{array} \tag{3}$$

Clearly, the fibre F of S is uniquely determined by f_n.

Definition 1.3. We say that the fibre $F \subset S$ is an *embedded fibre of level* n if

$$n = \min_i \{\text{there exists a blow–up } \sigma : \widetilde{S}_0 \to S_0 \text{ and a fibre } f_i \subset \widetilde{S}_0 \text{ of level } i \text{ such that } F = \Phi_{\widetilde{D}-B}(f_i)\}.$$

The purpose of this section is to describe some of the possible fibres of $\widetilde{S}_0$ obtained by a sequence of blow–ups of S_0 as before. We will call them "main fibres" since, in Section 3, we will prove that they are the only ones giving, by contractions, all the possible singularities on a surface S ruled by conics.

Notation. Consider the above sequence of blow–ups. If D is a bisecant divisor on S_0, then we denote by $\widetilde{D}$ the strict transform of D on $\widetilde{S}_0$ as well as on each surface S_i defined in (2). While the strict transform of a component e of a fibre will be denoted by $\tilde{e}$ and also (for simplicity) by e at each step of the sequence.

Remark 1.4. Let $\widetilde{D} \subset \widetilde{S}_0$ be a bisecant divisor without base locus. Clearly, $\Phi_{\widetilde{D}}$ is an isomorphism out of a finite number of fibres and assume, for simplicity, that this number is one and this fibre is f_n. Keeping the notation introduced in (3), we then have:

$$\Phi_{\widetilde{D}} : \quad \widetilde{S}_0 \setminus f_n \quad \xrightarrow{\cong} \quad S \setminus F.$$

Therefore $\Phi_{\widetilde{D}}$ contracts f_n to F. It is also clear that $\Phi_{\widetilde{D}}$ maps the divisor $\widetilde{D}$ to the hyperplane divisor $H_{\widetilde{D}}$ of S, which is a surface ruled by conics. Hence $H_{\widetilde{D}}$ meets each fibre of S in two points; in particular, $H_{\widetilde{D}} \cdot F = 2 = \widetilde{D} \cdot f_n$.

Definition 1.5. If $\widetilde{D} \subset \widetilde{S}_0$ is as before, then we call the "$\widetilde{D}$*–degree*" of a component e of f_n the integer

$$\deg_{\widetilde{D}}(e) = \widetilde{D} \cdot e.$$

The arguments in 1.4 can be easily generalized to the case of non–empty base locus B of $|\widetilde{D}|$, giving immediately the following:

Proposition 1.6. *Let $\widetilde{S}_0$, $\widetilde{D}$, B, f_n be as before and e be an irreducible component of f_n. Then:*
i) $\widetilde{D} \cdot f_n = 2$;
ii) *if* $e \not\subset B$ *and* $\deg_{\widetilde{D}}(e) = 0$ *then* $\Phi_{\widetilde{D}}$ *contracts* e *to a point of the fibre* F *of* S;
iii) *if* $\deg_{\widetilde{D}}(e) < 0$ *then* $e \subset B$. ◇

Remark 1.7. Throughout this section, devoted to the construction of "main" fibres, we will assume that the centre of each blow–up $\sigma_{P_i} : S_i \longrightarrow S_{i-1}$ is a point $P_i \in S_{i-1}$

- either belonging to $\widetilde{D} \subset S_{i-1}$ or infinitely near to it, i.e. P_i belongs to the total transform of D in S_{i-1};
- belonging to a component of positive $\widetilde{D}$–degree of $f_{i-1} \subset S_{i-1}$.

It is easy to see that the first condition is a consequence of the second one.

Remark 1.8. Let $\sigma_{P_i} : S_i \longrightarrow S_{i-1}$ be as before and let g_{i-1} be a component of $f_{i-1} \subset S_{i-1}$ containing P_i. Denote as usual by $\tilde{g}_{i-1}$ the strict transform of g_{i-1} in S_i and by e the exceptional divisor of the blow–up. Then $\deg_{\widetilde{D}}(\tilde{g}_{i-1}) = \deg_{\widetilde{D}}(g_{i-1}) - 1$ and $\deg_{\widetilde{D}}(e) = 1$. In other words

$$\deg_{\widetilde{D}}(\tilde{g}_{i-1}) + \deg_{\widetilde{D}}(e) = \deg_{\widetilde{D}}(g_{i-1}).$$

Notation. In the sequel we will draw the pictures of the fibres using the following agreement. Let $f_i \subset S_i$ be as before and let e be one of its irreducible components.

- The self–intersection of e is "represented" as its degree i.e. if $e^2 = -1, -2, -3\ldots$ then e will be drawn as a line, a conic, a cubic, etc. respectively.
- The $\widetilde{D}$–degree of e is represented by a continous line if $\deg_{\widetilde{D}}(e) = 1$, a dashed line if $\deg_{\widetilde{D}}(e) = 0$ and a dotted line if $\deg_{\widetilde{D}}(e) < 0$.
- The fibre f_0 of S_0 will be represented as a continous smooth conic, even if $f_0^2 = 0$ and $\deg_D(f_0) = 2$.

Remark 1.9. Let us describe the fibres of level 0, 1, 2 arising from blow–ups satisying the conditions in 1.7.

Level 0. By definition there is only one fibre of level 0 on $\widetilde{S}_0 = S_0$, which is f_0 itself.

Level 1. In this case also, there is only one fibre f_1 of $S_1 = \widetilde{S}_0$ obtained from f_0 by blowing–up S_0 at a point $P_1 \in f_0$. Denoting by e_1 the exceptional divisor, it is clear that $f_1 = f_0 + e_1$ and $e_1^2 = -1$. Since $f_1^2 = 0$, it is immediate to see that also the other component f_0 has self–intersection -1. Moreover, by 1.8, $\deg_{\widetilde{D}}(f_0) = \deg_{\widetilde{D}}(e_1) = 1$.

Level 2. In order to get a fibre of level 2, we can blow–up f_1 in a point P_2 which is either a smooth point (i.e. belonging to exactly one of the two components) or the singular point of f_1. These two cases are deeply different, so we denote the corresponding fibres in a different way: $f_2(A)$ and $f_2(D)$, respectively. Finally note that, in the case (A), we can assume that P_2 belongs to one specific component, since the other construction can be recovered from this one by an elementary transformation.

