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BACKWARD UNIQUENESS FOR PARABOLIC OPERATORS WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract

In this paper we study the backward uniqueness for parabolic equations with non-Lipschitz coefficients in time and space. The result presented here improves an old uniqueness theorem due to Lions and Malgrange [7] and some more recent results of Del Santo and Prizzi [5, 6].

1. Introduction

The question of uniqueness and non-uniqueness for solutions of partial differential equations has a fairly long history, starting form the classical works of Holmgren and Carleman. A good and rather complete survey about the results on this topic, until the early 1980's, can be found in the book of Zuily [16].

In this paper we are interested in a particular class of parabolic operators for which we consider the uniqueness property, backwards in time. Uniqueness for smooth solutions of parabolic and backward parabolic operators is not trivial. In [15] Tychonoff showed that a solution $u \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ of the Cauchy problem

$$\begin{cases} \partial_t u - \Delta_x u = 0, & (t, x) \in \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0, x) = 0, & x \in \mathbb{R}^n_x, \end{cases} \tag{1}$$

not necessarily vanishes. In particular, the example given by Tychonoff is such that the solution $u(t, x)$ to (1) satisfies

$$\sup_{x \in \mathbb{R}^n_x} \left(\max_{t \in [-T,T]} |u(t, x)| e^{-a|x|^2} \right) = +\infty, \tag{2}$$

for all $a > 0$. On the other hand Tychonoff proved that uniqueness to (1) can be obtained, for example, if one imposes $\max_{t \in [-T,T]} |u(t, x)| \le C e^{a|x|^2}$, for some $C, a > 0$. Other interesting examples of non-uniqueness for (1), under particular assumptions, can e.g. be found in [11].

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Here we consider the *backward parabolic* operator

$$Pu = \partial_t u + \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t,x)\partial_{x_k}u) + c(t,x)u, \tag{3}$$

defined on the strip $[0,T]\times\mathbb{R}^n_x$; all the coefficients are supposed to be measurable and bounded; the 0-order coefficient $c(t,x)$ is allowed to be complex valued and we assume that the matrix $(a_{jk}(t,x))^n_{j,k=1}$ is real and symmetric for all $(t,x)\in[0,T]\times\mathbb{R}^n_x$ and that there exists an $a_0\in(0,1]$ such that, for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^n_x\times\mathbb{R}^n_\xi$,

$$\sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \geq a_0|\xi|^2. \tag{4}$$

Under *uniqueness property in* $\mathcal{H}$ we will mean the following: let $\mathcal{H}$ be a space of functions (in which it makes sense to look for solutions u of the equation $Pu=0$). Then we say that the operator P has the uniqueness property in $\mathcal{H}$ if, whenever $u\in\mathcal{H}$, $Pu=0$ on $[0,T]\times\mathbb{R}^n_x$ and $u(0,x)=0$ in $\mathbb{R}^n_x$, then $u\equiv 0$ in $[0,T]\times\mathbb{R}^n_x$.

In [7] Lions and Malgrange proved the uniqueness property for (3) in the space

$$\mathcal{H} := L^2([0,T],H^2(\mathbb{R}^n_x))\cap H^1([0,T],L^2(\mathbb{R}^n_x)), \tag{5}$$

(note that this choice for $\mathcal{H}$ excludes the pathological situation of (2)) under the assumption that, for all $j,k=1,\dots,n$,

$$a_{jk}(t,x)\in \mathrm{Lip}([0,T],L^\infty(\mathbb{R}^n_x)).$$

An example of Miller in [10] showed that the regularity of the coefficients a_{jk} with respect to t should be taken under consideration, if one wants to have uniqueness in $\mathcal{H}$. In particular he constructed a nontrivial solution to the Cauchy problem for (3) with 0 initial data, for an operator having the coefficients a_{jk} in $C^\alpha([0,T],C^\infty_b(\mathbb{R}^n_x))$, for all $0<\alpha<1/6$.

The example of Miller was considerably improved by Mandache in [8], in the following way: consider a modulus of continuity μ which does not satisfy the Osgood condition, i.e.

$$\int_0^1 \frac{1}{\mu(s)}\,ds < +\infty,$$

then it is possible to construct an operator of type (3) having the regularity with respect to t of the coefficients of the principal part ruled by μ, such that this operator does not have the uniqueness property in $\mathcal{H}$.

In [5] Del Santo and Prizzi proved uniqueness for (3) in $\mathcal{H}$, under the condition that, for all $j,k=1,\dots,n$,

$$a_{jk}(t,x)\in C^\mu([0,T],L^\infty(\mathbb{R}^n_x))\cap L^\infty([0,T],C^2(\mathbb{R}^n_x)),$$

and with the modulus of continuity μ satisfying the Osgood condition

$$\int_0^1 \frac{1}{\mu(s)} ds = +\infty. \tag{6}$$

If the result in [5] was completely satisfactory from the point of view of the regularity with respect to t, the same cannot be said for the regularity with respect to the space variables: the C^2 regularity with respect to x was a consequence of a difficulty in obtaining the Carleman estimate from which the uniqueness was deduced.

In [4] Del Santo made the technique used in [5] more effective by using a theorem of Coifman and Meyer ([2, Theorem 35], see also [13, Section 3.6]) and he could lower the regularity assumption in x from C^2 to $C^{1+\varepsilon}$ for an arbitrary small $\varepsilon > 0$.

Refining this approach Del Santo and Prizzi got in [6] the uniqueness property for (3) with the coefficients of the principal part

$$a_{jk} \in C^{\mu}([0, T], L^{\infty}(\mathbb{R}^n_x)) \cap L^{\infty}([0, T], \mathrm{Lip}(\mathbb{R}^n_x)).$$

In the present paper we will lower the regularity assumption for the coefficients of the principal part with respect to the space variables, going beyond the Lipschitz-continuity. The regularity with respect to x will be controlled by a modulus of continuity linked to the Osgood modulus of continuity with respect to t. More precisely we will prove that the uniqueness property in $\mathcal{H}$ for (3) holds for principal part coefficients

$$a_{jk} \in C^{\mu}([0, T], L^{\infty}(\mathbb{R}^n_x)) \cap L^{\infty}([0, T], C^{\omega}(\mathbb{R}^n_x)),$$

where μ satisfies (6) and $\omega(s) = \sqrt{\mu(s^2)}$. The proof of this uniqueness result will use the Littlewood–Paley theory and the Bony's paraproduct and will be obtained exploiting a Carleman estimate. The Carleman estimate will be proved in H^{-s} with $s \in (0, 1)$ while the weight function in the Carleman estimate will be the same as that in [12].

The paper is organized as follows. First we state the uniqueness results and we give some remarks. Then we introduce the Littlewood–Paley theory and Bony's paraproduct. These tools are used in obtaining some estimates, presented in Subsection 3.3. Finally, Section 4 is devoted to the proof of the Carleman estimate needed to deduce our uniqueness theorem.

2. The uniqueness result

DEFINITION 1. A continuous function $\mu\colon [0, 1] \to \mathbb{R}$ is called *modulus of continuity* if it is strictly increasing, concave and satisfies $\mu(0) = 0$.

REMARK 1. The concavity of the modulus of continuity has a list of simple consequences: for all $s \in [0, 1]$ we have $\mu(s) \geq \mu(1)s$, the function $s \mapsto \mu(s)/s$ is decreasing on $(0, 1]$, the limit $\lim_{s\to 0+} \mu(s)/s$ exists, the function $\sigma \mapsto \mu(1/\sigma)/(1/\sigma)$ is increasing on $[1,+\infty)$ and the function $\sigma \mapsto 1/(\sigma^2\mu(1/\sigma))$ is decreasing on $[1,+\infty)$. Moreover, there exists a constant $C>0$ such that

$$\mu(2s) \le C\mu(s). \tag{7}$$

DEFINITION 2. Let Ω be a convex set in $\mathbb{R}^n$ and $f:\Omega\to\mathcal{B}$, where $\mathcal{B}$ is a Banach space. We will say that f belongs to $C^\mu(\Omega,\mathcal{B})$ if f is bounded and it satisfies

$$\sup_{\substack{0<|t-s|<1\\ t,s\in\Omega}} \frac{\|f(t)-f(s)\|_{\mathcal{B}}}{\mu(|t-s|)} < +\infty.$$

For $f\in C^\mu(\Omega,\mathcal{B})$ we set

$$\|f\|_{C^\mu(\Omega,\mathcal{B})} = \|f\|_{L^\infty(\Omega,\mathcal{B})} + \sup_{\substack{0<|t-s|<1\\ t,s\in\Omega}} \frac{\|f(t)-f(s)\|_{\mathcal{B}}}{\mu(|t-s|)}.$$

In case of no ambiguity we will omit the space $\mathcal{B}$ from the notation.

