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Giovanni Bellettini • Valentina Beorchia • Maurizio Paolini • Franco Pasquarelli

Shape Reconstruction from Apparent Contours

Theory and Algorithms

Springer

Giovanni Bellettini
Department of Mathematics
University of Rome Tor Vergata
Rome
Italy

Valentina Beorchia
University of Trieste Department
of Mathematics & Geosciences
Trieste
Italy

Maurizio Paolini
Università Cattolica del Sacro Cuore
Department of Mathematics & Physics
Brescia
Italy

Franco Pasquarelli
Università Cattolica del Sacro Cuore
Department of Mathematics & Physics
Brescia
Italy

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Contents

Introduction

Computer vision and image processing have become active areas of basic and applied mathematical research, due to their impact in the development of new technologies and to the related interesting theoretical problems. The vastness of applications requires a multi-disciplinary study; a selection of involved mathematical areas includes, in particular, calculus of variations, optimization and partial differential equations, probability and statistics, topology and differential topology, differential and discrete geometry, affine geometry, harmonic analysis, inverse problems and numerical analysis; see for instance [2, 9, 18, 19, 26, 28, 37, 57, 62, 63, 68, 70, 75, 89]. The areas of application in ordinary life are numerous and we could mention: medical imaging (image reconstruction, interpretation and aid to diagnostics), video processing and analysis, stereo vision, 3D reconstruction and shape recognition from image sequences, and the restoration and interpretation of satellite images; we refer the reader to [69] and references therein.[1] These subjects are mostly directed by applications, but they require solid grounded theories, appropriate for instance to ensure robustness of the related algorithms.

The aim of this book is to investigate one of the central problems of computer vision,[2] namely the *topological and algorithmical reconstruction of a three-dimensional scene* $E \subset \mathbb{R}^3$, composed of various smooth bounded solid objects (the connected components of E) starting from information on a generic orthogonal plane projection[3] of E. As explained in detail in Chap. 1, the original motivation that led us to this study came from the calculus of variations, in the effort of finding an action functional $\mathcal{F}$ (introduced in [12]) defined on plane graphs and whose minimization should give information on the depth ordering of the various objects composing the scene. Postponing the technical discussion on the variational aspects

[1] See also [17, 29, 51, 77, 78, 81].

[2] See for instance [30–32, 38, 39, 41–44, 55, 56, 65, 66, 76, 82, 83] and references therein.

[3] In this book we shall not consider the case when two or more simultaneous projections are involved; the case of shapes evolving in time is, instead, related to ambient isotopic deformations of the objects, an issue which will be treated in various chapters.

Fig. 1 Two partially overlapping objects: neither of them is in front of the other

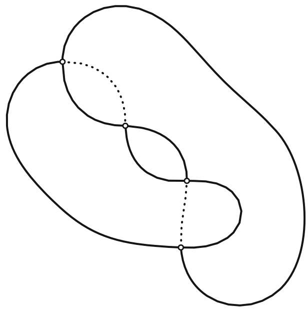
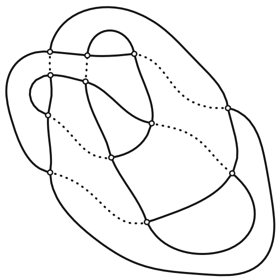


Fig. 2 A knotted torus



of the model to Chaps. 1 and 11, it is worth recalling that the functional $\mathcal{F}$ has been introduced with the purpose of removing a difficulty in a previous model by Nitzberg and Mumford [65, 66] related to self-occlusions: in particular, in Figs. 1 and 2, we draw two interesting and typical examples of self-occlusions, which can be analysed using the functional $\mathcal{F}$. We also remark that, as observed in [12], admissible configurations for $\mathcal{F}$, which can be arguably minimizers under a certain range of parameters, may give, as a result, the illusory contours[4] of the famous Kanizsa triangle [54], as discussed in Sect. 1.5; see, more specifically, Figs. 1.6, 1.7, and 1.8.