Case (A). Assume that $P_2 \in e_1$ and consider $\sigma_{P_2} : S_2 \longrightarrow S_1$. The fibre f_2 of S_2 consists of three components: $f_2 = f_0 + e_1 + e_2$, where e_2 is the exceptional divisor of P_2. From 1.8, it is clear that $\deg_{\widetilde{D}}(e_1) = 0$, while $\deg_{\widetilde{D}}(f_0) = \deg_{\widetilde{D}}(e_2) = 1$. On the other hand, the self–intersection of e_2 is -1, while the one of e_1 drops by one; briefly: $f_0^2 = e_2^2 = -1$ and $e_1^2 = -2$. We can then draw the picture of $f_2(A)$ using the agreement introduced before.

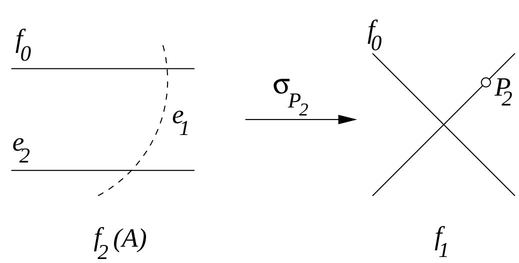


Figure 1

Case (D)**.** Assume that $P_2 = f_0 \cdot e_1$ and consider $\sigma_{P_2} : S_2 \longrightarrow S_1$. Now the exceptional divisor consists of a component of multiplicity 2 since P_2 is a double point of f_1. Therefore the fibre f_2 of S_2 has the form $f_2 = f_0 + e_1 + 2e_2$. Since $\widetilde{D}$ is still bisecant on the fibre f_2, from 1.8 we have: $\deg_{\widetilde{D}}(f_0) = \deg_{\widetilde{D}}(e_1) = 0$ and $\deg_{\widetilde{D}}(2e_2) = 2$. On the other hand $f_0^2 = e_1^2 = -2$ and $e_2^2 = -1$, as the following picture shows.

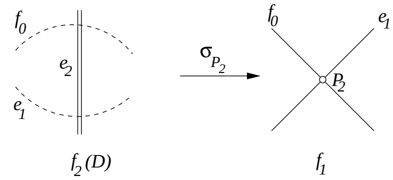


Figure 2

Note that, in all the previous cases, there are no components with negative $\widetilde{D}$–degree, hence 1.7 is automatically fulfilled. Moreover $|\widetilde{D}|$ is base–point–free both on S_1 and on S_2.

Definition 1.10. We define a *main fibre* of level n on f_0 recursively:
- f_1 is the *main fibre of level* 1 *on* f_0;
- if $n \geq 1$, then f_n is a *main fibre of level* n *on* f_0 if it is a fibre of $S_n = Bl_{P_n}(S_{n-1})$, where $P_n \in f_{n-1}$ and f_{n-1} is a main fibre of level $n-1$ on f_0.
- if $n \geq 3$ then P_n does not belong to components of $\widetilde{D}$–degree ≤ 0.

Level n**.** Now it is clear how to iterate the above construction. The "main fibres" of level n, for $n \geq 3$, either of type (A) and of type (D) have the shapes described in the following picture:

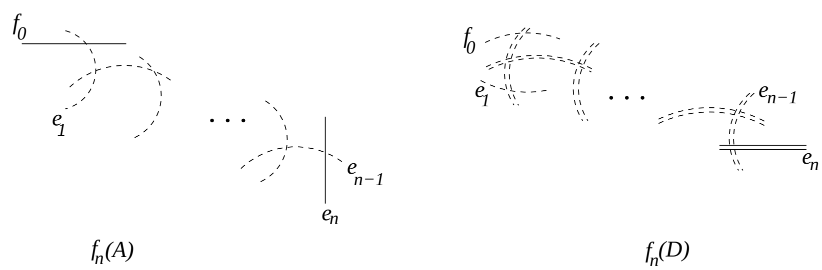


Figure 3

Let us describe the singularities of a surface S ruled by conics as in (3), where $\widetilde{S}_0 = Bl_{P_1,\dots,P_n}(S_0)$, the point P_1 belongs to a fibre f_0 of S_0 and $P_2, \dots, P_n$ are such that the corresponding fibre of $\widetilde{S}_0$ is a main fibre f_n of level n. Denote by F_n the corresponding fibre on S, i.e. $F_n := \Phi_{\widetilde{D}-B}(f_n)$.

Remark 1.11. Since f_n is a main fibre, it is clear that all its components have $\widetilde{D}$–degree 0, but one or two, having $\widetilde{D}$–degree 2 or 1, respectively (see Figure 3). Moreover, it is easy to see that the above construction of main fibres leads to a linear system $|\widetilde{D}|$ which is base–point–free. Finally, as noted in 1.6, the morphism $\Phi_{\widetilde{D}}$ contracts exactly the components of $\widetilde{D}$–degree zero.

Level 1. The fibre is $f_1 = f_0 + e_1$ and both have $\widetilde{D}$–degree one. So $\Phi_{\widetilde{D}}$ is an isomorphism and $F_1 \cong f_1$.

Level 2. Case (A)**.** There is only one component of $\widetilde{D}$–degree zero, namely e_1. Hence $\Phi_{\widetilde{D}} = con(e_1)$. Since $e_1^2 = -2$, then the fibre $F_2(A)$ consists of $f_0 + e_2$ and the point $f_0 \cap e_2$ is an ordinary double point of S.
Case (D)**.** Here two components have $\widetilde{D}$–degree zero, hence $\Phi_{\widetilde{D}} = con(f_0, e_1)$, and $f_0^2 = e_1^2 = -2$. Therefore the fibre $F_2(D)$ is $2e_2$ and its points $Q_0 := \Phi_{\widetilde{D}}(f_0)$, $Q_1 := \Phi_{\widetilde{D}}(e_1)$ are ordinary double points of S.