DEFINITION 3. We will say that a modulus of continuity μ satisfies the Osgood condition if

$$\int_0^1 \frac{1}{\mu(s)}\,ds = +\infty. \tag{8}$$

EXAMPLE 1. A simple example of a modulus of continuity is $\mu(s)=s^\alpha$, for $\alpha\in(0,1]$. If $\alpha\in(0,1)$ (Hölder-continuity) μ does not satisfies the Osgood condition, while if $\alpha=1$ (Lipschitz-continuity) μ satisfies the Osgood condition. Similarly $\mu(s)=s(1+|\log(s)|)^\alpha$, for $\alpha>0$, (Log^α-Lipschitz-continuity) satisfies the Osgood condition if and only if $\alpha\le 1$.

Now we state our main uniqueness result.

Theorem 1. *Let μ and ω be two moduli of continuity such that $\omega(s)=\sqrt{\mu(s^2)}$. Suppose that μ satisfies the Osgood condition. Suppose moreover that there exists a constant $C>0$ such that*

$$\int_0^h \frac{\omega(t)}{t}\,dt \le C\omega(h); \tag{9}$$

there exists a constant $C>0$ such that, for all $1\le p\le q-1$,

$$\frac{\omega(2^{-q})}{\omega(2^{-p})} \le C\omega(2^{p-q}); \tag{10}$$

for all $s \in (0, 1)$,

$$\sum_{k=0}^{+\infty} 2^{(1-s)k}\omega(2^{-k}) < +\infty. \tag{11}$$

Assume that, for all $j, k = 1, \ldots, n$,

$$a_{jk} \in C^{\mu}([0, T], L^{\infty}(\mathbb{R}^n_x)) \cap L^{\infty}([0, T], C^{\omega}(\mathbb{R}^n_x)).$$

Then the operator P *has the uniqueness property in* $\mathcal{H}$*, where* P *and* $\mathcal{H}$ *are defined in* (3) *and* (5) *respectively.*

REMARK 2. We don't know at the present stage whether the conditions (10) and (11) are purely technical or can be removed. They are necessary to the proof of some auxiliary remainder estimates (see Section 3.3, Lemma 1). Let us remark that (11) is implied by the following: for all $\sigma \in (0, 1)$, there exists $\delta_\sigma \in (0, 1)$ and c, $C > 0$ such that, for all $s \in [0, \delta_\sigma]$, we have $cs \le \omega(s) \le Cs^\sigma$.

REMARK 3. It would be possible to prove uniqueness for an operator with terms of order one, i.e. for

$$\tilde{P} = \partial_t + \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t, x)\partial_{x_k}) + \sum_{k=1}^{n} b_k(t, x)\partial_{x_k} + c(t, x),$$

assuming that $b_k(t, x)$ are $L^\infty([0, T], C^\sigma(\mathbb{R}^n_x))$ for some $\sigma > 0$. This is due to the fact that the Carleman estimate, which we are able to prove, is in H^{-s} with $s \in (0, 1)$. In [5] and [4] the Carleman estimate was proved in L^2 and this fact allowed to consider the coefficients $b_k(t, x)$ under no hypotheses on $b_k(t, x)$, apart boundedness.

EXAMPLE 2. A simple example of moduli of continuity μ and ω satisfying the hypotheses of Theorem 1 is $\mu(s) = s(1 + |\log(s)|)$ and $\omega(s) = s\sqrt{1 + |\log(s)|}$.

3. Littlewood–Paley theory and Bony's paraproduct

In this section we recall some well-known results of the Littlewood–Paley theory and Bony's paraproduct. These results will be fundamental tools in the proof of our Carleman estimate.

3.1. Littlewood–Paley theory. Let χ and φ be two functions in $C_0^\infty(\mathbb{R}^n_\xi)$, with values in [0, 1], such that

$$\operatorname{supp}(\varphi) \subseteq \left\{\xi \in \mathbb{R}^n_\xi : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},$$
$$\operatorname{supp}(\chi) \subseteq \left\{\xi \in \mathbb{R}^n_\xi : |\xi| \le \frac{4}{3}\right\}. \tag{12}$$

Let, for all $\xi \in \mathbb{R}^n_\xi$,

$$\chi(\xi) + \sum_{q\ge 0} \varphi(2^{-q}\xi) = 1,$$

i.e. $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. By these choices we have

$$\operatorname{supp}(\chi(2^{-q}\cdot)) \subseteq \left\{\xi \in \mathbb{R}^n_\xi : |\xi| \le \frac{4}{3}2^q\right\}$$

and therefore

$$\operatorname{supp}(\varphi(2^{-q}\cdot)) \subseteq \left\{\xi \in \mathbb{R}^n_\xi : \frac{3}{4}2^q \le |\xi| \le \frac{8}{3}2^q\right\}.$$

We get

$$\operatorname{supp}(\varphi(2^{-q}\cdot)) \cap \operatorname{supp}(\varphi(2^{-p}\cdot)) = \emptyset, \quad \text{for all} \quad |p-q| \ge 2. \tag{13}$$

With this preparations, we define the Littlewood–Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^n$ and by $\mathcal{F}^{-1}$ its inverse. Let Δ_q and S_q, for $q \in \mathbb{Z}$, be defined as follows:

$$\begin{aligned}
&\Delta_q u := 0 \quad \text{if} \quad q \le -2,\\
&\Delta_{-1} u := \chi(D_x)u = \mathcal{F}^{-1}(\chi(\cdot)\mathcal{F}(u)(\cdot)),\\
&\Delta_q u := \varphi(2^{-q}D_x)u = \mathcal{F}^{-1}(\varphi(2^{-q}\cdot)\mathcal{F}(u)(\cdot)), \quad q \ge 0
\end{aligned}$$

and

$$S_q u = \chi(2^{-q}D_x)u = \mathcal{F}^{-1}(\chi(2^{-q}\cdot)\mathcal{F}(u)(\cdot)) = \sum_{p\le q-1} \Delta_p u, \quad q \ge 0.$$

Furthermore we denote

$$\operatorname{spec}(u) := \operatorname{supp}(\mathcal{F}(u)).$$

For $u \in \mathcal{S}'(\mathbb{R}^n_x)$,

$$u = \sum_{q\ge -1} \Delta_q u$$

in the sense of $\mathcal{S}'(\mathbb{R}^n_x)$.

The following two propositions describe the decomposition and synthesis of the classical Sobolev spaces H^s, via Littlewood–Paley decomposition. A proof of these two propositions can be found in [9, Proposition 4.1.11 and Proposition 4.1.12].