In order to carry out our analysis on the topological and algorithmical aspects of the reconstruction problem of a three-dimensional shape E, let us briefly explain what is the information we need on one of its stable plane projections. Denote by Σ the boundary ∂E of E; for a given generic projection direction, let us consider the so-called *visible apparent contour* $\mathrm{vis}(G_\Sigma) \subset \mathbb{R}^2$ of Σ, an oriented plane graph which is the natural sketch of Σ that one usually draws by hand in order to represent the scene. For instance, for the solid shape in Fig. 3, the bold curves in Fig. 5 represent the visible apparent contour. In order to have a better picture of the various graphs involved, it is often useful to imagine the shape to be semi-transparent, as in Fig. 4. Due to the genericity assumption on the projection direction, it turns out that

[4]We shall treat contours without corners; as we shall explain in Chap. 1, a slight smoothing of the original Kanizsa image does not change the qualitative properties of the example, and does not alter the presence of illusory contours.

Fig. 3 The three-dimensional scene E producing the apparent contour of Fig. 5. Image taken from [14]

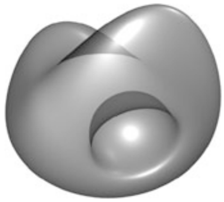


Fig. 4 The same three-dimensional scene as in Fig. 3, but made semi-transparent. Image taken from [14]

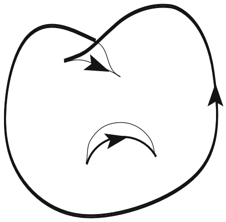


Fig. 5 The *bold graph* represents the visible part of the apparent contour of the three-dimensional scene in Fig. 3; the *whole graph* represents its apparent contour. Image taken from [14]

the singular points of $\mathrm{vis}(G_\Sigma)$, if any, are only of two types: terminal points and T-junctions. The example of Fig. 5 shows three terminal points and one T-junction. It is not difficult to realize that $\mathrm{vis}(G_\Sigma)$ is a subset (usually, but not always, a proper subset) of another oriented graph G_Σ, the so-called *apparent contour* of Σ[5]; see Fig. 5 again. The graph G_Σ has two types of singular points only: cusps, arising as local completions of terminal points of $\mathrm{vis}(G_\Sigma)$, and X-junctions (called crossings), local completions of T-junctions. Accordingly, the apparent contour of Fig. 5 has four cusps and one X-junction. It is useful to observe that G_Σ has here a geometric meaning: it is the plane projection of a finite set of smooth pairwise disjoint closed curves lying on Σ (and called *critical set*, or also *singular set*), obtained as the set of all points of Σ where the tangent plane contains the projection direction. Now, the crucial three-dimensional information carried by G_Σ is contained in a labelling [25, 53, 87, 88], which is a number $d_\Sigma(a) \in \mathbb{N}$ attached to any arc a, and representing the number of sheets of Σ in front of the part of the singular curve projecting on that arc. Accordingly, $\mathrm{vis}(G_\Sigma)$ is the closure of the set $\{d_\Sigma = 0\}$. For instance, in Figs. 1 and 2, we have $d_\Sigma = 2$ on the dotted arcs, and $d_\Sigma = 0$ on the visible arcs. The *labelled apparent contour*, namely the pair (G_Σ, d_Σ), is the starting point for the definition of the action functional $\mathcal{F}$, leading to the minimization principle described in Chaps. 1 and 11. We mention here that another

[5] Sometimes, we shall call G_Σ apparent contour of E.

symbol, denoted by f_Σ, appears in the domain of $\mathcal{F}$: for $x \notin G_\Sigma$ in the projection plane, the value $f_\Sigma(x)$ represents the total number of intersections between Σ and a light ray emanating from x (see Fig. 1.1 (right)). Such a locally constant function, which can be easily recovered as a doubled winding number with respect to G_Σ, is useful for various reasons, one of them being that it simplifies the presentation of the model.