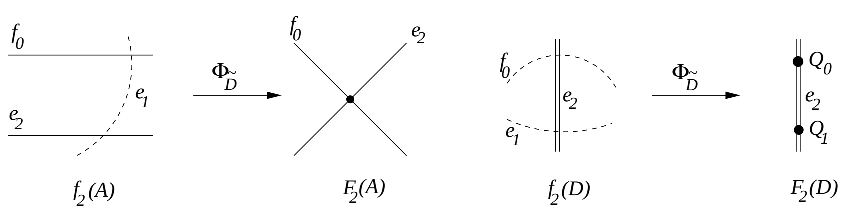


Figure 4

Level 3.
Case (A)**.** As before, since there are 2 components of $\widetilde{D}$–degree zero, $\Phi_{\widetilde{D}} = con(e_1, e_2)$ and $F_3(A) = f_0 + e_3$, where $Q := f_0 \cap e_3$ is a rational double point of S.
Case (D)**.** In this case there are 3 components of $\widetilde{D}$–degree zero: $f_0, e_1, 2e_2$, so $\Phi_{\widetilde{D}} = con(f_0, e_1, e_2)$ and $F_3(D) = 2e_3$, where $Q := \Phi_{\widetilde{D}}(f_0 + e_1 + 2e_2)$ is a rational double point of S.

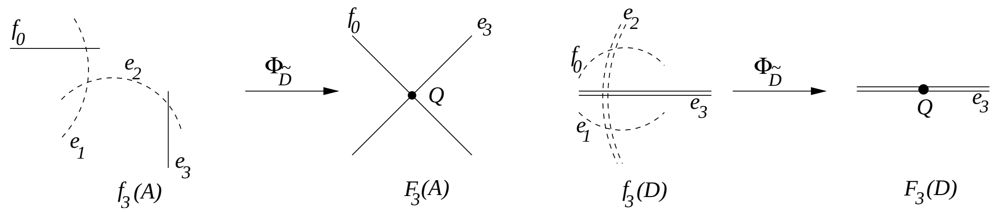


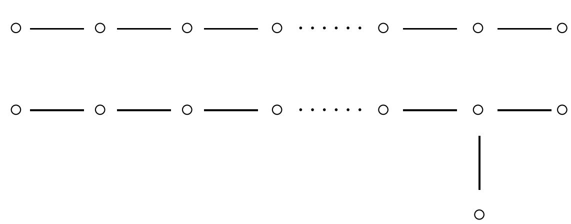
Figure 5

Level n**.** With the aid of the first cases and Figure 3, the shapes of the fibres F_n (where $n \geq 3$) of S and their singularities are clear. Namely the morphism $\Phi_{\widetilde{D}}$ is exactly the blow–up of S in a singular point and the union of the components with $\widetilde{D}$–degree zero is the exceptional divisor of the blow–up.

Let us briefly recall the notion of rational double point of a projective surface.

Definition 1.12. A *rational double point* Q of a surface S is a double point such that the exceptional set $E = E_1 + \cdots + E_n$ on the surface $\widetilde{S} = Bl_Q(S)$ fulfills the following requirements:
i) $E_1 \cup \cdots \cup E_n$ is connected;
ii) $E_i \cong \mathbb{P}^1$ for all $i = 1, \ldots, n$;
iii) $E_i^2 = -2$ for all $i = 1, \ldots, n$;
iv) the lattice $\langle E_1, \ldots, E_n \rangle$ generated by the irreducible components of E is negative definite.

It is well–known that a rational double point of a surface is of one of the following types: A_n, D_n, E_6, E_7, E_8 (see [1], 3.32). The dual graphs of the rational double points of type A_n and D_n are the following, where the n components are represented by the n vertices of the graph and two meeting components are connected by a segment (see for instance [5], Ch.3):



$A_n,\ n \geq 1$

$D_n,\ n \geq 4$

Note that an (A_1)–singularity is an ordinary double point.

We can summarize the above study of the singularities of S arising from main fibres:

Proposition 1.13. *Let $S_0 = \Phi_D(\mathbb{P}(\mathcal{F}))$ be a surface geometrically ruled by conics, $\widetilde{S}_0 \xrightarrow{\sigma} S_0$ be a blow–up which is an isomorphism out of the fibre $f_n \subset \widetilde{S}_0$ and assume that f_n is a main fibre of level n over $f_0 \subset S_0$. Then $\widetilde{D} = \sigma^*(D)$ is base–point–free and the morphism $\Phi_{\widetilde{D}} : \widetilde{S}_0 \longrightarrow S$ is an isomorphism on $\widetilde{S}_0 \setminus f_n$. Moreover the fibre $F_n := \Phi_{\widetilde{D}}(f_n)$ of S is of one of the following types:*
• $n = 1$: *F_1 is the union of two distinct lines and S is smooth (in this case $\Phi_{\widetilde{D}}$ is an isomorphism everywhere);*
• $n = 2$: *$F_2(A)$ is the union of two distinct lines, whose common point is an ordinary double point of S;*
$F_2(D)$ is the union of two coincident lines, containing exactly two ordinary double points of S;
• $n \geq 3$; *$F_n(A)$ is the union of two distinct lines, meeting in a rational double point of type (A_{n-1});*
$F_n(D)$ is the union of two coincident lines, containing exactly one rational double point of S; in particular, this point is of type (A_3), if $n = 3$, and of type (D_n), if $n \geq 4$. ⋄

2. Singularities arising from elementary transformations of any type

Remark 2.1. All the fibres F_n of S arising from main fibres f_n of $\widetilde{S}_0 = S_n$ (described in 1.13) are embedded fibres of level n. Namely, let F_n be as before and assume that $f_m \subset S_m$ is a fibre of level m which gives rise to F_n. Since S_m is smooth at each point of f_m (from 1.1), then f_m has to be obtained from F_n by at least n blow–ups. Therefore $m \geq n$, so (from definition 1.3) the level of the embedded fibre F_n is exactly n.

The purpose of this section is to show that a fibre $F \subset S$ arising from a *not main* fibre can be obtained from a suitable *main* fibre (possibly of different level).

As usual, let us begin by describing the first case.

Example 2.2. Consider the fibre $f_2(A)$. In order to obtain a *not main* fibre f_3 we have to blow–up S_2 in a point belonging to a component of $f_2(A)$ having D–degree ≤ 0 (see 1.5), hence either in the vertex $P = e_1 \cap e_2$ (or equivalently $f_0 \cap e_2$) or in a point $Q \in e_1$ which is not a vertex, as the following picture illustrates.