Proposition 1. *Let $s \in \mathbb{R}$. Then a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n_x)$ belongs to $H^s(\mathbb{R}^n_x)$ if and only if the following two conditions hold*:
(i) *for all $q \geq -1$, $\Delta_q u \in L^2(\mathbb{R}^n_x)$,*
(ii) *the sequence $(\delta_q)_{q\in\mathbb{Z}_{\geq -1}}$, where $\delta_q := 2^{qs}\|\Delta_q u\|_{L^2(\mathbb{R}^n_x)}$, belongs to $l^2(\mathbb{Z}_{\geq -1})$.*
Moreover, there exists $C_s \geq 1$ such that, for all $u \in H^s(\mathbb{R}^n_x)$, we have

$$\frac{1}{C_s}\|u\|_{H^s(\mathbb{R}^n_x)} \leq \|(\delta_q)\|_{l^2(\mathbb{Z}_{\geq -1})} \leq C_s\|u\|_{H^s(\mathbb{R}^n_x)}.$$

Proposition 2. *Let $s \in \mathbb{R}$ and $R \in \mathbb{R}_{>1}$. Suppose that a sequence $(u_q)_{q\in\mathbb{Z}_{\geq -1}}$ in $L^2(\mathbb{R}^n_x)$ satisfies*
(i) $\operatorname{spec}(u_{-1}) \subseteq \{\xi \in \mathbb{R}^n_\xi \colon |\xi| \leq R\}$ *and, for all $q \geq 0$,*

$$\operatorname{spec}(u_q) \subseteq \{\xi \in \mathbb{R}^n_\xi \colon R^{-1}2^q \leq |\xi| \leq 2R2^q\},$$

(ii) *the sequence $(\delta_q)_{q\geq -1}$, where $\delta_q := 2^{qs}\|u_q\|_{L^2(\mathbb{R}^n_x)}$, belongs to $l^2(\mathbb{Z}_{\geq -1})$.*
Then $u = \sum_{q\geq -1} u_q \in H^s(\mathbb{R}^n_x)$ and there exists $C_s \geq 1$ such that, for all $u \in H^s(\mathbb{R}^n_x)$, we have

$$\frac{1}{C_s}\|u\|_{H^s(\mathbb{R}^n_x)} \leq \|\delta_q\|_{l^2(\mathbb{Z}_{\geq -1})} \leq C_s\|u\|_{H^s(\mathbb{R}^n_x)}.$$

When $s > 0$ it is enough to assume, instead if (i), that, for all $q \geq -1$,

$$\operatorname{spec}(u_q) \subseteq \{\xi \in \mathbb{R}^n_\xi \colon |\xi| \leq R2^q\}.$$

The following result will be crucial in the sequel.

Proposition 3. *There exists a constant $C > 0$ such that the following estimates hold true*:
(i) (*Bernstein inequalities*) *for $u \in L^p(\mathbb{R}^n_x)$, $p \in [1, +\infty]$:*

$$\|\nabla_x S_q u\|_{L^p(\mathbb{R}^n_x)} \leq C2^q\|u\|_{L^p(\mathbb{R}^n_x)}, \quad q \geq 0,$$
$$\frac{1}{C}\|\Delta_q u\|_{L^p(\mathbb{R}^n_x)} \leq 2^{-q}\|\nabla_x \Delta_q u\|_{L^p(\mathbb{R}^n_x)} \leq C\|\Delta_q u\|_{L^p(\mathbb{R}^n_x)}, \quad q \geq 0.$$

For $q = -1$ only $\|\nabla_x \Delta_{-1} u\|_{L^p(\mathbb{R}^n_x)} \leq C\|\Delta_{-1} u\|_{L^p(\mathbb{R}^n_x)}$ holds.
(ii) (*Commutator estimate*) *for $a \in L^\infty(\mathbb{R}^n_x)$ and $u \in L^2(\mathbb{R}^n_x)$*:

$$\|[S_{q'}a, \Delta_q]\Delta_p u\|_{L^2(\mathbb{R}^n_x)} \leq C2^{-p}\|\nabla_x S_{q'}a\|_{L^\infty(\mathbb{R}^n_x)}\|\Delta_p u\|_{L^2(\mathbb{R}^n_x)}, \quad q' \geq 0,\ p, q \geq -1. \tag{14}$$

Proof. The proof of the Bernstein inequalities can be found in [9, Corollary 4.1.17]. The commutator estimate follows from [2, Theorem 35]. This result applied to our case reads

$$\|[a, \Delta_q]\partial_{x_k} u\|_{L^2(\mathbb{R}^n_x)} \leq C\|\nabla_x a\|_{L^\infty(\mathbb{R}^n_x)}\|u\|_{L^2(\mathbb{R}^n_x)} \tag{15}$$

for $a \in \mathrm{Lip}(\mathbb{R}^n_x)$ and $u \in H^1(\mathbb{R}^n_x)$. Estimate (14) follows from (15) writing $\Delta_p u$ as a sum of derivatives. □

The proof of the following proposition can be found in [14, Proposition 1.5].

Proposition 4. *Let ω be a modulus of continuity. Then, for all $u \in C^\omega(\mathbb{R}^n_x)$,*

$$\|\nabla_x S_q u\|_{L^\infty(\mathbb{R}^n_x)} \leq C2^q\omega(2^{-q}). \tag{16}$$

Conversely, given $u \in L^\infty(\mathbb{R}^n_x)$, if (16) *holds, then $u \in C^\sigma(\mathbb{R}^n_x)$, where $\sigma(h) = \int_0^h \omega(t)/t\, dt$.*

The main consequence of Proposition 4 is contained in the following corollary.

Corollary 1. *Let ω be a modulus of continuity satisfying condition* (9). *Then a function $u \in L^\infty(\mathbb{R}^n_x)$ belongs to $C^\omega(\mathbb{R}^n_x)$ if and only if*

$$\sup_{q\in\mathbb{N}_0} \frac{\|\nabla_x(S_q u)\|_{L^\infty(\mathbb{R}^n_x)}}{2^q\omega(2^{-q})} < +\infty. \tag{17}$$

Other interesting properties of the Littlewood–Paley decomposition are contained in the following proposition.

Proposition 5. *Let $a \in C^\omega(\mathbb{R}^n_x)$. Then, for all $q \geq -1$*

$$\|\Delta_q a\|_{L^\infty(\mathbb{R}^n_x)} \leq C\|a\|_{C^\omega(\mathbb{R}^n_x)}\omega(2^{-q}), \tag{18}$$

and, if additionally (9) *holds,*

$$\|a - S_q a\|_{L^\infty(\mathbb{R}^n_x)} \leq C\|a\|_{C^\omega(\mathbb{R}^n_x)}\omega(2^{-q}). \tag{19}$$

Proof. The proof of (18) is the same as [3, Proposition 3.4]. To prove the second estimate we note that

$$a - S_q a = \sum_{p\geq q} \Delta_p a$$

and therefore, from (18), we get

$$\|a - S_q a\|_{L^\infty(\mathbb{R}^n_x)} \leq \sum_{p\geq q} \|\Delta_p a\|_{L^\infty(\mathbb{R}^n_x)} \leq C\|a\|_{C^\omega(\mathbb{R}^n_x)} \sum_{p\geq q} \omega(2^{-p}).$$

An elementary computation gives that (9) is equivalent to $\sum_{p\geq q} \omega(2^{-p}) \leq \omega(2^{-q+1})$. This concludes the proof. □

REMARK 4. Estimate (19) implies that $(S_q a_{jk})^n_{j,k=1}$ is a positive matrix if $(a_{jk})^n_{j,k=1}$ is a positive matrix and q is sufficiently large.

For later use we introduce a weighted Sobolev space.

DEFINITION 4. Let $s \in \mathbb{R}$ and ω be a modulus of continuity. Let $\Omega(q) = 2^q \omega(2^{-q})$. We say that $u \in \mathcal{S}'(\mathbb{R}^n_x)$ belongs to $H^s_\Omega(\mathbb{R}^n_x)$ if

$$\|u\|_{H^s_\Omega(\mathbb{R}^n_x)} := \left(\sum_{q\geq -1} 2^{2sq}\Omega^2(q)\|\Delta_q u\|^2_{L^2(\mathbb{R}^n_x)}\right)^{1/2} < +\infty.$$

3.2. Bony's paraproduct. Let us now define Bony's paraproduct (see [1]) for tempered distributions u and v as

$$T_u v = \sum_{q\geq 1}\sum_{p\leq q-2} \Delta_p u \Delta_q v = \sum_{q\geq 1} S_{q-1} u \Delta_q v.$$

Let us define also

$$R(u, v) = \sum_{\substack{q\geq -1 \\ i\in\{0,\pm 1\}}} \Delta_q u \Delta_{q+i} v = \sum_{q\geq -1} \Delta_q u \tilde{\Delta}_q v, \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

With this we can (formally) decompose a product uv with $u, v \in \mathcal{S}'(\mathbb{R}^n_x)$ by

$$uv = T_u v + T_v u + R(u, v).$$

Proposition 6. *Let $a \in L^\infty(\mathbb{R}^n_x)$, $s \in \mathbb{R}$. Then the operator T_a maps $H^s(\mathbb{R}^n_x)$ continuously into $H^s(\mathbb{R}^n_x)$, i.e. there exist a constant $C_s > 0$ such that*

$$\|T_a u\|_{H^s(\mathbb{R}^n_x)} \leq C_s\|a\|_{L^\infty(\mathbb{R}^n_x)}\|u\|_{H^s(\mathbb{R}^n_x)}.$$

The proof of this proposition can be found in [9, Proposition 5.2.1]. Other mapping properties, especially of the remainder $R(u, v)$, will be proved in Section 3.3.