Several theoretical and practical questions arise in a natural way, and show beautiful and unexpected relations between calculus of variations [62], singularity theory [4–6, 20, 86], Morse theory [10] and knot theory [73]. Moreover, we stress that the techniques we use when dealing with most of such issues fit naturally in an algorithmic setting: as we shall see, it is one of our primary goals to analyse this algorithmic part, with implementations, experiments and computed examples.

We shall be interested in investigating:

(i) the completion problem, namely: the characterization of those plane graphs which are visible part of a labelled contour graph;
(ii) an algorithmic construction of a completion, to be implemented as a computer program;
(iii) the characterization of those labelled plane graphs which are apparent contours of some smooth stable three-dimensional scene;
(iv) an algorithmic reconstruction of the topology of a three-dimensional smooth shape starting from a labelled apparent contour, and its implementation on a computer;
(v) a list of topological invariants of three-dimensional shapes, which can be directly computed starting from the apparent contour, and that can be implemented on a computer;
(vi) the recognition of two labelled apparent contours which are apparent contours of two ambient isotopic shapes, using a finite sequence of elementary moves, taken from a complete finite set. In other words, what are the moves on the labelled apparent contours that relate two *embedded* surfaces, deformable into each other by a smooth path of embeddings?
(vii) a computer program aiming to implement the elementary moves on apparent contours, and, more in general, capable to manage labelled (or unlabelled) apparent contours from a structural/topological point of view;
(viii) the problem of elimination of cusps, namely: how to use the elementary moves in order to modify a labelled apparent contour into another one without cusps, representing a three-dimensional shape, ambient isotopic to the original one;
(ix) the generalization of some of the above problems, in particular the algorithmic parts, to more general situations, concerning for instance abstract closed (not necessarily orientable) surfaces;
(x) a variational study of the functional $\mathcal{F}$, such as an investigation of the properties of sequences of labelled apparent contours having a uniform bound on the action.

Problem (i), which is global in nature, addresses necessary and sufficient conditions on an oriented plane graph K with nonexterior terminal points and T-junctions, in order to be the visible part of what we shall call a *complete labelled contour graph* (G, d), having cusps and X-junctions. Here the function d, defined on the arcs of G, is a labelling, and must fulfill the same consistency properties[6] shared by the function d_Σ. We can roughly rewrite (i) with the statement

$$\text{given } K \quad \exists\,(G, d) \ \text{ such that } \ G \supseteq K \text{ and } \ K = \{d = 0\}. \tag{1}$$

Following [14], in Chap. 4 we give a constructive solution to this problem (see Theorem 4.3.1, that we call the *completion theorem*), based on a suitable *Morse description* of K and G. We note here that the aim of the completion theorem is not to provide the "simplest" completion of K, whatever simplest could mean[7]; the scope of the result is to show that the conditions[8] imposed on K are sharp, and allow us to construct at least one completion. The problem of returning, as output of the completion, a "simple" graph G is related to points (v) and (vi), and will be addressed below. An improvement of the results of [14] is given by Corollary 4.5.1, based on the introduction of the background; this represents a further degree of freedom, which allows us to fix a priori the regions of G where $f = 0$. Here the function f is defined on $\mathbb{R}^2 \setminus G$; it can be obtained as a doubled winding number with respect to G, and has to satisfy various consistency relations with the labelling d. Examples 4.6.2 and 4.6.4 clarify the interest in the use of Corollary 4.5.1, in connection with the reconstruction of the apparent contour of a standard torus; see also the example illustrated in Sect. 9.1 with the use of the `visible` program. We remark that the completion of a visible contour is inherently nonunique, even when forcing a priori the background (the region where $f = 0$); this is clarified with the example displayed in Fig. 9.10.

