On finite groups of isometries of handlebodies in arbitrary dimensions 
and finite extensions of Schottky groups

by

Mattia Mecchia and Bruno P. Zimmermann

Abstract. It is known that the order of a finite group of diffeomorphisms of a 3-
dimensional handlebody of genus $g > 1$ is bounded by the linear polynomial $12(g - 1)$, 
and that the order of a finite group of diffeomorphisms of a 4-dimensional handlebody 
(or equivalently, of its boundary 3-manifold), faithful on the fundamental group, is 
bounded by a quadratic polynomial in $g$ (but not by a linear one). In the present 
paper we prove a generalization for handlebodies of arbitrary dimension $d$, uniformizing 
handlebodies by Schottky groups and considering finite groups of isometries of such 
handlebodies. We prove that the order of a finite group of isometries of a handlebody 
of dimension $d$ acting faithfully on the fundamental group is bounded by a polynomial 
of degree $d/2$ in $g$ if $d$ is even, and of degree $(d + 1)/2$ if $d$ is odd, and that the degree 
d/2 for even $d$ is best possible. This implies then analogous polynomial Jordan-type 
bounds for arbitrary finite groups of isometries of handlebodies (since a handlebody of 
dimension $d > 3$ admits $S^1$-actions, there does not exist an upper bound for the order 
of the group itself).

2010 Mathematics Subject Classification: 57S17, 57S25, 57N16

Key words and phrases: handlebody, finite group action, Schottky group, Jordan-type 
bound.

1. Introduction. All finite group actions in the present paper will be faithful, smooth 
and orientation-preserving, and all manifolds will be orientable. We study finite group 
actions of large order on handlebodies of dimension $d \geq 3$ and genus $g > 1$.

An orientable handlebody $V^d_g$ of dimension $d$ and genus $g$ can be defined as a regular 
neighbourhood of a finite graph with free fundamental group of rank $g$ embedded in the 
sphere $S^d$; alternatively, it is obtained from the ball $B^d$ by attaching along its boundary 
g copies of a handle $B^{d-1} \times [0, 1]$ in an orientable way, or as the boundary-connected 
sum of $g$ copies of $B^{d-1} \times S^1$. The boundary of $V^d_g$ is a closed manifold $H^{d-1}_g$ which is 
the connected sum of $g$ copies of $S^{d-2} \times S^1$.

By [Z1] the order of a finite group of diffeomorphisms of a 3-dimensional handlebody 
$V^3_g$ of genus $g > 1$ is bounded by the linear polynomial $12(g - 1)$ (see also [MMZ, 
Theorem 7.2], [MZ]); also, a finite group $G$ acting faithfully on $V^3_g$ acts faithfully also 
on the fundamental group. On the other hand, since the closed 3-manifold $H^3_g$ admits 
$S^1$-actions, it has finite cyclic group actions of arbitrarily large order acting trivially on
the fundamental group, and the same is true also for handlebodies $V_g^d$ of dimensions $d > 3$. However it is shown in [Z4] that if a finite group of diffeomorphisms of $H_g^3 = \partial V_g^4$ acts faithfully on the fundamental group then the order of the group is bounded by a quadratic polynomial in $g$ (but not by a linear one), and hence the same holds also for 4-dimensional handlebodies $V_g^4$. As a consequence, each finite group $G$ acting on $H_g^3$ or $V_g^4$ has a finite cyclic normal subgroup $G_0$ (the subgroup acting trivially on the fundamental group) such that the order of $G/G_0$ is bounded by a quadratic polynomial in $g$ ([Z4]).

There arises naturally the question (as asked in [Z4]) whether there are analogous polynomial bounds also for the orders of finite groups acting on handlebodies $V_g^d$ of arbitrary dimension $d$. Whereas finite group actions in dimension 3 are standard by the recent geometrization of such actions after Thurston and Perelman, the situation in higher dimensions is more complicated and not well-understood. Hence one is led to consider some kind of standard actions also in higher dimensions. We will do so by uniformizing handlebodies $V_g^d$ by Schottky groups (groups of Möbius transformations of the ball $B^d$ acting by isometries on its interior, the Poincaré-model of hyperbolic space $\mathbb{H}^d$), thus realizing their interiors as complete hyperbolic manifolds, and then considering finite groups of isometries of such hyperbolic (Schottky) handlebodies (see section 2 for the definition of Schottky groups).

Our main results are as follows.