Let now a and b be tempered distributions sufficiently regular such that ab makes sense. Then we have

$$\Delta_q(ab) = \Delta_q T_a b + \Delta_q T_b a + \Delta_q R(a,b) = \Delta_q T_a b + \Delta_q \tilde{R}(a,b),$$

where

$$\tilde{R}(a,b) = T_b a + R(a,b) = \sum_{q' \geq -1} S_{q'+2} b \Delta_{q'} a. \tag{20}$$

From the definition of Δ_q and S_q it is easy to verify that

$$\Delta_q(S_{q'-1} a \Delta_{q'} b) = 0 \quad \text{if} \quad |q'-q| \geq 5, \tag{21}$$

and similarly

$$\Delta_q(S_{q'+2} a \Delta_{q'} b) = 0 \quad \text{if} \quad q' \leq q-4, \tag{22}$$

so that

$$\begin{aligned}
\Delta_q(ab) &= \sum_{|q'-q|\leq 4} \Delta_q(S_{q'-1} a \Delta_{q'} b) + \sum_{q'>q-4} \Delta_q(S_{q'+2} b \Delta_{q'} a) \\
&= \sum_{|q'-q|\leq 4} [\Delta_q, S_{q'-1} a] \Delta_{q'} b + \sum_{|q'-q|\leq 4} S_{q'-1} a \Delta_q \Delta_{q'} b \\
&\quad + \sum_{q'>q-4} \Delta_q(S_{q'+2} b \Delta_{q'} a) \\
&= \sum_{|q'-q|\leq 4} [\Delta_q, S_{q'-1} a] \Delta_{q'} b + \sum_{|q'-q|\leq 4} (S_{q'-1} a - S_{q-1} a) \Delta_q \Delta_{q'} b \\
&\quad + \sum_{q'>q-4} \Delta_q(S_{q'+2} b \Delta_{q'} a) + \underbrace{\sum_{|q'-q|\leq 4} S_{q-1} a \Delta_q \Delta_{q'} b}_{= S_{q-1} a \Delta_q b}.
\end{aligned}$$

Consequently,

$$\Delta_q(ab) = S_{q-1} a \Delta_q b + \mathcal{R}_q(a,b), \tag{23}$$

where

$$\begin{aligned}
\mathcal{R}_q(a,b) &= \sum_{|q'-q|\leq 4} [\Delta_q, S_{q'-1} a] \Delta_{q'} b + \sum_{|q'-q|\leq 4} (S_{q'-1} a - S_{q-1} a) \Delta_q \Delta_{q'} b \\
&\quad + \sum_{q'>q-4} \Delta_q(S_{q'+2} b \Delta_{q'} a) \\
&= \mathcal{R}_q^{(1)}(a,b) + \mathcal{R}_q^{(2)}(a,b) + \mathcal{R}_q^{(3)}(a,b).
\end{aligned}$$

Let us remark that a consequence of (23) is that

$$\operatorname{spec}\mathcal{R}_q(a,b)\subseteq\left\{\xi\in\mathbb{R}^n_\xi:\ |\xi|\le\frac{10}{3}2^q\right\}. \tag{24}$$

3.3. Auxiliary estimates for $\mathcal{R}_q(a,b)$. In this section we prove an estimate about $\mathcal{R}_q(a,b)$ which we will use in the sequel.

Lemma 1. *Let ω be a modulus of continuity satisfying* (9), $s\in\mathbb{R}$, $\Omega(q)$ *as in* Definition 4. *Let $a\in C^\omega(\mathbb{R}^n_x)$ and $b\in H^{-s}_\Omega(\mathbb{R}^n_x)$. Then*

$$\left(\sum_{q\ge -1}2^{2(1-s)q}\|\mathcal{R}^{(i)}_q(a,b)\|^2_{L^2(\mathbb{R}^n_x)}\right)^{1/2}\le C_{s,i}\|a\|_{C^\omega(\mathbb{R}^n_x)}\|b\|_{H^{-s}_\Omega(\mathbb{R}^n_x)},\quad i=1,2. \tag{25}$$

Suppose moreover that $s\in(0,1)$ and ω satisfies (10) *and* (11). *Then the estimate* (25) *holds also for $i=3$.*

Proof. Let us start with the inequality (25), for $i=1$. We have

$$\begin{aligned}\mathcal{R}^{(1)}_q(a,b)&=\sum_{|q'-q|\le 4}[\Delta_q,S_{q'-1}a]\Delta_{q'}b\\&=[\Delta_q,S_{q-5}a]\Delta_{q-4}b+[\Delta_q,S_{q-4}a]\Delta_{q-3}b+\cdots+[\Delta_q,S_{q+3}a]\Delta_{q+4}b.\end{aligned} \tag{26}$$

Consider the first term of this sum. We have, from (14) and (17),

$$\begin{aligned}\|[\Delta_q,S_{q-5}a]\Delta_{q-4}b\|_{L^2(\mathbb{R}^n_x)}&\le C2^{-(q-4)}\|\nabla_xS_{q-5}a\|_{L^\infty(\mathbb{R}^n_x)}\|\Delta_{q-4}b\|_{L^2(\mathbb{R}^n_x)}\\&\le\frac{C}{2}\omega(2^{-(q-5)})\|a\|_{C^\omega(\mathbb{R}^n_x)}\|\Delta_qb\|_{L^2(\mathbb{R}^n_x)}.\end{aligned}$$

Since $b\in H^{-s}_\Omega(\mathbb{R}^n_x)$ we have that

$$\|\Delta_{q-4}b\|_{L^2(\mathbb{R}^n_x)}\le\frac{2^{s(q-4)}}{\Omega(q-4)}\varepsilon_q=\frac{2^{(s-1)(q-4)}}{\omega(2^{-(q-4)})}\varepsilon_q,$$

where $(\varepsilon_q)_{q\in\mathbb{Z}_{\ge-1}}$ is a sequence in $l^2(\mathbb{Z}_{\ge-1})$ and there exists $c_s\ge 1$ such that

$$\frac{1}{c_s}\|b\|_{H^{-s}_\Omega(\mathbb{R}^n_x)}\le\|(\varepsilon_q)\|_{l^2(\mathbb{Z}_{\ge-1})}\le c_s\|b\|_{H^{-s}_\Omega(\mathbb{R}^n_x)}. \tag{27}$$

We get

$$\begin{aligned}2^{(1-s)q}\|[\Delta_q,S_{q-5}a]\Delta_{q-4}b\|_{L^2(\mathbb{R}^n_x)}&\le C2^{3-4s}\frac{\omega(2^{-(q-5)})}{\omega(2^{-(q-4)})}\|a\|_{C^\omega(\mathbb{R}^n_x)}\varepsilon_{q-4}\\&\le C_s\|a\|_{C^\omega(\mathbb{R}^n_x)}\varepsilon_{q-4}.\end{aligned} \tag{28}$$

For all the other terms in (26) we obtain an estimate similar to (28) and the inequality (25) follows.