The Morse description, explained in Sect. 2.5, is a convenient way to encode all topological information of a graph, and fits well for practical purposes. This is clearly seen when dealing with problem (ii): the software code described in Chap. 9, in particular the `visible` program, is an actual implementation of the constructive proof given in the completion theorem. The input of the program is a Morse description of a drawing of a visible contour, see for instance Figs. 9.1, 9.9 and 9.11. The output, provided the graph is completable (namely, it satisfies the necessary and sufficient conditions of the completion theorem) is a complete labelled contour graph, still identified using a textual Morse description, and next graphically reconstructed as a drawing, using the visualization program `showcontour`. The `visible` program also recognizes those graphs K which are not completable (called "impossible graphs"), such as those described in Figs. 9.12 and 9.13.

[6] See Definition 4.2.5.

[7] For instance, a graph with a minimal number of vertices, or without cusps.

[8] See Definition 4.1.8 and Figs. 3.15 and 3.16.

The algorithmic reconstruction of the completion theorem (and of the `visible` program) strongly depends on the Morse description of the visible contour: different Morse descriptions of the same visible contour will in general lead to different structurally non-equivalent 3D shape reconstructions. Conversely, different (but structurally equivalent) visible contours described by the same Morse description will clearly lead to the same reconstruction.

The usefulness of the existence of a consistent labelling d on an oriented complete graph G is that it *characterizes* the apparent contours of smooth *stable* 3D shapes. Following closely the proof of [12] (see also [58, 87, 88]), in Theorem 5.1.1 (called the *reconstruction theorem*) we show how to reconstruct a smooth, not necessarily connected, 3D shape E starting from the labelled graph (G, d); namely, how to find a smooth closed surface $\Sigma := \partial E$ such that

$$G = G_\Sigma, \qquad d = d_\Sigma. \tag{2}$$

The notion of stability[9] employed in this theorem (Definition 2.1.2) goes back to the pioneering works of Whitney [86] and Thom [79], (see also Arnold [3] and Wall [84]) on singularity theory; stability turns out to be a crucial concept, and its generalizations and ramifications are of central importance in the whole book. Just to mention a few consequences of this assumption,[10] it guarantees that all graphs that we consider have a finite number of nodes, that the self-intersections (X-junctions and T-junctions) are double and transverse, and that the cusps are ordinary cusps. Remarkably, stable maps from a closed two- or three-dimensional manifold to a two- or three-dimensional manifold are dense, and their singularities have been classified (see for instance [40] and references therein): these results are the cornerstone for the completeness result illustrated in Chap. 6 and, as a consequence, for a large part of the algorithms described in Chap. 10.

The cut-and-paste proof of the reconstruction theorem is topological in character and constructive. Deferring the technical details to Chap. 5, a couple of related comments are in order. The reconstructed ∂E is unique, up to transformations which do not change the order and the number of intersections of the manifold with the light rays emanating from the projection plane (and therefore do not modify the corresponding labelled apparent contour): this sort of uniqueness result is proven in Theorem 5.1.4. The proof of the reconstruction theorem furnishes an embedded smooth manifold ∂E, but not the "roundest" way to embed it in the ambient space $\mathbb{R}^3$; investigation of this latter problem is beyond the scope of the present book.

Summarizing the discussion concerning points (i)–(iii), we conclude that, starting from a visible contour graph K, we can construct a complete labelled contour graph (G, d) satisfying (1), which, in turn, provides a three-dimensional scene

[9] See, e.g., [40] and references therein.

[10] We assume that the boundary of the scene E is in general position with respect to the projection: using the concept of stability, this means that the restriction to ∂E of the projection is stable, see Sect. 3.2 for the details.

E satisfying (2) and fulfilling the natural sort of uniqueness for such kind of problems. The completion and the cut-and-paste procedure are automatized in a computer program. Moreover, problem (iv) is one of the issues considered in Chap. 10, which aims to be a self-contained user's guide to an original and rather complex computer program for the reconstruction of three-dimensional shapes, based on an analysis of apparent contours. The reconstruction problem is completely solved from an algorithmic point of view; the program `appcontour` reconstructs the topological structure of ∂E, in particular information such as the number of connected components of ∂E and the Euler–Poincaré characteristic of each of them can be obtained, together with information about the relative position in space allowing to distinguish, e.g., between two concentric spheres (E is a hollow sphere) and two mutually external spheres (E is a pair of solid spheres), with the commands "`contour countcc`", "`contour extractcc`", "`contour characteristic`", "`contour ccordering`", "`contour ccparent`" (see Sect. 10.10.1).

