CANONICAL CONSTRUCTION OF POLYTOPE BARABANOV NORMS AND ANTINORMS FOR SETS OF MATRICES

N. GUGLIELMI* AND M. ZENNARO†

Abstract. Barabanov norms have been introduced in Barabanov (1988) and constitute an important instrument to analyze the joint spectral radius of a family of matrices and related issues. However, although they have been studied extensively, even in very simple cases it is very difficult to construct them explicitly (see, e.g., Kozyakin (2010)). In this paper we give a canonical procedure to construct them exactly, which associates a polytope extremal norm - constructed by using the methodologies described in Guglielmi, Wirth and Zennaro (2005) and Guglielmi and Protasov (2013) - to a polytope Barabanov norm. Hence, the existence of a polytope Barabanov norm has the same genericity of an extremal polytope norm. Moreover, we extend the result to polytope antinorms, which have been recently introduced to compute the lower spectral radius of a finite family of matrices having an invariant cone.

Key words. joint spectral radius, lower spectral radius, invariant norm, polytope extremal norm, polytope extremal antinorm, polytope Barabanov norm, polytope Barabanov antinorm, unit antiball

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1. Introduction.

When considering the stability under arbitrary switching of a discrete-time linear switched system

$$x(k+1) = A_{\sigma(k)}x(k), \qquad \sigma : \mathbb{N} \to \{1, \ldots, m\}, \quad A_1, \ldots, A_m \quad \text{given matrices,}$$

one is mainly interested in determining the most unstable switching law (MUSL). This is equivalent to computing the so-called joint spectral radius of the underlying set of matrices (see e.g. the recent survey monography by Jungers [J]). If the solution of the switched system corresponding to the MUSL converges to zero, then the switched system is stable for any switching law.

It is well-known that the MUSL can be characterized using optimal control techniques. As mentioned in Teichner and Margaliot [TM], such variational approach leads to a Hamilton–Jacobi–Bellman equation describing the behavior of the switched system under the MUSL. The solution of this equation is sometimes referred to as a Barabanov norm of the switched system. Quoting [TM]: "Although the Barabanov norm was studied extensively, it seems that there are few examples where it was actually computed in closed form".

Barabanov norms are widely studied as they provide a very important tool in the analysis of the joint spectral radius of a set of matrices (for example they played a key role in the disprovement of the well-known finiteness conjecture), as well as in the analysis of the behaviour of the solutions of switched systems. Their duality to extremal norms has been studied, e.g., by Plischke, Wirth and Barabanov [PW, PWB] in the analysis of semigroups generated by linear inclusions.

It appears that in the literature an explicit computation of a Barabanov norm has been provided only in a few cases (see e.g. [M1, TM]).

This is due to the fact that they are defined only implictly and there do not exist procedures that are able to construct them in an exact way. Nevertheless iterative

*Dipartimento Matematica, Università de L'Aquila, I-67100 L'Aquila (Italy), *e-mail:* guglielm@univaq.it

†Dipartimento di Matematica e Geoscienze, Università di Trieste, I-34100 Trieste (Italy), *e-mail:* zennaro@units.it

methods able to approximate Barabanov norms have been recently introduced by Kozyakin [K1, K2], based on a so-called max-relaxation procedure. Such iterative methods allow us to obtain a sequence of norms converging to a Barabanov norm, whose unit ball is not a polytope in general.

1.1. Contribution of the paper. In this paper we provide a methodology, which is supplementary to the one presented by Guglielmi, Wirth and Zennaro [GWZ] and by Guglielmi and Protasov [GP], for the computation of polytope extremal norms of sets of matrices (and hence of switched systems), which allows us to construct a polytope Barabanov norm in a canonical way.

Similarly we consider a dual framework and study the problem of determining a most stable switching law (MSSL) and the related problem of stabilizability of a switched system. Also in this case, but restricted to nonnegative systems, we are able to provide a canonical way, which is dual to the one presented by Guglielmi and Protasov [GP], to explicitly construct a so-called Barabanov antinorm, which plays an analogous role to the Barabanov norm.

2. Joint spectral radius. We let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ a finite family of matrices with $A_i \in \mathbb{C}^{d,d}$ (or $\mathbb{R}^{d,d}$) for $i = 1, \ldots, m$.

Moreover let $\|\cdot\|$ be a norm on $\mathbb{C}^d$ ($\mathbb{R}^d$) and $\|A\| = \max_{\|x\|=1} \|Ax\|$ the corresponding induced matrix norm.