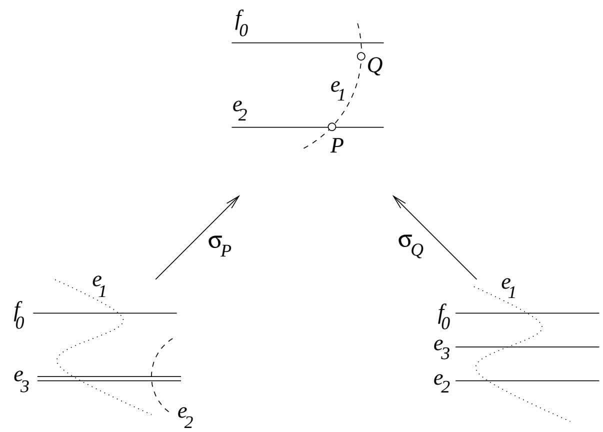


Figure 6

In the first case, the fibre is $f_3 := f_0 + e_1 + e_2 + 2e_3$. Note that the intersections are described in the above picture and the following relations hold:

$$f_0^2 = -1, \quad e_1^2 = -3, \quad e_2^2 = -2, \quad e_3^2 = -1$$
$$\widetilde{D} \cdot f_0 = 1, \quad \widetilde{D} \cdot e_1 = -1, \quad \widetilde{D} \cdot e_2 = 0, \quad \widetilde{D} \cdot e_3 = 1.$$

Therefore $e_1 \subset B$. Then we have to compute the $(\widetilde{D} - e_1)$–degree of the components:

$$(\widetilde{D} - e_1) \cdot f_0 = 0, \quad (\widetilde{D} - e_1) \cdot e_1 = 2, \quad (\widetilde{D} - e_1) \cdot e_2 = 0, \quad (\widetilde{D} - e_1) \cdot e_3 = 0.$$

This proves that $\Phi_{\widetilde{D}-e_1}$ contracts all the components of f_3 but e_1, hence $\Phi_{\widetilde{D}-e_1}(e_1)$ is a smooth conic: the main fibre of level 0.
In the second case, the fibre is $f_3 := f_0 + e_1 + e_2 + e_3$. It can be easily shown that:

$$f_0^2 = -1, \quad e_1^2 = -3, \quad e_2^2 = -1, \quad e_3^2 = -1$$
$$\widetilde{D} \cdot f_0 = 1, \quad \widetilde{D} \cdot e_1 = -1, \quad \widetilde{D} \cdot e_2 = 1, \quad \widetilde{D} \cdot e_3 = 1.$$

As before, $e_1 \subset B$. So we have to compute the $(\widetilde{D} - e_1)$–degree of the components:

$$(\widetilde{D} - e_1) \cdot f_0 = 0, \quad (\widetilde{D} - e_1) \cdot e_1 = 2, \quad (\widetilde{D} - e_1) \cdot e_2 = 0, \quad (\widetilde{D} - e_1) \cdot e_3 = 0.$$

As in the previous case, $\Phi_{\widetilde{D}-e_1}$ contracts f_0, e_2, e_3, while $\Phi_{\widetilde{D}-e_1}(e_1)$ is a smooth conic, so isomorphic to f_0.

The case $f_2(D)$ is analogous. Both cases show that, if f_3 is any possible fibre of level 3, but not a main fibre, then $F_3 = \Phi_{\widetilde{D}-B}(f_3)$ is of type f_0. Concerning the general case, we want to show the following:

Claim: Let $F \subset S$ be a fibre obtained as $\Phi_{\widetilde{D}-B}(f_n)$, where f_n is a fibre of level n of $\widetilde{S}_0 = S_n$. If f_n is *not* a main fibre, then F is an embedded fibre of level m, where $m < n$, i.e. it can be obtained as $F = \Phi_{\widetilde{D}-B}(f_m)$, where f_m is a main fibre of level m on S_m for a suitable $m < n$.

Remark 2.3. It i is enough to show the claim when f_n is not a main fibre, but comes from a main fibre by a blow–up in a "not admissible" single point, in the sense of 1.10. So we can assume that:
- f_{n-1} is a main fibre of level $n-1$ on S_{n-1};
- $P_n \in f_{n-1}$;
- P_n belongs to a component of f_{n-1} having $\widetilde{D}$–degree zero.

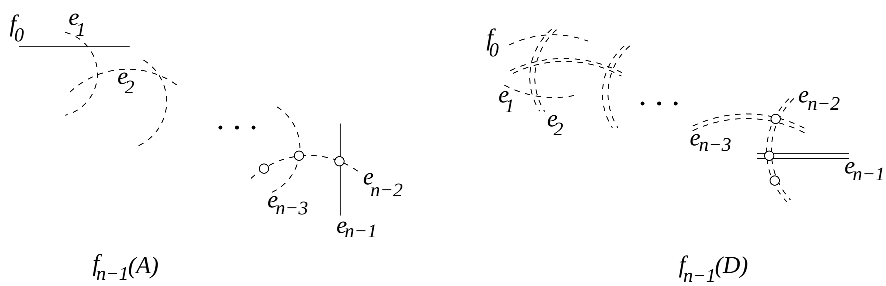


Figure 7

Looking at the picture above, it is clear that the cases to be considered are the following:
- $P_n \in f_{n-1}(A)$
  (1) P_n is a vertex and belongs also to a component of self–intersection -1 (e.g. $P_n = e_{n-1} \cap e_{n-2}$);
  (2) P_n is not a vertex (e.g. $P_n \in e_{n-2}$ and no other component);
  (3) P_n is a vertex belonging to two components having both self–intersection -2 (e.g. $P_n = e_{n-2} \cap e_{n-3}$).
- $P_n \in f_{n-1}(D)$
  (4) P_n is a vertex and belongs also to a component of self–intersection -1 (e.g. $P_n = e_{n-1} \cap e_{n-2}$);
  (5) P_n is not a vertex (e.g. $P_n \in e_{n-2}$ and no other component);
  (6) P_n is a vertex belonging to two components having both self–intersection -2 (e.g. $P_n = e_{n-2} \cap e_{n-3}$).