When proving results such as the reconstruction theorem, or also when analysing topological invariants of apparent contours, one realizes that a basic idea is to consider the more general concept of apparent contour of a map from a manifold into another manifold.[11] This is a classical topic in differential topology: see for instance [50, 60, 79, 86]. In particular, given a two-dimensional smooth closed (abstract) manifold M and a smooth stable map $\varphi : M \to \mathbb{R}^2$, the *apparent contour* $\mathrm{appcon}(\varphi)$ of φ is the subset of $\mathbb{R}^2$ where the function counting the number of preimages of φ has a jump. It can be equivalently defined as the image in $\mathbb{R}^2$ of the *critical set* (or *singular set*) of φ in M, where the rank of the differential of φ is not maximal. The previously discussed labelled apparent contour of an embedded surface is a special case: in particular, the reconstruction theorem can be restated in terms of *factorization of maps*, as follows. Let (G, d) be a complete labelled contour graph. Then

$$G = \mathrm{appcon}(\varphi) = G_\Sigma \qquad \text{and} \qquad d = d_\Sigma,$$

where φ is a map from a smooth closed two-manifold M to the plane, $\Sigma := \mathrm{e}(M)$ for a smooth embedding e of M into $\mathbb{R}^2 \times \mathbb{R}$, and φ factorizes as

$$\varphi = \pi \circ \mathrm{e}, \tag{3}$$

where π is an orthogonal projection $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, with Σ in general position with respect to π. Indeed, the core of the proof of the reconstruction theorem consists in producing the manifold M as a quotient, and next in embedding it in $\mathbb{R}^3$; the same theoretical procedure is next implemented in the `appcontour` program, as a starting point of the computation of the first fundamental group, as we shall see.

[11] As we shall see, this abstract viewpoint is essential also in Chap. 6.

According to this more general viewpoint, in Chap. 2 we recall a few well-known facts from singularity theory[12] (see, e.g., [7, 8, 40] and references therein), for a stable map φ from a closed smooth manifold $\mathscr{X}$ to a manifold $\mathscr{Y}$, and give some examples. It is worth recalling here that knot theory is the study of stable maps from the circle $\mathbb{S}^1$ to $\mathbb{R}^3$. The choices

$$\mathscr{X} = M, \qquad \mathscr{Y} = \mathbb{R}^2 \text{ or } \mathbb{R}^3$$

(in particular $\dim(\mathscr{X}) = 2$) can be applied to the study of apparent contours of closed not necessarily embeddable (or even not immersible) manifolds in $\mathbb{R}^3$; we quickly touch these issues (point (ix) of the above list) in Sect. 10.17, with the Boy surface (a standard immersion of the real projective plane), the Klein bottle, and examples from the literature such as the Haefliger sphere, the Millet projection of the real projective plane and the Milnor sphere. These examples lead to consider the interesting problem of apparent contours possibly without labelling, a subject that we do not want to further deepen in the present book.