Let $\mathcal{I} = \{1, \ldots, m\}$. Then, for $k = 1, 2, \ldots$, consider the set of all products of *length* (or *degree*) k

$$\Sigma_k(\mathcal{F}) = \{A_{i_k} \cdots A_{i_1} \ \mid \ i_1, \ldots, i_k \in \mathcal{I}\} \tag{2.1}$$

and the number

$$\hat{\rho}_k(\mathcal{F}) = \max_{P \in \Sigma_k(\mathcal{F})} \|P\|^{1/k}. \tag{2.2}$$

Definition 2.1 (joint spectral radius, Rota and Strang [RS] and Strang [S]). *The number*

$$\hat{\rho}(\mathcal{F}) = \limsup_{k \to \infty} \hat{\rho}_k(\mathcal{F}) \tag{2.3}$$

is said to be the joint spectral radius (j.s.r.) of the family $\mathcal{F}$. Remark that the limit in (2.3) always exists and does not depend on the considered norm.

Analogously, let $\rho(\cdot)$ denote the spectral radius of a $d \times d$-matrix and then, for each $k = 1, 2, \ldots$, consider the number

$$\bar{\rho}_k(\mathcal{F}) = \sup_{P \in \Sigma_k(\mathcal{F})} \rho(P)^{1/k}. \tag{2.4}$$

Definition 2.2 (generalized spectral radius, Daubechies and Lagarias [DL]). *The number*

$$\bar{\rho}(\mathcal{F}) = \limsup_{k \to \infty} \bar{\rho}_k(\mathcal{F}) \tag{2.5}$$

is said to be the generalized spectral radius (g.s.r.) of the family $\mathcal{F}$.

Daubechies and Lagarias [DL] also proved that

$$\bar{\rho}_k(\mathcal{F}) \le \bar{\rho}(\mathcal{F}) \le \hat{\rho}(\mathcal{F}) \le \hat{\rho}_k(\mathcal{F}) \quad \text{for all } k \ge 1. \tag{2.6}$$

The fundamental equality

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$$

has been proved later by Berger and Wang [BW]. Consequently we can simply denote the *spectral radius* of $\mathcal{F}$ by $\rho(\mathcal{F})$.

An important characterization of the spectral radius $\rho(\mathcal{F})$ of a matrix family is the generalization of Gelfand's formula. In order to state this characterization, we define the *norm of the family* $\mathcal{F}$ as

$$\|\mathcal{F}\| = \hat{\rho}_1(\mathcal{F}) = \max_{i \in \mathcal{I}} \|A_i\|.$$

PROPOSITION 2.3 (Rota and Strang [RS], Elsner [E]). *The spectral radius of a bounded family $\mathcal{F}$ of $d \times d$-matrices is characterized by*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathrm{Op}} \|\mathcal{F}\|, \tag{2.7}$$

where Op *denotes the set of operator norms.*

In connection with the possibility for the infimum in (2.7) to be a minimum or not, we recall the following definitions.

DEFINITION 2.4 (Extremal norm). *We say that a norm $\|\cdot\|$ satisfying*

$$\|\mathcal{F}\| = \rho(\mathcal{F})$$

is extremal for the family $\mathcal{F}$.

DEFINITION 2.5 (Barabanov norm). *We say that an extremal norm $\|\cdot\|$ for the family $\mathcal{F}$ is a Barabanov norm (or equivalently an invariant norm) if*

$$\max_{i \in \mathcal{I}} \|A_i x\| \; = \; \rho(\mathcal{F})\|x\| \quad \forall x \in \mathbb{C}^d(\mathbb{R}^d). \tag{2.8}$$

Before stating the well-known result by Barabanov we need the following definition.

DEFINITION 2.6. *A family $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ of $d \times d$-matrices is said to be* reducible *if there exist a nonsingular $d \times d$-matrix M and two integers $d_1, d_2 \geq 1$, $d_1 + d_2 = d$, such that, for all $i \in \mathcal{I}$, it holds that*

$$M^{-1} A_i M = \begin{bmatrix} A_i^{(11)} & A_i^{(12)} \\ O & A_i^{(22)} \end{bmatrix}, \tag{2.9}$$

where the blocks $A_i^{(11)}$, $A_i^{(12)}$, $A_i^{(22)}$ are $d_1 \times d_1$-, $d_1 \times d_2$- and $d_2 \times d_2$-matrices, respectively. If a family $\mathcal{F}$ is not reducible, then it is said to be irreducible.

The following theorem establishes an existence result for a Barabanov norm.

THEOREM 2.7 (Barabanov [B]). *Assume that a family of matrices $\mathcal{F}$ is irreducible. Then there exists an operator norm $\|\cdot\|$ such that* (2.8) *holds true, i.e. $\|\cdot\|$ is a Barabanov norm.*

Note that extremal norms as well as Barabanov norms are not unique. For uniqueness results we refer the reader to the recent publications by Morris [M1, M2], where a sufficient condition for the uniqueness is presented. However, the property of having a unique Barabanov norm can be very sensitive to small perturbations to the family $\mathcal{F}$.