In order to compute the singularities arising in the above cases, let us introduce some notation. Let D be the bisecant divisor on the surface S_{n-1}, so $\deg_D(f_{n-1}) = 2$. Let $C_0 := f_0$, $C_i := e_i$, for $i = 1, \dots, n-1$. Set also $I_{n-1} := (C_i \cdot C_j)_{i,j=0,\dots,n-1}$ the intersection matrix of the main fibre f_{n-1} of level $n-1$ and D_{n-1} the vector of the D–degrees of the components, i.e. $D_{n-1} := (\deg_D(C_0), \deg_D(C_1), \dots, \deg_D(C_{n-1}))$. Finally, let us introduce the vector consisting of the multiplicities of the components in the fibre f_{n-1} (and of the other fibres arising from the blow–ups). Set

$$\overline{\mu}(f_{n-1}) := (\mu(C_0), \mu(C_1), \dots, \mu(C_{n-1})).$$

Looking at Figure 7, it is clear that: $\overline{\mu}(f_{n-1}(A)) = (1, 1, \dots, 1))$ and $\overline{\mu}(f_{n-1}(D)) = (1, 1, 2, \dots, 2))$. Let us begin from the data of the two fibres $f_{n-1}(A)$ and $f_{n-1}(D)$:

$$\underbrace{\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \vdots \\ \\ 1 \\ 1 \\ 1 \end{vmatrix}}_{\overline{\mu}(f_{n-1}(A))} \qquad \underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ \vdots & & & & \ddots & & & \vdots \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}}_{I_{n-1}(A)} \qquad \underbrace{\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \\ \vdots \\ \\ 0 \\ 0 \\ 1 \end{vmatrix}}_{D_{n-1}(A)}$$

and

$$\underbrace{\begin{vmatrix} 1 \\ 1 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 2 \\ 2 \end{vmatrix}}_{\overline{\mu}(f_{n-1}(D))} \quad \underbrace{\begin{pmatrix} -2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}}_{I_{n-1}(D)} \quad \underbrace{\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{vmatrix}}_{D_{n-1}(D)}$$

Note that, in both cases (A) and (D), the scalar product of $\overline{\mu}(f_{n-1})$ and each row of I_{n-1} is zero, while $\overline{\mu}(f_{n-1}) \cdot D_{n-1} = 2$.

Consider now the blow–up at P_n and let $\widetilde{D}$ be the strict transform of D on S_n. Let us denote again by $C_0, \ldots, C_{n-1}, C_n$ the components of the fibre f_n, where C_n is the exceptional divisor of the blow–up σ_{P_n}. We denote by $\widetilde{I}_n$ the intersection matrix of f_n and by $\deg_{\widetilde{D}}$ the vector of the $\widetilde{D}$–degrees of its components, i.e. $\deg_{\widetilde{D}} := (\deg_{\widetilde{D}}(C_0), \deg_{\widetilde{D}}(C_1), \ldots, \deg_{\widetilde{D}}(C_n))$. Finally set, for each h and k: $\Sigma_h^k = \sum_{i=h}^k C_i$.

For sake of brevity, we examine explicitly only the first case.

Case (1): $f_{n-1}(A) \ni P_n = C_{n-1} \cap C_{n-2}$
Since $\deg_{\widetilde{D}}(C_{n-2}) = -1$, then C_{n-2} is contained in the base locus of $\widetilde{D}$. So we compute the degrees of the components of f_n with respect to $\widetilde{D} - C_{n-2}$. It is immediate to see that $\deg_{\widetilde{D}-C_{n-2}}(C_{n-3}) = -1$, so also C_{n-3} is contained in the base locus of $\widetilde{D}$. We then iterate this computation up to a degree vector whose components are all non–negative. This is the intersection matrix $\widetilde{I}_n$ and the list of the degree vectors:

$$\underbrace{\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{vmatrix}}_{\overline{\mu}} \underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{\widetilde{I}_n} \underbrace{\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{vmatrix}}_{\deg_{\widetilde{D}}} \underbrace{\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{vmatrix}}_{(*_1)} \underbrace{\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}}_{(*_2)} \underbrace{\begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}}_{(*_3)} \cdots \underbrace{\begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}}_{(*_4)} \underbrace{\begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}}_{(*_5)}$$

where

$$(*_1) : \deg_{\widetilde{D}-C_{n-2}} \quad (*_2) : \deg_{\widetilde{D}-\Sigma_{n-3}^{n-2}} \quad (*_3) : \deg_{\widetilde{D}-\Sigma_{n-4}^{n-2}} \quad (*_4) : \deg_{\widetilde{D}-\Sigma_2^{n-2}} \quad (*_5) : \deg_{\widetilde{D}-\Sigma_1^{n-2}}.$$

Hence set $B := \Sigma_1^{n-2} = \sum_{i=1}^{n-2} C_i$. First note that $\Phi_{\widetilde{D}-B}$ contracts C_n, hence the blow–up in P_n is somewhat irrelevant and the image of $\Phi_{\widetilde{D}-B}$ is a surface which can be recovered from a main fibre of lower level.
In order to precisely understand the kind of the fibre we obtain, let us examine the contracted locus.
As appears from the vector $\deg_{\widetilde{D}-\Sigma_1^{n-2}}$, it consists of three connected components: C_0, $C_{n-1} \cup C_n$ and $C_2 \cup \ldots \cup C_{n-3}$. The first two components give rise to two smooth points, respectively. Moreover, as the following picture shows, $\Phi_{\widetilde{D}-B}$ factorizes as

$$\Phi_{\widetilde{D}-B} = con(C_0, C_2, C_3, \ldots, C_{n-3}, C_{n-1}, C_n) = con(C_2, C_3, \ldots, C_{n-3}) \circ con(C_{n-1}) \circ con(C_0, C_n).$$

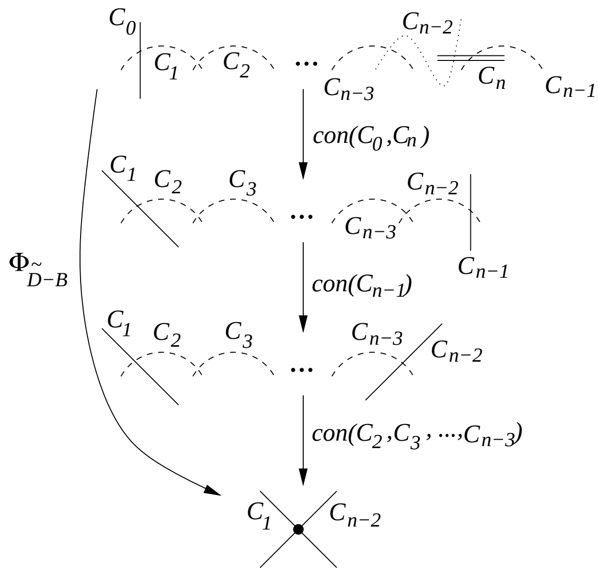


Figure 8

On the other hand, the last component $C_2 \cup \ldots \cup C_{n-3}$ of the contracted locus gives rise to a rational double point of type (A_{n-4}). Therefore $F \subset S = \Phi_{\widetilde{D}-B}(\widetilde{S}_0)$ is an embedded fibre of level $n-3$.