Concerning point (v), in Chap. 7, we study some *invariants* of an apparent contour for a map $\varphi : M \to \mathbb{R}^2$. In the first part of the chapter (Sects. 7.1–7.3), we analyse invariants under diffeomorphisms of the target space $\mathbb{R}^2$. Besides the number of cusps and of crossings of the apparent contour appcon(φ), a third invariant has been considered in [67], and called Bennequin-type invariant, denoted by $BL(\text{appcon}(\varphi))$. This invariant, based on the Bennequin's construction for Legendrian knots [16, 52], does not have an immediate interpretation.[13] Following [13], in Theorem 7.3.1, we show that such an invariant can be obtained solely looking at the apparent contour, without resorting to a Legendrian lift (see, in particular, Definition 7.1.2): indeed, it turns out that the invariant can be computed only taking into account the nodes, the cusps, the extremal points with respect to some height function and the orientation of the apparent contour. Here, again, Morse descriptions of appcon(φ) play a central role. In this way the computation can be implemented into a computer program, and this is done by the program `appcontour`, command "`contour info`". In the last part of the chapter (Sect. 7.4), we suppose that the map φ factorizes through an embedding in $\mathbb{R}^3$ and an orthogonal projection as in (3). Then, we analyse some invariants of the apparent contour under diffeomorphisms of $\mathbb{R}^3$. The computation [12] of the total Euler–Poincaré characteristic $\chi(\Sigma)$ of the surface ∂E of the corresponding solid shape is given in Theorem 7.4.1, in terms only of the apparent contour G_Σ: interestingly, and

[12] such as the notion of stratification [45], see Sect. 6.2.

[13] It is defined as an appropriate linking number of the Legendrian lift of appcon(φ) in the projectivized cotangent bundle $PT^*\mathbb{R}^2$, and its computation for a given apparent contour is not trivial. More precisely, $BL(\text{appcon}(\varphi))$ is defined by taking the sum of the self-linking numbers of the liftings of the components of appcon(φ) and the linking numbers between the liftings of two different components. The self-linking number is itself defined by also taking into account the twisting of a strip constructed by shifting points of the lifted curve by a small amount in the normal direction to the contact plane; we refer to [67] for the precise definition.

a posteriori not surprisingly, the formula for $\chi(\Sigma)$ is independent of the labelling d_Σ on G_Σ; see the discussion in Sect. 7.4.

The chapter concludes with a number of remarks concerning the first fundamental group of E and $\mathbb{R}^3 \setminus E$. In particular, we discuss the Alexander polynomial (focusing mainly on Fox differential calculus [35, 36]) and some invariants of fundamental groups, applied to surfaces with genus two; see Sects. 7.6–7.9. These issues, as well as the actual computation of $\chi(\Sigma)$, are implemented in Chap. 10 (see Sects. 10.7 and 10.9).

Now, let us discuss the (huge) problems listed in points (vi) and (vii) and their consequences (for instance, the solution to point (viii)). To this aim, it is useful to start by recalling the well-known result [1, 21, 64, 71] of Reidemeister in knot theory, asserting that two *link diagrams*[14] represent ambient isotopic links if and only if they can be related by a finite number of local Reidemeister moves or their inverses. We are interested in a similar question for two-dimensional smooth closed manifolds M *embedded* in $\mathbb{R}^3$. Following [15],[15] in Chap. 6 we prove that two generic embeddings of a closed surface M in $\mathbb{R}^3$ are ambient isotopic if and only if their apparent contours can be connected using only a finite set of *elementary moves* (also called *rules*, or *Reidemeister-type moves*) on labelled apparent contours and a finite number of smooth planar isotopies. We refer to Sect. 6.3 for an informal presentation of this result, which is proven in Sect. 6.5, and addressed as the *completeness theorem*. It turns out that there are six basic moves[16] on an apparent contour (see Fig. 6.2 for a graphical representation) originated from a general deformation of the corresponding embedded surface; they can be used in exactly the same way as the Reidemeister moves on link diagrams. The essence of the result is that this set of moves is complete. This means that two embedded

[14]The diagram of a knot, or more generally of a link, is an orthogonal projection of the image of the link onto some generic plane, with the addition of the knowledge of which strand goes over at each crossing. Stability implies that transversal crossings are the only possible singularities of the diagram.

[15]The proof has some similarities with the one described in [24] for the embedding of surfaces in $\mathbb{R}^4$.