**Theorem 1.** Let $G$ be a finite group of isometries of a hyperbolic handlebody $V_g^d$ of dimension $d \geq 3$ and of genus $g > 1$ which acts faithfully on the fundamental group. Then the order of $G$ is bounded by a polynomial of degree $d/2$ in $g$ if $d$ is even, and of degree $(d+1)/2$ if $d$ is odd. The degree $d/2$ is best possible in even dimensions whereas in odd dimensions the optimal degree is at least $(d-1)/2$.

By hypothesis such a group $G$ injects into the outer automorphism group of the fundamental group of $V_g^d$, a free group of rank $g$. We note that by [WZ] the optimal upper bound for the order of an arbitrary finite subgroup of the outer automorphism group Out($F_g$) of a free group $F_g$ of rank $g > 2$ is $2^g g!$ (i.e., exponential in $g$). It is shown in [Z2] that every finite subgroup of Out($F_g$) can be induced (or realized in the sense of the Nielsen realization problem) by an isomorphic group of isometries of a handlebody $V_g^d$ of sufficiently high dimension $d$.

Without the hypothesis that $G$ acts faithfully on the fundamental group, the proof of Theorem 1 gives the following polynomial Jordan-type bound for finite groups of isometries of $V_g^d$.

**Corollary.** Let $G$ be a finite group of isometries of a hyperbolic handlebody $V_g^d$ of genus $g > 1$, and let $G_0$ denote the normal subgroup of $G$ acting trivially on the fundamental group. Then the following holds.
i) $G_0$ is isomorphic to subgroup of the orthogonal group $\text{SO}(d-2)$, and the order of the factor group $G/G_0$ is bounded by a polynomial as in Theorem 1.

ii) $G$ has a normal abelian subgroup, a subgroup of $G_0$, whose index in $G$ is bounded by a polynomial as in Theorem 1.

By the classical Jordan bound, each finite subgroup $G$ of a complex linear group $\GL(d, \mathbb{C})$ has a normal abelian subgroup whose index in $G$ is bounded by a constant depending only on the dimension $d$ (see [C] for the optimal bound for each $d$; see also [Z5] and its references for generalizations of the Jordan bound in the context of diffeomorphism groups of manifolds).

In more algebraic terms, Theorem 1 is equivalent to the following:

**Theorem 2.** Let $E$ be a group of Möbius transformations of $S^{d-1}$ which is a finite effective extension of a Schottky group $S_g$ of rank $g > 1$. Then the order of the factor group $E/S_g$ is bounded by a polynomial in $g$ as in Theorem 1.

Here effective extension means that no element of $E$ acts trivially on $S_g$ by conjugation. By [Z2] every finite effective extension of a Schottky group can be realized by a group of Möbius transformations in some sufficiently high dimension $d$.

As a consequence of the geometrization of finite group actions in dimension three, using the methods of [RZ, section 2] every finite group $G$ of diffeomorphisms of a 3-dimensional handlebody $V^3_g$ is conjugate to a group of isometries, uniformizing $V^3_g$ by a suitable Schottky group (which depends on $G$). This is no longer true in higher dimensions; however, if $G$ is a finite group of diffeomorphisms of a 4-dimensional handlebody $V^4_g$ then, uniformizing $V^4_g$ by a suitable Schottky group, $G$ acts also as a group of isometries of $V^4_g$ inducing the same action on the fundamental group (applied methods of [Z4] to the boundary 3-manifold $H^3_g$ of $V^4_g$). This raises naturally the following:

**Questions.** i) Is every finite group $G$ of diffeomorphisms of a handlebody $V^d_g$ isomorphic to a group of isometries of a hyperbolic handlebody $V^d_g$ (inducing the same action on the fundamental group)?

ii) Is every finite group $G$ of diffeomorphisms of a ball $B^d$ (i.e., a handlebody of genus zero) or of a sphere $S^{d-1}$ isomorphic to a subgroup of the orthogonal group $\text{SO}(d)$?

In general, such a finite group $G$ of diffeomorphisms is not conjugate to a group of isometries of a handlebody resp. to a group of orthogonal maps; we note that ii) is not true for finite groups $G$ of homeomorphisms of $B^d$ or $S^{d-1}$, see [GMZ, section 7].

In section 2 we prove the first part of Theorem 1. In section 3 we present examples of finite isometric group actions on handlebodies which show that the degree $d/2$ of the polynomial bound in Theorem 1 is best possible in even dimensions (even for finite
cyclic groups $G$), and that a lower bound for the degree in odd dimensions is $(d - 1)/2$. Note that for $d = 3$ the bound $(d + 1)/2$ is not best possible since it gives a quadratic bound instead of the actual linear bound $12(g - 1)$; for odd dimensions $d > 3$ we have no intuition at present if the optimal bound should be $(d - 1)/2$ or $(d + 1)/2$.