With similar arguments, the whole claim can be proved. Hence, using 1.13 and the claim, we have shown in this way the following result:

Theorem 2.4. *Let $S \subset \mathbb{P}^N$ be a projective surface ruled by conics over a smooth irreducible curve. Then the degenerate fibres and the singular points of S are as follows:*

$i)$ *If a fibre is the union of two distinct lines then S is either smooth along this fibre or singular only at the common point of the lines; in this case the singularity is a rational double point of type (A_n), $n \geq 1$.*

$i)$ *A fibre which is the union of two coincident lines contains either exactly two ordinary double points of S or exactly one singular point of S; in this case the singularity is a rational double point either of type (A_3) or of type (D_n), $n \geq 4$.* ◇

Corollary 2.5. *Let $S \subset \mathbb{P}^N$ be a projective surface ruled by conics over a smooth irreducible curve. Then the singular points of S are rational double points of type (A_n) or (D_n).* ◇

Remark 2.6. By 0.3 and 0.10, $(i)-(ii)$, every projective surface S ruled by conics can have at worst (finitely many) normal Gorenstein singularities. Assume, for sake of simplicity, that S has exactly one singularity. Clearly, the exceptional divisor of the resolution of singularities $\widetilde{S}_0$ of S is contained in the degenerated fibre of the composition

$$\pi := p \circ \sigma : \ \widetilde{S}_0 \longrightarrow C$$

where $\sigma : \widetilde{S}_0 \to S$ is as in diagram (1) and $p : \widetilde{S}_0 = \Phi_D(\mathbb{P}(\mathcal{F})) \longrightarrow C$ is the canonical projection.

Let us observe that a result of Badescu concerning nonrational ruled surfaces (see [1], Lemma 14.35) holds also in a slightly different situation: namely, if $\pi : X \to B$ is a surjective morphism from a smooth projective surface X to a smooth rational curve B, whose general fibre is a smooth rational curve, then $H^1(\mathcal{O}_Z) = 0$ for every positive divisor Z of support contained in a degenerated fibre of π and such that the intersection matrix of Z is negative definite.

Hence, one can apply this result to the morphism $\pi : \widetilde{S}_0 \to C$, obtaining in particular that $H^1(\mathcal{O}_Z) = 0$ for every positive divisor Z of support contained in the exceptional divisor of $\sigma : \widetilde{S}_0 \to S$.

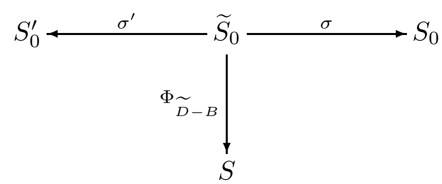
Therefore, performing the same argument for each singular point of S, one obtains via a criterion of M. Artin (see [1], Lemma 3.8 and Definition 3.17) that every singularity of S is rational.

Finally, (using [1], Corollary 4.19) any rational surface singularity which is Gorenstein is a rational double point.

These general arguments show that every projective surface ruled by conics can have only rational double points as singularities. However, the main result of this paper (beside the constructive procedure performed in sections 1 and 2) gives a more precise description of the singularities: namely, it tells us that every projective surface ruled by conics can have only rational double points of type $\mathbf{A}_n$ ($n \geq 1$), or of type $\mathbf{D}_n$ ($n \geq 4$), as singularities. In other words, the rational double points of type $\mathbf{E}_n$, with $n = 6, 7, 8$, cannot occur on such a surface.

3. Birational models of a surface ruled by conics

We want to describe the surfaces geometrically ruled by conics which give rise to a surface S ruled by conics, i.e. we want to find the surfaces $S_0, S'_0, \ldots$ such that for each of them diagram (1) holds, as follows:



Since this is a local study, we can assume that S has only one singular embedded fibre F_n of level n. Clearly $F_n \subset S$ can be obtained in a unique way from a main fibre $f_n \subset \widetilde{S}_0$ having the same level n (see 2.1). Therefore the initial problem can be reduced to the following:
For each n, choose a main fibre f_n of $\widetilde{S}_0$. Describe all the surfaces S_0 and the blow–ups $\sigma : \widetilde{S}_0 \to S_0$ such that f_n maps to a suitable fibre f_0 of S.

Example 3.1. The unique fibre of level 1 is $f_1 \subset \widetilde{S}_0$, obtained from f_0 simply by blowing–up S_0 at a point $P_1 \in f_0$. Clearly, $f_1 = f_0 \cup e_1$ can be obtained in exactly one other way. If we contract its component f_0, then we obtain a surface $S'_0 = elm_{P_1}(S_0)$, whose fibre corresponding to f_1 is a smooth conic e_1 and the component f_0 contracts to a (smooth) point, say F_0, of e_1.
Finally, it is also clear that the above contraction is the blow–up of S'_0 at F_0. The following picture describes this situation:

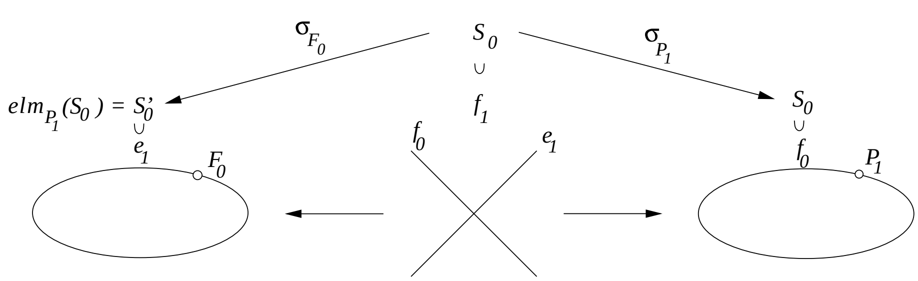


Figure 9

Example 3.2. Consider now the main fibre $f_2(A) \subset \widetilde{S}_0$, arising from f_0 via the blowing–up $\sigma_{P_1P_2}$, where $P_1 \in f_0$ and P_2 is a point infinitely near to P_1 along a direction which is transversal to f_0.
It is clear that $\sigma_{P_1P_2}$ contracts the components e_1 and e_2 of $f_2(A)$. The following figure illustrates the other two surfaces, S'_0 and S''_0 say, obtained by contracting f_0 and e_2 (in the middle) and f_0 and e_1 (on the left hand side), respectively.