[16]Namely, K (from the Russian word *kasanie* = *tangency*), L (lips), B (beak-to-beak), C (cusp-fold), S (swallow's tail) and T (triple point). This list of moves is essentially the same found in the literature for the related subject of maps from two-manifolds into $\mathbb{R}^2$ (see, e.g., [67]), even if the addition of the labelling entails a different classification of the list of moves. Similar classifications appear in various contexts, in particular in Thom's catastrophe theory [80] and in Cerf's theory [27], and in the paper [61] of Mond; see also the papers [59, 72, 85]. Concerning a complete set of Reidemeister moves relating two equivalent knotted surfaces in $\mathbb{R}^4$, we refer to the set of moves found by Roseman [74], to the papers of Carter and Saito [22, 23] where generic embedded surfaces in $\mathbb{R}^4$ are considered, projected in $\mathbb{R}^3$ (diagram) and projected further in $\mathbb{R}^2$, and to the papers [46–49] of Goryunov. We refer to [7, 8, 11, 24, 25] for further information. The results illustrated in Chap. 6 treat the case of embeddings, which are usually not considered in the literature. Considering paths of embeddings concretely means that one has to take into account the behaviour of the labelling at the "critical times" corresponding to the intersection with the strata of the so-called discriminant hypersurface.

surfaces in general position with respect to the projection, that can be deformed into each other, have apparent contours that can be connected using solely a finite sequence of such moves (and a finite number of planar isotopies). A relevant part of Chap. 10 consists in the *implementation* of the above-mentioned moves, which are essential for the results related to Chaps. 4, 7 and 9; see Sects. 10.1 and 10.2. Among the various interesting features of the program `appcontour`, the implementation of the moves allows, in several situations, to "simplify" the apparent contour, thus making possible to recognize the topology of the actual three-dimensional shape to which it corresponds (via the reconstruction theorem). It is however worth recalling that, in the simpler case of knots, there is at the moment no algorithm (and no invariant) which is capable to recognize equivalent knots. The knot group (fundamental group of the complement in $\mathbb{R}^3$), a powerful invariant that can be computed by `appcontour` via a presentation, is capable of distinguishing the unknot; however, it is not a manageable invariant and the problem is shifted to that of recognizing equivalent presentations of the same finitely presented group.[17]

A typical situation is when considering a completion of a visible contour graph provided by the completion theorem; if the completion is so complicated that the corresponding three-dimensional shape is not recognizable, we can resort to the `appcontour` program in order to try to simplify it: in several cases, this makes possible to figure out the scene (see for instance Example 4.6.4).

One interesting application of the completeness theorem is given in Chap. 8, where we give a solution to point (viii): in Theorem 8.3.2, we show that, up to $\mathbb{R}^3$-ambient isotopies, any smooth closed surface embedded in $\mathbb{R}^3$ has an apparent contour without cusps.[18] The proof of this result is based on the judicious application of various combinations of the elementary moves and their inverses. This result is, in some case, another example showing a possible way to simplify an apparent contour. Notice carefully, however, that this is not always the case: indeed, there are situations in which the elimination of all cusps is obtained at the expenses of increasing the number of crossings.

The book concludes with Chap. 11 where, following [12], we analyse some variational properties of the action functional $\mathcal{F}$ discussed at the beginning (and described in Chap. 1). In order to minimize $\mathcal{F}$ it is useful to deepen the study of its lower semicontinuous envelope, and this amounts in taking limits of sequences of labelled apparent contours with a uniform bound on the action. In this passage to the limit, many nice properties of labelled apparent contours (consequences of the stability assumptions) are lost. In particular, in Sect. 11.3.2 we produce some

[17] There are a number of software codes for the study of knots and their invariants and for the manipulation of three-manifolds, such as SnapPea, SnapPy, Orb, and Knotscape; we refer to the link http://www.math.uiuc.edu/~nmd/computop/index.html for further information. See also [33, 34].

[18] Probably the more common example is represented by the apparent contour of a torus with four cusps and two crossings, which can be modified into two concentric circles.

examples which illustrate the difficulties in identifying a notion of limit labelling defined on a limit graph.

Finally, the reference list of this book is far from being complete; we apologize for this incompleteness.

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