2. Schottky groups and the Proof of Theorem 1. A Schottky group $S_g$ of rank or genus $g$ is a group of M"obius transformations acting on a sphere $S^{d-1} = \partial B^d$ defined in the following way (analogously to the Schottky groups in dimension two acting on $S^2$, see [L],[M] or [R, p. 584]; see also [Z2] for the following). Let $S_1, T_1, \ldots, S_g, T_g$ be spheres of dimension $d - 2$ in $S^{d-1}$ which bound disjoint balls $B_1, D_1, \ldots, B_g, D_g$ of dimension $d - 1$; choose M"obius transformations $f_1, \ldots, f_g$ such that $f_i(S_i) = T_i$ and $f_i$ maps the exterior of $B_i$ to the interior of $D_i$. Then it is easy to see that $f_1, \ldots, f_g$ are free generators of a free group $S_g$ of M"obius transformations. The complement in $S^{d-1}$ of the interiors of the balls $B_i$ and $D_i$ is a fundamental domain for the action of $S_g$ on $S^{d-1} - \Lambda(S_g)$ where $\Lambda(S_g)$ denotes the set of limit points of $S_g$ in $S^{d-1}$ (a Cantor set). In this definition, one may consider round spheres $S_1, T_1, \ldots, S_g, T_g$ (thus defining a so-called classical Schottky group), or just topological spheres (and it is known that non-classical Schottky groups exist); however this is not relevant for the present paper, in particular in the examples constructed in section 3 the Schottky subgroups will be always classical).

The group of M"obius transformations of $S^{d-1}$ extends naturally to the interior of the ball $B^d$ ("Poincaré extension") where it becomes the group of orientation-preserving isometries of the Poincaré-model of hyperbolic space $\mathbb{H}^d$. The action of $S_g$ is free and properly discontinuous on the interior $\mathbb{H}^d$ of $B^d$, and a fundamental domain for this action is the region of $\mathbb{H}^d$ bounded by all hyperbolic hyperplanes defined by the spheres $S_i$ and $T_i$ (i.e., half-spheres of dimension $d - 1$ orthogonal to $S^{d-1}$ along these spheres). The quotient $(B^d - \Lambda(S_g))/S_g$ is a handlebody $V_g^d$ whose interior $\mathbb{H}^d/S_g$ has the structure of a complete hyperbolic manifold, and we say that the Schottky group $S_g$ uniformizes the handlebody $V_g^d$. When speaking of a finite group $G$ of isometries of a handlebody $V_g^d$ we then intend that $V_g^d$ can be uniformized by a Schottky group $S_g$ such that $G$ acts by hyperbolic isometries on the interior of $V_g^d$.

Let $V_g^d$ be a handlebody uniformized by a Schottky group $S_g$. Let $G$ be a finite group of isometries of $V_g^d$ which induces a faithful action on the fundamental group. The group of all lifts of elements of $G$ to the universal covering $B^d - \Lambda(S_g)$ of $V_g^d$ defines a group $E$ of M"obius transformations of $B^d$, with factor group $E/S_g \cong G$, so we have a finite extension

$$1 \rightarrow S_g \hookrightarrow E \rightarrow G \rightarrow 1;$$

by general covering space theory, this extension is effective since $G$ acts faithfully on the fundamental group of $V_g^d$ (isomorphic to the group $S_g$ of covering transformations).
Lemma 1. The extension $1 \to S_g \to E \to G \to 1$ is effective if and only if $E$ has no non-trivial finite normal subgroups.

Proof. Let $F$ be a finite normal subgroup $E$. Since the intersection of $F$ with the normal torsionfree subgroup $S_g$ of $E$ is trivial, the normal subgroups $F$ and $S_g$ of $E$ commute elementwise (any commutator $fsf^{-1}s^{-1}$ of elements $f \in F$ and $s \in S_g$ is an element of both $F$ and $S_g$ and hence trivial). Hence if the extension is effective, $F$ has to be trivial.

Conversely, suppose that every finite normal subgroup of $E$ is trivial. The subgroup of elements of the finite extension $E$ of $S_g$ inducing by conjugation the trivial automorphism of $S_g$ is clearly finite (since the center of $S_g$ is trivial), normal and hence trivial, so the extension is effective.

This completes the proof of Lemma 1.