Let us now consider the inequality (25), for $i = 2$. We have

$$\|\mathcal{R}_q^{(2)}(a, b)\|_{L^2(\mathbb{R}^n_x)} = \|(S_{q-2} - S_{q-1})a\Delta_q\Delta_{q-1}b + (S_q - S_{q-1})a\Delta_q\Delta_{q+1}b\|_{L^2(\mathbb{R}^n_x)}.$$

Since $S_{q-2} - S_{q-1} = -\Delta_{q-2}$ and $S_q - S_{q-1} = \Delta_{q-1}$, we deduce from (18),

$$\begin{aligned}\|\mathcal{R}_q^{(2)}(a, b)\|_{L^2(\mathbb{R}^n_x)} &\le (\|\Delta_{q-2}a\|_{L^\infty(\mathbb{R}^n_x)} + \|\Delta_{q-1}a\|_{L^\infty(\mathbb{R}^n_x)})\|\Delta_q b\|_{L^2(\mathbb{R}^n_x)}\\ &\le 2C\|a\|_{C^\omega(\mathbb{R}^n_x)}\omega(2^{-q})\frac{2^{sq}}{\Omega(q)}\varepsilon_q,\end{aligned}$$

where we have used the fact that $\|\Delta_q b\|_{L^2(\mathbb{R}^n_x)} \le (2^{qs}/\Omega(q))\varepsilon_q$, where $(\varepsilon_q)_{q\in\mathbb{Z}_{\ge -1}}$ is a sequence in $l^2(\mathbb{Z}_{\ge -1})$ satisfying (27). Therefore, remembering that $\Omega(q) = 2^q\omega(2^{-q})$, we get

$$2^{(1-s)q}\|\mathcal{R}_q^{(2)}(a, b)\|_{L^2(\mathbb{R}^n_x)} \le 2C\|a\|_{C^\omega(\mathbb{R}^n_x)}\varepsilon_q.$$

Thus, inequality (25), for $i = 2$, follows. Let now $s \in (0, 1)$. We have

$$\begin{aligned}R_q^{(3)}(a, b) &= \sum_{q'>q-4} \Delta_q(S_{q'+2}b\Delta_{q'}a)\\ &= \sum_{q'>q-4} (\Delta_q(S_{q'-1}b\Delta_{q'}a) + \Delta_q(\Delta_{q'-1}b\Delta_{q'}a + \Delta_{q'}b\Delta_{q'}a + \Delta_{q'+1}b\Delta_{q'}a)).\end{aligned}$$

From (21) and (22) we obtain

$$\begin{aligned}R_q^{(3)}(a, b) &= \Delta_q(S_{q-4}b\Delta_{q-3}a + \cdots + S_{q+4}b\Delta_{q+5}a)\\ &\quad + \sum_{q'\ge -1}(\Delta_q(\Delta_{q'-1}b\Delta_{q'}a + \Delta_{q'}b\Delta_{q'}a + \Delta_{q'+1}b\Delta_{q'}a)).\end{aligned} \tag{29}$$

The nine terms in the first line in (29) are essentially of the form $\Delta_q(S_{q-1}b\Delta_q a)$ and can be treated as follows:

$$\begin{aligned}\sum_{q\ge -1} 2^{2(1-s)q}\|\Delta_q(S_{q-1}b\Delta_q a)\|^2_{L^2(\mathbb{R}^n_x)} &\le \sum_{q\ge -1} 2^{2(1-s)q}\|S_{q-1}b\Delta_q a\|^2_{L^2(\mathbb{R}^n_x)}\\ &\le \sum_{q\ge -1} 2^{2(1-s)q}\|S_{q-1}b\|^2_{L^2(\mathbb{R}^n_x)}\|\Delta_q a\|^2_{L^\infty(\mathbb{R}^n_x)}\\ &\le \sum_{q\ge -1} 2^{2(1-s)q}\Bigg(\sum_{p\le q-2}\|\Delta_p b\|_{L^2(\mathbb{R}^n_x)}\Bigg)^2\|\Delta_q a\|^2_{L^\infty(\mathbb{R}^n_x)}\\ &\le \sum_{q\ge -1} 2^{2(1-s)q}\Bigg(\sum_{p\le q-2}\frac{2^{ps}}{\Omega(p)}\varepsilon_p\Bigg)^2 2^{-2q}\Omega^2(q)\|a\|^2_{C^\omega(\mathbb{R}^n_x)}\\ &\le \sum_{q\ge -1}\Bigg(\sum_{p\le q-2} 2^{-s(q-p)}\frac{\Omega(q)}{\Omega(p)}\varepsilon_p\Bigg)^2\|a\|^2_{C^\omega(\mathbb{R}^n_x)},\end{aligned}$$

where $(\varepsilon_j)_{j\in\mathbb{Z}_{\geq -1}}$ is a sequence in $l^2(\mathbb{Z}_{\geq -1})$ with (27). From (10) and the definition of Ω we get

$$\tilde{\varepsilon}_q := \sum_{p\leq q-2} 2^{-s(q-p)}\frac{\Omega(q)}{\Omega(p)}\varepsilon_p \leq \sum_{p\leq q-2} 2^{(1-s)(q-p)}\omega(2^{-(q-p)})\varepsilon_p.$$

Then (11) and the Young inequality for convolution in l^p spaces give that the sequence $(\tilde{\varepsilon}_j)_{j\in\mathbb{Z}_{\geq -1}}$ is in $l^2(\mathbb{Z}_{\geq -1})$ and there exists $C_s > 0$ such that

$$\|(\tilde{\varepsilon}_j)\|_{l^2(\mathbb{Z}_{\geq -1})} \leq \tilde{C}_s \|(\varepsilon_j)\|_{l^2(\mathbb{Z}_{\geq -1})}.$$

From (27) we conclude that

$$\begin{aligned}\sum_{q\geq -1} 2^{2(1-s)q}\|\Delta_q(S_{q-1}b\Delta_q a)\|^2_{L^2(\mathbb{R}^n_x)} &\leq \tilde{C}_s^2\|(\varepsilon_j)\|^2_{l^2(\mathbb{Z}_{\geq -1})}\|a\|^2_{C^\omega(\mathbb{R}^n_x)} \\ &\leq C_s^2\|b\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)}\|a\|^2_{C^\omega(\mathbb{R}^n_x)}.\end{aligned}$$

The second line of (29) is a sum of three terms of the form $\sum_{q'\geq -1}\Delta_q(\Delta_{q'}b\Delta_{q'}a)$. We have

$$\sum_{q\geq -1} 2^{2(1-s)q}\left\|\sum_{q'\geq -1}\Delta_q(\Delta_{q'}b\Delta_{q'}a)\right\|^2_{L^2(\mathbb{R}^n_x)} = \sum_{q\geq -1} 2^{2(1-s)q}\left\|\Delta_q\left(\sum_{q'\geq -1}\Delta_{q'}b\Delta_{q'}a\right)\right\|^2_{L^2(\mathbb{R}^n_x)}.$$

Thanks to the result of Proposition 1, this last quantity coincides with $\left\|\sum_{q'\geq -1}\Delta_{q'}b\Delta_{q'}a\right\|^2_{H^{1-s}(\mathbb{R}^n_x)}$. To compute the $H^{1-s}(\mathbb{R}^n_x)$ of $\sum_{q'\geq -1}\Delta_{q'}b\Delta_{q'}a$ we use Proposition 2. In fact $1-s>0$,

$$\operatorname{spec}(\Delta_{q'}b\Delta_{q'}a) \subseteq \left\{\xi \in \mathbb{R}^n_\xi : |\xi| \leq \frac{16}{3}2^{q'}\right\},$$

and

$$2^{(1-s)q'}\|\Delta_{q'}b\Delta_{q'}a\|_{L^2(\mathbb{R}^n_x)} \leq 2^{(1-s)q'}\|\Delta_{q'}b\|_{L^2(\mathbb{R}^n_x)}\|\Delta_{q'}a\|_{L^\infty(\mathbb{R}^n_x)} \leq \varepsilon_{q'}\|a\|_{C^\omega(\mathbb{R}^n_x)}.$$

Again (27) gives $\left\|\sum_{q'\geq -1}\Delta_{q'}b\Delta_{q'}a\right\|^2_{H^{1-s}(\mathbb{R}^n_x)} \leq C_s\|b\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)}\|a\|^2_{C^\omega(\mathbb{R}^n_x)}$. The proof of the lemma is concluded. □