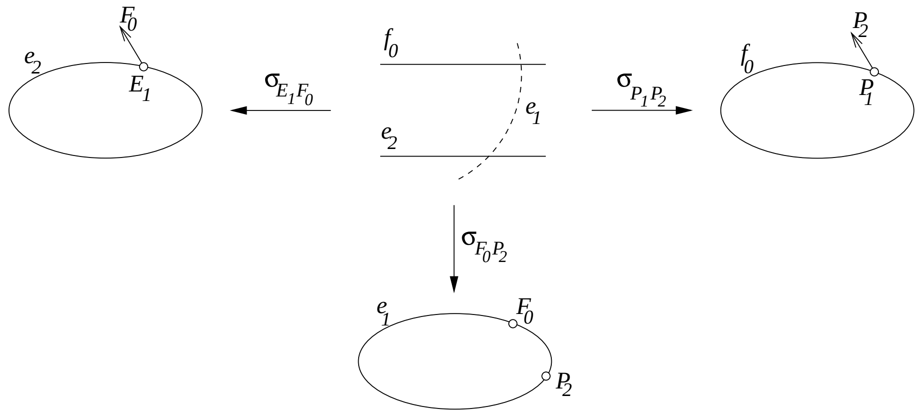


Figure 10

In the first case, we contract f_0 and e_2 to two distinct points, say F_0 and E_2, of the fibre $e_1 \subset S'_0$.
In the second case, we obtain a smooth conic $e_2 \subset S''_0$ and the contraction turns out to be the blowing–up of S''_0 at $E_1 \in e_2$ and at F_0, a point infinitely near to E_1. The exceptional divisor of σ_{E_1} is e_1, moreover $F_0 \in e_1$ and its exceptional divisor is e_2. Hence $\sigma_{P_1P_2} = con(e_1, e_2)$, $\sigma_{F_0P_2} = con(f_0, e_2)$, $\sigma_{EF_0} = con(e_1, f_0)$.
Finally note that the surfaces S_0, S'_0, S''_0 are related by elementary transformations, as in the following picture:

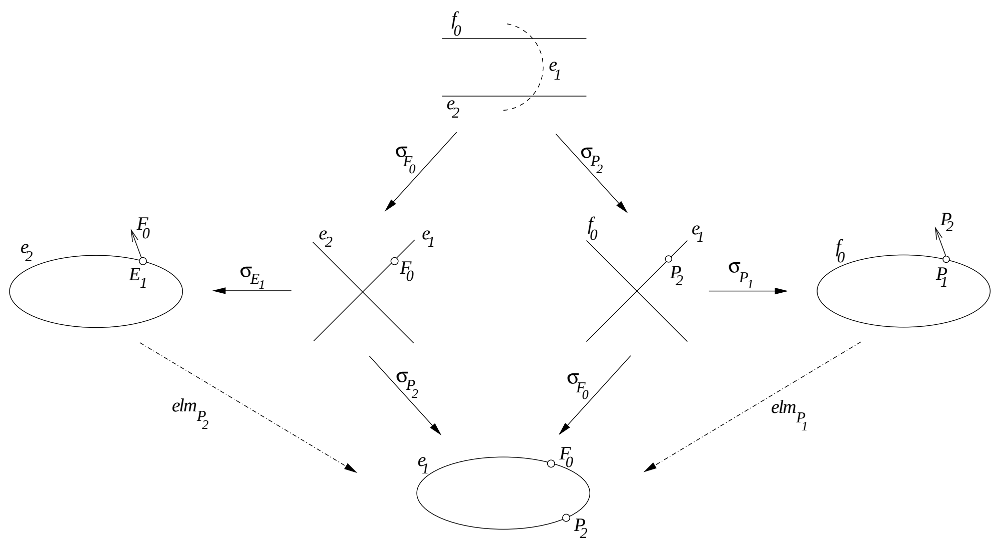


Figure 11

Therefore $S'_0 = elm_{P_1}(S_0)$ and $S''_0 = elm_{P_2}(elm_{P_1}(S_0))$.

From the two examples above it is clear that, given a main fibre $f_n(A) = \ f_0 \cup e_1 \cup e_2 \cup \ldots \cup e_n$, we obtain $n+1$ geometrically ruled surfaces by contracting n components of it. Moreover, if one of them is S_0, then each other is obtained from S_0 by a chain of elementary transformations.
From an analogous procedure, one can see the general behaviour also in case $f_n(D)$: the only possibilities to contract it to a smooth conic are $con(e_n, e_{n-1}, \ldots, e_2, e_1)$ and $con(e_n, e_{n-1}, \ldots, e_2, f_0)$. Therefore, the only geometrically ruled surfaces giving rise to $\widetilde{S}_0$ are S_0 and $elm_{P_1}(S_0)$.