As a consequence of Stallings’s structure theorem for groups with infinitely many ends, a finite extension $E$ of a free group is the fundamental group $\pi_1(\Gamma, G)$ of a finite graph of finite groups $(\Gamma, G)$ ([KPS]); here $\Gamma$ denotes a finite graph, and to its vertices $v$ and edges $e$ are associated finite vertex groups $G_v$ and edge groups $G_e$, with inclusions of the edge groups into the adjacent vertex groups. The fundamental group $\pi_1(\Gamma, G)$ of the finite graph of finite groups $(\Gamma, G)$ is the iterated free product with amalgamation and HNN-extension of the vertex groups amalgamated over the edge groups, first taking the iterated free product with amalgamation over a maximal tree of $\Gamma$ and then associating an HNN-generator to each of the remaining edges. We note that each finite subgroup of $E = \pi_1(\Gamma, G)$ is conjugate into a vertex group of $(\Gamma, G)$, and that the vertex groups are maximal finite subgroups of $E$ (see [ScW], [Se] or [Z3] for the standard theory of graphs of groups and their fundamental groups).

We will assume in the following that the graph of groups $(\Gamma, G)$ has no trivial edges, i.e. no edges with two different vertices such that the edge group coincides with one of the two vertex groups (by collapsing trivial edges, i.e. amalgamating the two vertex groups into a single vertex group); we say that such a graph of groups is in normal form.

We denote by

$$\chi(\Gamma, G) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

the Euler characteristic of the graph of groups $(\Gamma, G)$ (the sum is taken over all vertex groups $G_v$, resp. edge groups $G_e$ of $(\Gamma, G)$); then, by multiplicativity of Euler characteristics under finite coverings of graphs of groups,

$$g - 1 = -\chi(\Gamma, G) |G|$$

(see [ScW] or [Z3]); note that this is positive since we are assuming that $g > 1$. 

5
The finite extension $E = \pi_1(\Gamma, G)$ of the Schottky group $S_g$ is a group of Möbius transformations of $B^d$ and acts as a group of hyperbolic isometries on its interior $\mathbb{H}^d$. Each finite group of isometries of hyperbolic space $\mathbb{H}^d$ has a global fixed point in $\mathbb{H}^d$ and is conjugate to a finite group of orthogonal transformations of $B^d$ (which are exactly the isometries of $\mathbb{H}^d$ which fix the origin in $B^d$). In particular each finite vertex group $G_v$ of $E = \pi_1(\Gamma, G)$ has a fixed point in $\mathbb{H}^d$ and is isomorphic (conjugate) to a subgroup of the orthogonal group $O(d)$, and different vertex groups of $(\Gamma, G)$ have different fixed points (since the vertex groups are maximal finite subgroups of $E$ and the action of $E$ is properly discontinuous in $\mathbb{H}^d$); also, if a vertex group fixes a point in $\mathbb{H}^d$ then it is the maximal finite subgroup of $E$ fixing this point.

Consider a non-closed edge $e$ of $(\Gamma, G)$, i.e. with two distinct vertices $v_1$ and $v_2$, with edge group $G_e$ and vertex groups $G_1$ and $G_2$ (which we consider as subgroups of $E$), with $G_e = G_1 \cap G_2$. Let $P_1 \neq P_2$ be fixed points of $G_1$ resp. $G_2$ in $\mathbb{H}^d$; then $P_1$ and $P_2$ define a hyperbolic line $L$ which is fixed pointwise by the edge group $G_e = G_1 \cap G_2$. The line $L$ intersects $S^{d-1} = \partial B^d$ in two points which are fixed by $G_e$; moreover no subgroup of $G_1$ larger than $G_e$ can fix one of these two points since otherwise it would fix pointwise the line $L$ and hence $P_2$, so it would be contained also in $G_2$.

Now let $e$ be a closed edge of $(\Gamma, G)$, i.e. an edge with only one vertex $v$. There are two inclusions of the edge group $G_e$ into the vertex group $G_v$ defining two subgroups $G_e$ and $G'_e$ of $G_v$; denoting by $t$ an HNN-generator corresponding to the edge $e$, we have that $t^{-1}G'_e t = G_e$, and $G_e = G_v \cap (t^{-1}G_v t)$. Note that $t$ has infinite order so it does not fix any point in $\mathbb{H}^d$. Let $P$ be a fixed point of the finite subgroup $G_v$ of $E$ in $\mathbb{H}^d$; then $t^{-1}G_v t$ fixes the point $t(P) \neq P$, and its subgroup $G_e = t^{-1}G'_e t$ fixes the hyperbolic line $L$ defined by $P$ and $t(P)$. As before, the hyperbolic line $L$ intersects $S^{d-1} = \partial B^d$ in two points which are fixed by $G_e$, and $G_e$ is the maximal subgroup of $G_v$ fixing these two points.