4. The Carleman estimate

4.1. The weight function. The idea of constructing a weight function which is linked to the modulus of continuity is due to Tarama ([12], see also [5, 4, 6]). Let μ be a modulus of continuity satisfying (8). We set

$$\varphi(t) := \int_{1/t}^1 \frac{1}{\mu(s)}\,ds.$$

The function φ is strictly increasing and $C^1([1,+\infty[)$. We have $\varphi([1,+\infty)) = [0,+\infty)$ and $\varphi'(t) = 1/(t^2\mu(1/t)) > 0$ for all $t \in [1,+\infty)$. We define

$$\Phi(\tau) := \int_0^\tau \varphi^{-1}(s)\, ds. \tag{30}$$

From this we get $\Phi'(t) = \varphi^{-1}(t)$ and therefore $\lim_{\tau\to+\infty}\Phi'(\tau) = +\infty$. Moreover we have

$$\Phi''(\tau) = (\Phi'(\tau))^2\mu\left(\frac{1}{\Phi'(\tau)}\right) \tag{31}$$

for all $\tau \in [0,+\infty)$ and, since the function $\sigma \mapsto \sigma\mu(1/\sigma)$ is increasing on the interval $[1,+\infty)$, we obtain that

$$\lim_{\tau\to+\infty}\Phi''(\tau) = \lim_{\tau\to+\infty}(\Phi'(\tau))^2\mu\left(\frac{1}{\Phi'(\tau)}\right) = +\infty.$$

4.2. The Carleman estimate. The uniqueness result of Theorem 1 will be a consequence of the following Carleman estimate.

Proposition 7. *Let μ and ω be two moduli of continuity such that $\omega(s) = \sqrt{\mu(s^2)}$. Suppose that μ and ω satisfy* (8) *and* (9), (10), (11) *respectively. Suppose that, for all $j,k = 1,\dots,n$,*

$$a_{jk} \in C^\mu([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], C^\omega(\mathbb{R}^n_x)),$$

and let (4) *hold. Let Φ and $H^s_\Omega(\mathbb{R}^n_x)$ defined in* (30) *and* Definition 4 *respectively. Let $s \in (0,1)$. Then there exist $\gamma_0 \ge 1$, $C > 0$, such that, for all $\gamma \ge \gamma_0$ and all $u \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ with* $\operatorname{supp}(u) \subseteq [0,T/2]\times\mathbb{R}^n_x$,

$$\begin{aligned}&\int_0^{T/2} e^{(2/\gamma)\Phi(\gamma(T-t))}\left\|\partial_t u + \sum_{j,k=1}^n \partial_{x_j}(a_{jk}(t,x)\partial_{x_k}u)\right\|^2_{H^{-s}(\mathbb{R}^n_x)} dt\\ &\ge C\gamma^{1/4}\int_0^{T/2} e^{(2/\gamma)\Phi(\gamma(T-t))}\left(\|\nabla u\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)} + \gamma^{3/4}\|u\|^2_{L^2(\mathbb{R}^n_x)}\right) dt.\end{aligned} \tag{32}$$

Setting

$$u(t,x) = e^{-(1/\gamma)\Phi(\gamma(T-t))}v(t,x),$$

the Carleman estimate (32) becomes

$$
\begin{aligned}
(33)\qquad &\int_0^{T/2} \Big\| \partial_t v + \sum_{j,k=1}^n \partial_{x_j}(a_{jk}(t,x)\partial_{x_k} v) + \Phi'(\gamma(T-t))v \Big\|^2_{H^{-s}(\mathbb{R}^n_x)} dt \\
&\geq C\gamma^{1/4} \int_0^{T/2} (\|\nabla v\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)} + \gamma^{3/4}\|v\|^2_{L^2(\mathbb{R}^n_x)})\, dt.
\end{aligned}
$$

The proof of such inequality is divided in several steps which we will present in the subsequent subsections.

4.3. Regularization in *t*. In our proof of the Carleman estimate we need to perform some integrations by part with respect to t and if the coefficients a_{jk} are not sufficiently regular this is not possible. We will avoid this difficulty regularizing the a_{jk}'s with respect to t and to this end we will use Friedrichs mollifiers. We take a $\rho \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\rho) \subseteq [-1/2, 1/2]$ and $\int_{\mathbb{R}} \rho(\tau)d\tau = 1$ and $\rho(\tau) = \rho(-\tau)$ and we define

$$
a^\varepsilon_{jk}(t,x) := \frac{1}{\varepsilon}\int_{\mathbb{R}^n} a(s,x)\rho\left(\frac{t-s}{\varepsilon}\right) ds.
$$

We have easily

$$
|a^\varepsilon_{jk}(t,x) - a_{jk}(t,x)| \leq C\mu(\varepsilon)
$$

and

$$
|\partial_t a^\varepsilon_{jk}(t,x)| \leq C\frac{\mu(\varepsilon)}{\varepsilon}.
$$

where C depends only on $\|a_{j,k}\|_{C^\mu([0,T],L^\infty(\mathbb{R}^n_x))}$.

4.4. Estimates for the microlocalized operator. Using the characterization of Sobolev spaces given in Proposition 1 we have that the left hand side part of (33) reads

$$
(34)\qquad \sum_{q\geq -1} 2^{-2sq} \int_0^{T/2} \Big\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k} v)) + \Phi'(\gamma(T-t))v_q \Big\|^2_{L^2(\mathbb{R}^n_x)} dt,
$$

where we set $\Delta_q v := v_q$. We use formula (23) and we replace $a_{jk}(t,x)\partial_{x_k} v$ with $(S_{q-1}a_{jk}(t,x))\partial_{x_k} v_q + \mathcal{R}_q(a_{jk}, \partial_{x_k} v)$. We deduce that (34) is bounded from below by

$$
\begin{aligned}
&\sum_{q\geq -1} 2^{-2sq-1} \int_0^{T/2} \Big\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) + \Phi'(\gamma(T-t))v_q \Big\|^2_{L^2(\mathbb{R}^n_x)} dt \\
&- \sum_{q\geq -1} 2^{-2sq} \int_0^{T/2} \Big\| \partial_{x_j}(\mathcal{R}_q(a_{jk}, \partial_{x_k} v)) \Big\|^2_{L^2(\mathbb{R}^n_x)} dt.
\end{aligned}
$$

We use now (24), the Bernstein inequalities and the result of Lemma 1 and we get

$$\sum_{q\geq -1} 2^{-2sq} \int_0^{T/2} \|\partial_{x_j}(\mathcal{R}_q(a_{jk}, \partial_{x_k} v))\|^2_{L^2(\mathbb{R}^n_x)}\, dt \leq C \int_0^{T/2} \|\nabla v\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)}\, dt,$$

where C depends only on s and on $\max_{j,k} \|a_{j,k}\|_{L^\infty([0,T],C^\omega(\mathbb{R}^n_x))}$. Finally, (33) will be a consequence of

$$\sum_{q\geq -1} 2^{-2sq} \int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(S_{q-1} a_{jk}(t,x) \partial_{x_k} v_q)) + \Phi'(\gamma(T-t)) v_q \right\|^2_{L^2(\mathbb{R}^n_x)} dt$$
$$\geq C\gamma^{1/4} \int_0^{T/2} (\|\nabla v\|^2_{H^{-s}_\Omega(\mathbb{R}^n_x)} + \gamma^{3/4} \|v\|^2_{L^2(\mathbb{R}^n_x)})\, dt.$$

We have

$$\begin{aligned}
&\int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(S_{q-1} a_{jk}(t,x) \partial_{x_k} v_q) + \Phi'(\gamma(T-t)) v_q \right\|^2_{L^2(\mathbb{R}^n_x)} dt \\
&= \int_0^{T/2} \|\partial_t v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt \\
&\quad + \int_0^{T/2} \left\| \sum_{j,k=1}^n \partial_{x_j}(S_{q-1} a_{jk}(t,x) \partial_{x_k} v_q) + \Phi'(\gamma(T-t)) v_q \right\|^2_{L^2(\mathbb{R}^n_x)} dt \\
&\quad + 2\,\mathrm{Re} \int_0^{T/2} \langle \partial_t v_q \mid \Phi'(\gamma(T-t)) v_q \rangle_{L^2(\mathbb{R}^n_x)}\, dt \\
&\quad + 2\,\mathrm{Re} \sum_{j,k=1}^n \int_0^{T/2} \langle \partial_t v_q \mid \partial_{x_j}\big(S_{q-1} a_{jk}(t,x) \partial_{x_k} v_q\big) \rangle\, dt.
\end{aligned}$$