The above observations lead to the following result:

Theorem 3.3. *Let S be a surface ruled by conics and assume that it has a unique singular fibre F_n embedded of level n. Let S_0 and S'_0 be two distinct surfaces geometrically ruled by conics giving rise to S with a minimal number of blow–ups and contractions. The following facts hold:*
- *if F_n is of type $F_n(A)$ then $S'_0 \in \{elm_{P_1}(S_0), elm_{P_1P_2}(S_0), \ldots, elm_{P_1P_2\ldots P_n}(S_0)\}$, where $P_1 \in S_0$ is a suitable point and each P_i is a suitable point, infinitely near to P_1 of order $i-1$;*
- *if F_n is of type $F_n(D)$ then $S'_0 = elm_{P_1}(S_0)$, where $P_1 \in S_0$ is a suitable point.* ⋄

Remark 3.4. Clearly, if S has more than one degenerate fibre, then the previous theorem can be generalized in a obvious way. For instance, if S has two degenerate fibres $F_n(A)$ and $F_m(A)$ and S_0, S_0' are two distinct geometrically ruled by conics surfaces as in 3.3, then S_0' belongs to the set

$$\{elm_\Sigma(S_0) \mid \Sigma \subseteq \{P_1, \ldots, P_n, Q_1, \ldots, Q_m\}\}$$

where $P_1, Q_1 \in S_0$ are suitable points, $P_2, \ldots, P_n$ are suitable points, infinitely near to P_1, and $Q_2, \ldots, Q_m$ are suitable points, infinitely near to Q_1. Clearly, Σ has to fulfil the requirement: if P_i (resp. Q_i) $\in \Sigma$ then P_h (resp. Q_h) $\in \Sigma$ for all $h < i$.

Definition 3.5. Let $F^{(1)}, \ldots, F^{(p)}$ be the degenerate fibres of S and let l_i be the level of $F^{(i)}$, for $i = 1, \ldots, p$. If $\sum_{i=1}^{p} l_i = L$, we say that S is of *level* L.

The results concerning one singular fibre (3.3) and two singular fibres (3.4) can be easily generalized as follows.

Remark 3.6. Let S be a surface, ruled by conics of level L and S_0, S_0' be two distinct surfaces, geometrically ruled by conics as in 3.3. Then S_0' belongs to the set

$$\{elm_\Sigma(S_0) \mid \Sigma \text{ is a suitable set of points and } |\Sigma| \leq L\}.$$

Conversely, note that each surface in the above set is geometrically ruled by conics and gives rise to S with exactly L blow–ups (followed by contraction). Therefore, the above set coincides with the following:

$$\{S_0 \mid S_0 \text{ is a g.r.s. and } S \text{ can be obtained from it by a sequence of } L \text{ blow–ups and contractions }\}.$$

Definition 3.7. We denote the above set of geometrically ruled surfaces of level L by $\mathbf{GRC}_L(S)$.

All the previous constructions can be interpreted in terms of projections. More precisely, we show that $S = \Phi_{\widetilde{D}-B}(\widetilde{S}_0) \subset \mathbb{P}^M$ can be obtained from $S_0 = \Phi_D(\mathbb{P}(\mathcal{F})) \subset \mathbb{P}^N$ by a linear projection from a suitable centre. In the following pictures, Φ stands for $\Phi_{\widetilde{D}-B}$.

Example 3.8. A fibre $F_2(A) \subset S$ can be obtained as the projection of f_0 from two points, either distinct or coincident. Let us consider the first case: $P_1, P_2 \in f_0$ and $P_1 \neq P_2$. The following picture illustrates that $\pi_{P_1P_2}$ factors trough the projection π_{P_1} (giving a fibre of type F_1).

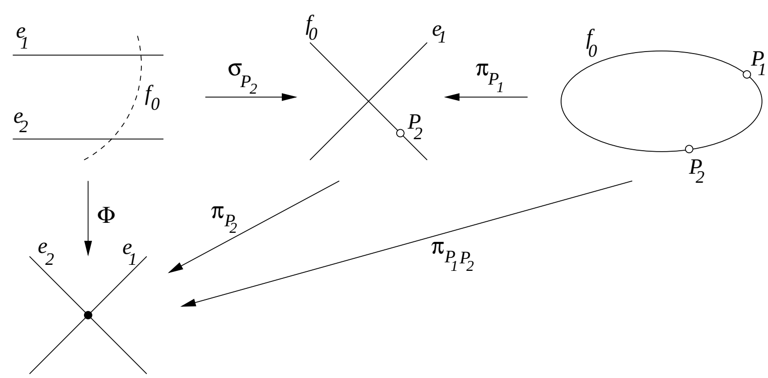


Figure 12

It is clear that $\pi_{P_1P_2}$ contracts f_0 to a point. This is due to the fact that $\pi_{P_1P_2}$ is the projection centered in the line P_1P_2 lying on the plane spanned by the conic f_0.
The same argument runs in the case $P_1 \in f_0$ and P_2 infinitely near point *along a transversal direction.*

Similar construction can be performed in the general case $F_n(A)$ as the following picture shows.

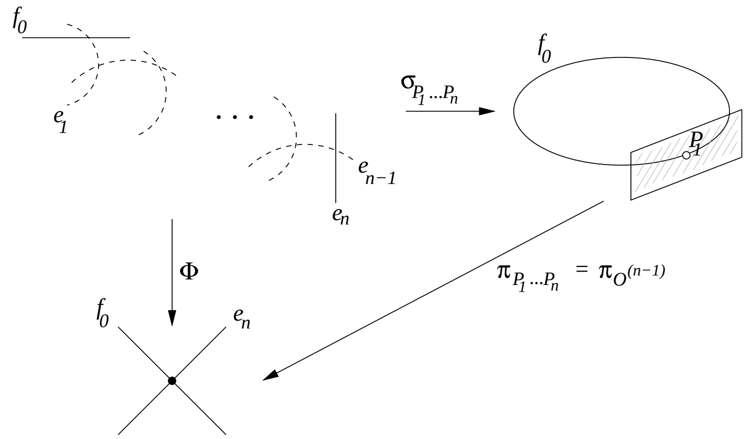


Figure 13

and an analogous argument holds for $F_n(D)$. In these cases, $O^{(n-1)}$ denotes the osculating space of dimension $n-1$ to a suitable unisecant curve $U \subset S_0$ passing through P_1 and such that $P_2, \dots, P_n$ are points infinitely near to P_1 along U. In this way, it is immediate to show the following fact:

Proposition 3.9. *Let $S \subset \mathbb{P}^M$ be a surface ruled by conics of level L and $S_0 \subset \mathbb{P}^N$ be a surface in $\mathbf{GRC}_L(S)$. Then there exists a projection $\pi : \mathbb{P}^N \longrightarrow \mathbb{P}^M$ such that $S = \pi(S_0)$ and $\deg(S) = \deg(S_0) - L$.* $\diamond$

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The referee of an earlier version of [3] pointed out that a certain argument we gave in that paper was insufficiently clear. His observations led us to begin this further, more complete study: we are grateful to that referee for his accuracy.

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