Note also that, since $G_e$ fixes a point in $S^{d-1}$, it is in fact isomorphic (conjugate) to a subgroup of the orthogonal group $\text{SO}(d - 1)$. Summarizing, we have:

**Lemma 2.** Let $G_v \subset E$ be a vertex group of the graph of groups $(\Gamma, G)$, and let $G_e \subset G_v$ be an adjacent edge group. Then $G_v$ has a global fixed point in $\mathbb{H}^d$, and $G_e$ has a global fixed point in $S^{d-1} = \partial B^d$ which is not fixed by any other element of $G_v$. In particular, every vertex group is isomorphic to a subgroup of the orthogonal group $\text{SO}(d)$, and every edge group is isomorphic to a subgroup of $\text{SO}(d - 1)$.

We need also the following lemma which is contained in [Z4, proof of Theorem 1]; since its proof is short, we present it for the convenience of the reader. Let $\chi = \chi(\Gamma, G)$ denote the Euler characteristic of $(\Gamma, G)$; note that $-\chi > 0$ since $g > 1$, and that for any graph of groups in normal form one has $-\chi \geq 0$ unless it consists of a single vertex.
Lemma 3. Let $e$ be an edge of $\Gamma$. Denote by $n$ the order of $G$ and by $a$ the order of the edge group $G_e$. Then
\[ \frac{n}{a} \leq 6(g - 1). \]

Proof. Suppose first that $e$ is a closed edge. If $e$ is the only edge of $(\Gamma, G)$ then
\[ -\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}, \quad g - 1 = -\chi n \geq \frac{n}{2a}, \quad \frac{n}{a} \leq 2(g - 1). \]
If $e$ is closed and not the only edge then
\[ -\chi \geq \frac{1}{a}, \quad g - 1 = -\chi n \geq \frac{n}{a}, \quad \frac{n}{a} \leq g - 1. \]
Suppose that $e$ is not closed. If $e$ is the only edge of $(\Gamma, G)$ then both vertices of $e$ are isolated and
\[ -\chi \geq \frac{1}{a} - \frac{1}{2a} - \frac{1}{3a} = \frac{1}{6a}, \quad g - 1 = -\chi n \geq \frac{n}{6a}, \quad \frac{n}{a} \leq 6(g - 1). \]
If $e$ is not closed, not the only edge and has exactly one isolated vertex then
\[ -\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}, \quad g - 1 = -\chi n \geq \frac{n}{2a}, \quad \frac{n}{a} \leq 2(g - 1). \]
Finally, if $e$ is not closed, not the only edge and has no isolated vertex then
\[ -\chi \geq \frac{1}{a}, \quad g - 1 = -\chi n \geq \frac{n}{a}, \quad \frac{n}{a} \leq g - 1. \]
Concluding, in all cases the inequality of Lemma 3 holds, completing the proof of the lemma.

After these preparations, we can now start with the actual:

Proof of Theorem 1. Let $e$ be any edge of the finite graph of finite groups $(\Gamma, G)$ given by the $G$-action. By Lemma 2, $G_e$ has a global fixed point in $S^{d-1} = \partial B^d$ and is isomorphic to a subgroup of the orthogonal group $\text{SO}(d - 1)$. By the classical Jordan bound for subgroups of $\text{GL}(d - 1, \mathbb{C})$, the edge group $G_e$ has an abelian subgroup $A_1$ whose index in $G_e$ is bounded by a constant $c$ depending only on the dimension. We will find a polynomial upper bound in $g$ for the order $a_1$ of the abelian group $A_1$; this will imply then a polynomial bound of the same degree also for the order $a \leq c a_1$ of $G_e$, and finally for the order $n$ of $G$ since, by Lemma 3,
\[ n \leq 6(g - 1) a \leq c 6(g - 1) a_1. \]
Let $E_1$ be the subgroup of $E$ generated by $S_g$ and $A_1$ (which is again an effective extension of $S_g$, with factor group $A_1$). Then also $E_1$ is the fundamental group of a finite graph of finite groups in normal form which we denote again by $(\Gamma, G)$. Since the finite group $A_1$ has a fixed point in $\mathbb{H}^d$, up to conjugation it is the vertex groups $G_v$ of $(\Gamma, G)$, and its fixed point set in $S^{d-1}$ is a sphere $S^{d_1}$ of dimension $d_1 \geq 0$ (since $G_e$ has a global fixed point in $S^{d-1}$). Since $(\Gamma, G)$ has no trivial edges and $E_1$ has no non-trivial finite normal subgroups by Lemma 1, some edge adjacent to $v$ has an edge group $A_2$ of order $a_2 < a_1$ (i.e., properly contained in $A_1$). By Lemma 3,

$$a_1 \leq 6(g-1)a_2.$$ 

By Lemma 2, the edge group $A_2$ has a fixed point in $S^{d-1} = \partial B^d$ which is not fixed by any other element of the vertex group $A_1$, hence the fixed point set of $A_2$ in $S^{d-1}$ is a sphere $S^{d_2}$ of dimension $d_2 > d_1$.