We compute by integration by parts

$$2\,\mathrm{Re} \int_0^{T/2} \langle \partial_t v_q \mid \Phi'(\gamma(T-t)) v_q \rangle_{L^2(\mathbb{R}^n_x)}\, dt = \gamma \int_0^{T/2} \Phi''(\gamma(T-t)) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt.$$

To handle the second scalar product we use the regularization from Section 4.3. In particular

$$|S_q a^\varepsilon_{jk}(t,x) - S_q a_{jk}(t,x)| \leq C\mu(\varepsilon), \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R}^n_x \tag{35}$$

and

$$|\partial_t S_q a^\varepsilon_{jk}(t,x)| \leq C\frac{\mu(\varepsilon)}{\varepsilon}, \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R}^n_x, \tag{36}$$

where C depends only on $\max_{j,k}\|a_{j,k}\|_{C^\mu([0,T],L^\infty(\mathbb{R}^n_x))}$. Adding and subtracting $\partial_{x_j}(S_{q-1}a^\varepsilon_{jk}(t,x)\partial_{x_k}v_q)$ we get

$$
\begin{aligned}
&2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt\\
&= 2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}a^\varepsilon_{jk}(t,x)\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt\\
&\quad + 2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}(a_{jk}(t,x)-a^\varepsilon_{jk}(t,x))\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt.
\end{aligned}
\tag{37}
$$

By integration by parts we get

$$
\begin{aligned}
&2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}a^\varepsilon_{jk}(t,x)\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt\\
&= \sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_{x_j} v_q \mid \partial_t(S_{q-1}a^\varepsilon_{jk}(t,x))\partial_{x_k}v_q\rangle_{L^2(\mathbb{R}^n_x)}.
\end{aligned}
$$

From (36) we obtain

$$
\begin{aligned}
&\left|2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}a^\varepsilon_{jk}(t,x)\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt\right|\\
&\le \sum_{j,k=1}^{n}\int_0^{T/2}\|\partial_{x_j}v_q\|_{L^2(\mathbb{R}^n_x)}\|\partial_t(S_{q-1}a^\varepsilon_{jk}(t,x))\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)}\,dt\\
&\le C_1\frac{\mu(\varepsilon)}{\varepsilon}2^{2q}\int_0^{T/2}\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt,
\end{aligned}
$$

where we have used the fact that $\|\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)} \le C2^q\|v_q\|_{L^2(\mathbb{R}^n_x)}$ and

$$
\begin{aligned}
\|\partial_t(S_{q-1}a^\varepsilon_{jk}(t,x))\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)} &\le \|\partial_t(S_{q-1}a^\varepsilon_{jk}(t,x))\|_{L^\infty(\mathbb{R}^n_x)}\|\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)}\\
&\le C2^q\frac{\mu(\varepsilon)}{\varepsilon}\|v_q\|_{L^2(\mathbb{R}^n_x)}.
\end{aligned}
$$

Remark that C_1 depends only on $\max_{j,k}\|a_{j,k}\|_{C^\mu([0,T],L^\infty(\mathbb{R}^n_x))}$. For the second term in (37) we perform one integration by parts in x and the we use the Cauchy–Schwarz inequality. We get

$$2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_t v_q \mid \partial_{x_j}(S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\partial_{x_k}v_q)\rangle_{L^2(\mathbb{R}^n_x)}\,dt$$
$$= -2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_{x_j}\partial_t v_q \mid (S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\partial_{x_k}v_q\rangle_{L^2(\mathbb{R}^n_x)}\,dt,$$

and then

$$\begin{aligned}
&\left|2\,\mathrm{Re}\sum_{j,k=1}^{n}\int_0^{T/2}\langle\partial_{x_j}\partial_t v_q \mid (S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\partial_{x_k}v_q\rangle_{L^2(\mathbb{R}^n_x)}\,dt\right|\\
&\le \sum_{j,k=1}^{n}\int_0^{T/2}\|\partial_{x_k}\partial_t v_q\|_{L^2(\mathbb{R}^n_x)}\|(S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)}\,dt\\
&\le C\sum_{j,k=1}^{n}\int_0^{T/2}2^{2q}\|\partial_t v_q\|_{L^2(\mathbb{R}^n_x)}\mu(\varepsilon)\|v_q\|_{L^2(\mathbb{R}^n_x)}\,dt\\
&\le \int_0^{T/2}\|\partial_t v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt + C_2 2^{4q}\mu(\varepsilon)\int_0^{T/2}\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt,
\end{aligned}$$

where we used (see (35))

$$\begin{aligned}
&\|(S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)}\\
&\le \|(S_{q-1}(a_{jk}(t,x)-a^{\varepsilon}_{jk}(t,x))\|_{L^\infty(\mathbb{R}^n_x)}\|\partial_{x_k}v_q\|_{L^2(\mathbb{R}^n_x)}\\
&\le C2^q\mu(\varepsilon)\|v_q\|_{L^2(\mathbb{R}^n_x)}
\end{aligned}$$

and the fact that $\mu^2(\varepsilon)\le\mu(1)\mu(\varepsilon)$; remark that here the constant C_2 depends only on μ and on $\max_{j,k}\|a_{j,k}\|_{C^\mu([0,T],L^\infty(\mathbb{R}^n_x))}$.

Resuming, we have

$$\begin{aligned}
&\int_0^{T/2}\left\|\partial_t v_q+\sum_{j,k=1}^{n}\partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k}v))+\Phi'(\gamma(T-t))v_q\right\|^2_{L^2(\mathbb{R}^n_x)}\,dt\\
&\ge \int_0^{T/2}\left\|\sum_{j,k=1}^{n}\partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k}v_q)+\Phi'(\gamma(T-t))v_q\right\|^2_{L^2(\mathbb{R}^n_x)}\,dt\\
&\quad+\gamma\int_0^{T/2}\Phi''(\gamma(T-t))\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt\\
&\quad-C_3\left(\frac{\mu(\varepsilon)}{\varepsilon}2^{2q}+2^{4q}\mu(\varepsilon)\right)\int_0^{T/2}\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt,
\end{aligned}\tag{38}$$

where C_3 depends only on $\max_{j,k}\|a_{j,k}\|_{C^\mu([0,T],L^\infty(\mathbb{R}^n_x))}$.

4.5. End of the proof: high frequencies. We detail the end of the proof, starting with the high frequencies. We follow the lines of [5, 6]. By Remark 4 there exist $q_0 \geq -1$ and a constant $C_4 > 0$ such that, for all $q \geq q_0$,

$$\begin{aligned}
&\Big\| \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) \Big\|_{L^2(\mathbb{R}^n_x)} \|v_q\|_{L^2(\mathbb{R}^n_x)} \\
&\geq \Big| \Big\langle \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) \,\Big|\, v_q \Big\rangle_{L^2(\mathbb{R}^n_x)} \Big| \\
&\geq \frac{a_0}{2} \|\nabla_x v_q\|^2_{L^2(\mathbb{R}^n_x)} \geq C_4 a_0 2^{2q} \|v_q\|^2_{L^2(\mathbb{R}^n_x)},
\end{aligned}$$

where a_0 is the constant in (4).