We iterate the construction and consider the subgroup $E_2$ of $E_1$ generated by $S_g$ and $A_2$, obtaining an edge group $A_3$ for $E_2$ which fixes a sphere $S^{d_3}$ of dimension $d_3 > d_2$ in $S^{d-1}$, of order

$$a_2 \leq 6(g-1)a_3.$$

Hence, after at most $d-1$ steps, we end up with a trivial edge group fixing all of $S^{d-1}$. Collecting, we obtain the polynomial bound

$$n \leq c6^d(g-1)^d$$

of degree $d$ in $g$ for the order of $G$.

In order to obtain a polynomial bound of the degree given in Theorem 1 we argue as follows. Suppose that the fixed point set of the normal subgroup $A_2$ of $A_1$ is a sphere $S^{d_1+1}$ of dimension $d_2 = d_1 + 1$; note that $S^{d_1+1}$ is invariant under the action of $A_1$. Let $A'_1$ denote the subgroup of index one or two of $A_1$ which acts orientation-preservingly on $S^{d_1+1}$. Then $A'_1$ fixes $S^{d_1+1}$ pointwise since otherwise the fixed point set of $A'_1$ would be a sphere of codimension at least two in $S^{d_1+1}$; this is not possible since already $A_1$ has fixed point set $S^{d_1}$ of dimension $d_1$. Continuing now with $A'_1$ in the place of $A_1$, we can assume that the dimensions $d_i$ increase by at least two in each step. Hence the number of steps is at most $d/2$ if $d$ is even, and $(d+1)/2$ if $d$ is odd, and this gives the degree of the polynomial upper bound as stated in Theorem 1.

This completes the proof of the first part of Theorem 1; the second part on the optimality of the degree $d/2$ for even $g$ and the lower bound $(d-1)/2$ for odd $g$ will follow from the examples of finite group actions on handlebodies constructed in the next section.

Proof of the Corollary. The proof proceeds along the lines of the proof of Theorem 1, with the following difference. In the proof of Theorem 1 we consider the sequence of
abelian subgroups $A_1, A_2, \ldots$ of $G$; after finitely many steps, this ended with the trivial group, using the effectiveness of the corresponding extensions $E_1, E_2, \ldots$ of $S_g$. Without effectiveness, the sequence $A_1, A_2, \ldots$ of $G$ ends with an abelian group $A_m$ which is a normal subgroup of the corresponding extension $E_m$; in particular, $A_m$ acts trivially on $S_g$ and is a subgroup of $G_0$. The index of $A_m$ in $G$ is bounded by a polynomial as in the proof of Theorem 1, hence also the index of $G_0$ in $G$ is bounded by such a polynomial.

The group $G_0$ lifts to an isomorphic normal subgroup of the extension $E$ of $S_g$ which we denote also by $G_0$. The finite group $G_0$ has a fixed point in $\mathbb{H}^d$; we can assume that it fixed the origin $O \in B^d$ and hence is isomorphic to a subgroup of $SO(d)$. Since $G_0$ is normal in $E$, it is contained (up to conjugation) in each edge group of the graph of groups $(\Gamma, G)$. By Lemma 2, $G_0$ has a global fixed point also in $S^{d-1} = \partial B^d$, hence it fixes pointwise a great sphere of dimension at least zero in $S^{d-1}$, and a linear subspace $B$ of dimension at least one in $B^d$. Since $G_0$ commutes elementwise with $S_g$, the Schottky group $S_g$ acts on $B$. Since the action of $S_g$ is properly discontinuous and $g > 1$, $B$ has dimension at least two. Now $G_0$ acts also on the orthogonal complement of $B$ in $O \in B^d$, a linear subspace of codimension at least two, so $G_0$ is isomorphic to a subgroup of the orthogonal group $SO(d-2)$.

Finally, by the classical Jordan bound for linear groups, the subgroup $G_0$ of $SO(d-2)$ contains a normal abelian subgroup whose index is bounded by a constant depending only on the dimension $d$. By taking the intersection of this normal abelian subgroup with all isomorphic normal subgroups of $G_0$ we obtain a characteristic abelian subgroup $A$ of $G_0$ whose index in $G_0$ is also bounded by a constant depending only on the dimension $d$. Hence the indices of $A$ and $G_0$ in $G$ are bounded by polynomials in $g$ of the same degree.

This completes the proof of the Corollary.

3. Examples. We construct isometric actions of finite groups $G$ on handlebodies which realize the lower bounds for the degrees of the polynomial bounds in Theorem 1; specifically, we prove the following:

Proposition. For a fixed $k \geq 2$ and all $m \geq 2$, the finite group $G = (\mathbb{Z}_m)^k$ admits an action, faithful on the fundamental group, on a handlebody $V_g^d$ of genus $g = mk - k$ and dimension $d = 2k$ and $2k + 1$; in particular, the order $n = m^k$ of $G$ is given by the polynomial

$$n = (g + k)^k / k^k = (1 + g/k)^k$$

of degree $k = d/2$ in $g$ if $d$ is even, and $k = (d - 1)/2$ if $d$ is odd.

Proof. For $k > 1$, let $G = C_1 \times \ldots \times C_k \cong (\mathbb{Z}_m)^k$, of order $n = m^k$, be the product of $k$ cyclic groups $C_i \cong \mathbb{Z}_m$ of order $m$. Choose an orthogonal action of $G$ on the closed ball $B^{2k} \subset \mathbb{R}^{2k}$ of dimension $d = 2k$ as follows. Decomposing $\mathbb{R}^{2k} = P_1 \times \ldots \times P_k$ as
the product of $k$ orthogonal planes $P_i$, each $C_i$ acts on $P_i$ faithfully by rotations and trivially on the $k-1$ orthogonal planes.

Define a finite graph of finite groups $(\Gamma, G)$ as follows. The graph $\Gamma$ is a star-shaped graph with one central vertex $v$ with vertex group $G_v = G = C_1 \times \ldots \times C_k$ and $k$ non-closed edges $e_1, \ldots, e_k$ each having $v$ as a vertex, with edge groups

$$G_{e_1} = C_2 \times \ldots \times C_k, \ G_{e_2} = C_1 \times C_3 \times \ldots \times C_k, \ldots, \ G_{e_k} = C_1 \times \ldots \times C_{k-1}$$

(i.e., exactly $C_i$ is missing in $G_{e_i}$). Hence $\Gamma$ has $k+1$ vertices, by definition all with vertex group $G = C_1 \times \ldots \times C_k$, and the Euler characteristic of $(\Gamma, G)$ is

$$\chi = (k+1) \frac{1}{m^k} - k \frac{1}{m^{k-1}}.$$

There is an obvious projection of the fundamental group $E = \pi_1(\Gamma, G)$ of the graph of groups $(\Gamma, G)$ onto $G$; its kernel is a free group $F_g$ of some rank $g$, and we have an extension

$$1 \to F_g \hookrightarrow E \to G \to 1$$

which by construction of $(\Gamma, G)$ is effective (has no nontrivial finite normal subgroups, see Lemma 1). The rank $g$ is given by

$$g - 1 = (-\chi)n = (-\chi)m^k = mk - (k + 1), \ g = mk - k,$$

hence

$$n = m^k = (g + k)^k / k^k$$

which is a polynomial of degree $k = d/2$ in $g$ and gives the maximal possibility for the degree in Theorem 1 for even dimensions $d$.

We realize $E = \pi_1(\Gamma, G)$ as a group of Möbius transformations of $B^d$, $d = 2k$, such that its subgroup $F_g$ corresponds to a Schottky group $S_g$. Then the quotient $(B^d - \Lambda(S_g))/S_g$ is a handlebody $V^d_g$ of genus $g$, and $E$ projects to an action of the factor group $E/S_g \cong G$ on $V^d_g$ which is faithful on the fundamental group. In particular, the degree $d/2$ in Theorem 1 is best possible for even dimensions $d = 2k$.

The realization of $E = \pi_1(\Gamma, G)$ as a group of Möbius transformations of $B^d$ proceeds inductively by standard combination methods (similar as in [Z2, section 3]). Starting with the orthogonal group $G$ described above, we realize first the free product with amalgamation