Suppose first that $\Phi'(\gamma(T-t)) \leq (1/2)C_4 a_0 2^{2q}$. Then, from the last inequality, we deduce

$$\Big\| \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) \Big\|_{L^2(\mathbb{R}^n_x)} - \Phi'(\gamma(T-t))\|v_q\|_{L^2(\mathbb{R}^n_x)} \geq \frac{1}{2} C_4 a_0 2^{2q}.$$

We choose $\varepsilon = 2^{-2q}$ in such a way that the quantities $2^{4q}\mu(\varepsilon)$ and $2^{2q}\mu(\varepsilon)/\varepsilon$ are equal. Using the fact that $\Phi''(\gamma(T-t)) \geq 1$ (this is a consequence of the nonrestrictive hypothesis that $\mu(1) = 1$; if it is not so, the modifications of the subsequent lines are easy), we obtain that

$$\begin{aligned}
&\int_0^{T/2} \Big\| \partial_t v_q + \sum_{j,k=1}^{n} \partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k} v)) + \Phi'(\gamma(T-t))v_q \Big\|^2_{L^2(\mathbb{R}^n_x)} dt \\
&\geq \int_0^{T/2} \bigg(\Big\| \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) \Big\|_{L^2(\mathbb{R}^n_x)} - \Phi'(\gamma(T-t))\|v_q\|_{L^2(\mathbb{R}^n_x)} \bigg)^2 \\
&\quad + \gamma \int_0^{T/2} \Phi''(\gamma(T-t))\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt - 2C_3 2^{4q}\mu(2^{-2q}) \int_0^{T/2} \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt \\
&\geq \int_0^{T/2} \bigg(\Big(\frac{1}{2}C_4 a_0\Big)^2 2^{4q} + \gamma - 2C_3 2^{4q}\mu(2^{-2q}) \bigg) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt \\
&\geq \int_0^{T/2} \bigg(\Big(\frac{1}{2}\Big(\frac{1}{2}C_4 a_0)^2 - 2C_3(\mu(2^{-2q}))\Big) 2^{4q} + \frac{\gamma}{3} \bigg) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt \\
&\quad + \int_0^{T/2} \bigg(\frac{1}{2}\Big(\frac{1}{2}C_4 a_0\Big)^2 2^{4q} + \frac{2}{3}\gamma \bigg) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt.
\end{aligned}$$

Since we have $\lim_{q\to+\infty} \mu(2^{-2q}) = 0$, there exists $\gamma_0 > 0$ such that

$$\left(\frac{1}{2}\left(\frac{1}{2}C_4 a_0\right)^2 - 2C_3\right)\mu(2^{-2q})\Big)\Big) 2^{4q} + \frac{\gamma}{3} \geq 0$$

for $\gamma \geq \gamma_0$ and all $q \geq q_0$. Consequently, for $\gamma \geq \gamma_0$,

$$\int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k} v)) + \Phi'(\gamma(T-t))v_q \right\|^2_{L^2(\mathbb{R}^n_x)} dt$$
$$\geq \int_0^{T/2} \left(\frac{1}{2}\left(\frac{1}{2}C_4 a_0\right)^2 2^{4q} + \frac{2}{3}\gamma\right) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt.$$

Recall now (11). Using it with $s = 1/2$, we have that there exists $C_0 > 0$ such that, for all $q \geq -1$, we have $\mu(2^{-2q}) \leq C_0 2^{-q}$. Then, for all $q \geq -1$ and for all $\gamma \geq \gamma_0$,

$$\frac{1}{2}\left(\frac{1}{2}C_4 a_0\right)^2 2^{4q} + \frac{1}{6}\gamma \geq C_5 \gamma^{\frac{1}{4}} 2^{3q} \geq C_6 \gamma^{\frac{1}{4}} 2^{4q} \mu(2^{-2q}),$$

for some $C_5, C_6 > 0$. Finally

$$\int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^n \partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k} v)) + \Phi'(\gamma(T-t))v_q \right\|^2_{L^2(\mathbb{R}^n_x)} dt$$
$$\geq \int_0^{T/2} \left(\frac{\gamma}{2} + C_6 \gamma^{1/4} 2^{4q} \mu(2^{-2q})\right) \|v_q\|^2_{L^2(\mathbb{R}^n_x)}\, dt. \tag{39}$$

Suppose now $\Phi'(\gamma(T-t)) \geq (1/2)C_4 a_0 2^{2q}$. Again we choose $\varepsilon = 2^{-2q}$. Then, using (31), the fact that $a_0 \leq 1$ and the properties of μ, we get

$$\begin{aligned}\Phi''(\gamma(T-t)) &= (\Phi'(\gamma(T-t)))^2 \mu\left(\frac{1}{\Phi'(\gamma(T-t))}\right)\\ &\geq \left(\frac{1}{2}C_4 a_0\right)^2 2^{4q} \mu\left(\frac{2}{C_4 a_0} 2^{-2q}\right)\\ &\geq \left(\frac{1}{2}C_4 a_0\right)^2 2^{4q} \mu(2^{-2q}).\end{aligned}$$

Hence there exist γ_0 and constants $C_7, C_8 > 0$ such that, for $\gamma \geq \gamma_0$,

$$
\begin{aligned}
&\int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^{n} \partial_{x_j}(\Delta_q(a_{jk}(t,x)\partial_{x_k} v)) + \Phi'(\gamma(T-t))v_q \right\|_{L^2(\mathbb{R}^n_x)}^2 dt \\
&\geq \int_0^{T/2} \left(\left\| \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) \right\|_{L^2(\mathbb{R}^n_x)} - \Phi'(\gamma(T-t))\|v_q\|_{L^2(\mathbb{R}^n_x)} \right)^2 \\
&\quad + \gamma \int_0^{T/2} \Phi''(\gamma(T-t))\|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt - 2C_3 2^{4q}\mu(2^{-2q}) \int_0^{T/2} \|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt \\
&\geq \int_0^{T/2} \left(\frac{\gamma}{2} + \left(\frac{\gamma}{2}(\frac{1}{2}C_4 a_0)^2 - 2C_3 \right) 2^{4q}\mu(2^{-2q}) \right) \|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt \\
&\geq \int_0^{T/2} \left(\frac{\gamma}{2} + C_7 \gamma 2^{4q}\mu(2^{-2q}) \right) \|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt \\
&\geq \int_0^{T/2} \left(\frac{\gamma}{2} + C_8 \gamma^{1/4} 2^{4q}\mu(2^{-2q}) \right) \|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt.
\end{aligned} \tag{40}
$$

Recall now that $2^{2q}\mu(2^{-2q}) = 2^{2q}\omega^2(2^{-q}) = \Omega^2(q)$. From (39) and (40) we immediately obtain

$$
\begin{aligned}
&\sum_{q \geq q_0} 2^{-2sq} \int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) + \Phi'(\gamma(T-t))v_q \right\|_{L^2(\mathbb{R}^n_x)}^2 dt \\
&\geq \sum_{q \geq q_0} 2^{-2sq} \int_0^{T/2} \left(\frac{\gamma}{2} + C\gamma^{1/4}\Omega^2(q) 2^{2q} \right) \|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt.
\end{aligned} \tag{41}
$$

4.6. End of the proof: low frequencies. In this section we complete the proof for low frequencies. We sum (38) multiplied with 2^{-2qs} for $q \leq q_0 - 1$ (q_0 is the same as in the previous section). We set $\varepsilon = 2^{-2q_0}$ and we obtain

$$
\begin{aligned}
&\sum_{q \leq q_0 - 1} 2^{-2sq} \int_0^{T/2} \left\| \partial_t v_q + \sum_{j,k=1}^{n} \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k} v_q) + \Phi'(\gamma(T-t))v_q \right\|_{L^2(\mathbb{R}^n_x)}^2 dt \\
&\geq \sum_{q \leq q_0 - 1} 2^{-2sq} \int_0^{T/2} (\gamma - 2C_3\mu(2^{-2q_0})2^{2(q+q_0)})\|v_q\|_{L^2(\mathbb{R}^n_x)}^2 \, dt.
\end{aligned}
$$

Taking γ_0 large enough we can absorb the negative term. We easily obtain

$$\begin{aligned}(42)\quad &\sum_{q\le q_0-1} 2^{-2sq}\int_0^{T/2}\Big\|\partial_t v_q+\sum_{j,k=1}^n \partial_{x_j}(S_{q-1}a_{jk}(t,x)\partial_{x_k}v_q)+\Phi'(\gamma(T-t))v_q\Big\|^2_{L^2(\mathbb{R}^n_x)}\,dt\\ &\ge \sum_{q\le q_0-1} 2^{-2sq}\int_0^{T/2}\Big(\frac{\gamma}{2}+C\gamma^{1/4}\Omega^2(q)2^{2q}\Big)\|v_q\|^2_{L^2(\mathbb{R}^n_x)}\,dt.\end{aligned}$$

Summing (41) and (42) we obtain (33). The proof is completed.

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