$$G_v *_{G_{e_1}} G_{v_1} = G *_{G_e} G_1$$

where $e = e_1$ denotes the first edge of $\Gamma$, with vertices $v$ and $v_1$ and vertex groups $G = G_v$ and $G_1 = G_{v_1} \cong G$. By construction, the fixed point set of the subgroup $G_e$ of $G$ is a 2-ball $B_1$ in $B^d$ defining a hyperbolic plane in $\mathbb{H}^d$ which will be denoted also
by $B_1$. Let $L_1$ be a hyperbolic half-line in $B_1$ starting from its center 0 and ending in a point $R_1$ in $S^{d-1} = \partial B^d$. Let $V_1$ be a neighbourhood of $R_1$ in $B^d$ bounded by a hyperbolic hyperplane $H_1$ in $\mathbb{H}^d$ orthogonal to $L_1$; choose $V_1$ sufficiently small such that $f(V)$ is disjoint form $V$ for all $f \in G - G_e$ (note that $G_e$ fixes $L_1$ pointwise but that no larger subgroup of $G$ fixes $L_1$ by construction of $G$). The reflection $\tau_1$ in the hyperbolic hyperplane $H_1$ commutes elementwise with $G_e \subset G$ and, considering $G_1 = \tau_1 G \tau_1^{-1}$, we have that $G \cap G_1 = G_e$. Similar as for Schottky groups it is now easy to see that the group of Möbius transformations generated by $G$ and $G_1$ is isomorphic to the free product with amalgamation $G \ast_{G_e} G_1$, and that every torsionfree subgroup of finite index is in fact a Schottky group (cf. [Z2] and the combination theorems in [M]).

We iterate the construction and adjoin $G_{e_2}$. Let $L_2$ be a hyperbolic half-line starting in the center 0 and ending in a point $R_2$ of $S^{d-1} = \partial B^d$ such that $R_2$ does not lie in $G(V_1)$. Let $V_2$ be a small neighbourhood of $R_2$ in $B^d$, bounded by a hyperbolic hyperplane $H_2$ orthogonal to $L_2$ which does not intersect $G(V_1)$. With $G_2 = \tau_2 G \tau_2^{-1}$ where $\tau_2$ denotes the reflection in $H_2$, this realizes the free product with amalgamation

$$G_{v_2} \ast_{G_{e_2}} G_v \ast_{G_{e_1}} G_{v_1}$$

as a group of Möbius transformations. Continuing in this way, after $k$ steps $E$ is realized as a group of Möbius transformations, with $F_g$ corresponding to a Schottky group $S_g$.

Finally, in odd dimensions $d = 2k + 1$, we extend the orthogonal action of $G$ on $B^{2k}$ described above to an orthogonal action on $B^{2k+1}$ (trivial on the last coordinate) and then proceed as before. We get a polynomial of degree $k = (d-1)/2$ in $g$ for the order $n$ of $G$ whereas Theorem 1 gives a polynomial bound of degree $(d+1)/2$. As noted in the introduction, the optimal degree in dimension $d = 3$ is in fact 1, but for odd dimensions $d > 3$ it remains open.

This completes the proof of the Proposition, and also of Theorem 1.

The examples given in the Proposition are for finite abelian groups $G$. By suitably modifying the construction, one obtains also examples for finite cyclic groups as follows.

Let $d = 2k$ be a fixed even dimension, and let $p > k$ be any prime. For $i = 1, \ldots, k$, the $k$ integers $q_i = p + i k!$, are pairwise coprime: in fact, if a prime $p'$ divides $q_i$ then $p' > k$; if $p'$ divides also $q_j$, for some $j > i$, then $p'$ divides $q_j - q_i = (j - i) k!$ which is a contradiction. Then $G = \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_k}$ is a cyclic group of order $n = q_1 \ldots q_k$. In analogy with the proof of the Proposition, let $(\Gamma, \mathcal{G})$ be a star-shaped graph of groups with $k + 1$ vertices all with vertex group $G$, and with $k$ edges where in each edge group is missing exactly one of the factors $\mathbb{Z}_{q_i}$ of $G$, with

$$\chi = \chi(\Gamma, \mathcal{G}) = \frac{k + 1}{n} - \frac{q_1}{n} - \ldots - \frac{q_k}{n}.$$
There is an obvious surjection of $\pi_1(\Gamma, G)$ onto $G$, its kernel is a free group of rank $g$ with
\[
g - 1 = (-\chi)n = -(k + 1) + q_1 + \ldots + q_k,
g = -k + kp + (1 + \ldots + k)k!,
p = (g + c_k)/k,
\]
for a constant $c_k$ depending only on $k$. Now
\[
|G| = n = q_1 \ldots q_k \geq p^k \geq (g + c_k)^{k/k^k},
\]
so the order of $G$ is bounded from below by a polynomial of degree $k = d/2$ in $g$.

Finally, the geometric realizations of $G$ and $E = \pi_1(\Gamma, G)$ are exactly as in the proof of the Proposition.

Acknowledgment. The authors were supported by a FRV grant from Università degli Studi di Trieste.

References


Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste
34127 Trieste, Italy
E-mail: meccia@dmi.units.it, zimmer@units.it