The following definition is necessary to the subsequent discussion.

Definition 2.8. *We say that a set $\mathcal{Y} \subset \mathbb{C}^d$ is* absolutely convex *if, for all $y_1, y_2 \in \mathcal{Y}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $|\lambda_1| + |\lambda_2| \leq 1$, it holds that $\lambda_1 y_1 + \lambda_2 y_2 \in \mathcal{Y}$. Let $\mathcal{S} \subset \mathbb{C}^d$. Then the intersection of all absolutely convex sets containing $\mathcal{S}$ is called the* absolutely convex hull *of $\mathcal{S}$ and is denoted by* $\mathrm{absco}(\mathcal{S})$.

We conclude this section by defining a so-called Protasov norm.

Definition 2.9. *An extremal norm for a family $\mathcal{F}$ is said to be a Protasov norm if its unit ball $\mathcal{B}$ is such that*

$$\rho(\mathcal{F})\mathcal{B} = \mathrm{absco}\left(\bigcup_{1 \leq i \leq m} A_i \mathcal{B} \right), \tag{2.10}$$

where $A_i\mathcal{B}$ denotes the set $\{z = A_i x \mid x \in \mathcal{B}\}$.

The previous definition, restricted to real norms and matrices, has been given by Protasov [P1] and Plische and Wirth [PW].

Remark 2.1. *For a generic extremal norm the equality sign "$=$" in* (2.10) *is replaced by the inclusion sign "$\supseteq$".*

For a real irreducible family $\mathcal{F}$, a Protasov norm always exists (see [P1]). Moreover, it turns out that a Protasov norm is dual to a Barabanov norm (see [PW]).

These results can be extended to families of complex matrices in a strightforward way.

3. Complex polytope and adjoint complex polytope norms. A class of norms which is particularly interesting for us, due to the possibility of a finite computation and representation, is the one of polytope norms. The forthcoming definition extends the usual definition of symmetric polytope in the real space $\mathbb{R}^d$.

As commonly done, the notation "$\subset$" will mean "strict inclusion".

Definition 3.1. *We say that a bounded set $\mathcal{P} \subset \mathbb{C}^d$ is a* balanced complex polytope *(in short b.c.p.) if there exists a finite set $\mathcal{X} = \{x_i\}_{1 \leq i \leq m}$ of vectors such that* $\mathrm{span}(\mathcal{X}) = \mathbb{C}^d$ *and*

$$\mathcal{P} = \mathrm{absco}(\mathcal{X}). \tag{3.1}$$

Moreover, if $\mathrm{absco}(\mathcal{X}') \subset \mathrm{absco}(\mathcal{X})$ *for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ is called an* essential system of vertices *for $\mathcal{P}$, whereas any vector $u x_i$ with $u \in \mathbb{C}$, $|u| = 1$, is called a* vertex *of $\mathcal{P}$.*

Definition 3.2. *We say that a bounded set $\mathcal{P}^* \subset \mathbb{C}^d$ is a balanced complex polytope of* adjoint *type (in short a.b.c.p.) if there exists a finite set $\mathcal{X} = \{x_i\}_{1 \leq i \leq m}$ of vectors such that* $\mathrm{span}(\mathcal{X}) = \mathbb{C}^d$ *and*

$$\mathcal{P}^* = \mathrm{adj}(\mathcal{X}) = \Big\{ y \in \mathbb{C}^d \;\Big|\; |\langle y, x_i \rangle| \leq 1,\ i = 1, \ldots, m \Big\}. \tag{3.2}$$

Moreover, if $\mathrm{adj}(\mathcal{X}') \supset \mathrm{adj}(\mathcal{X})$ *for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ is called an* essential system of facets *for $\mathcal{P}^*$, whereas any vector $u x_i$ with $u \in \mathbb{C}$, $|u| = 1$, is called a* facet *of $\mathcal{P}^*$.*

According to [GZ1], any b.c.p. $\mathcal{P}$ is the unit ball of a (complex polytope) norm $\|\cdot\|_{\mathcal{P}}$ on $\mathbb{C}^d$ and any a.b.c.p. $\mathcal{P}^*$ is the unit ball of a (adjoint complex polytope) norm $\|\cdot\|_{\mathcal{P}^*}$.

The following result establishes a useful relationship between complex polytope and adjoint complex polytope norms.

Lemma 3.3. *Let $\mathcal{P}$ be a balanced complex polytope and let $\|\cdot\|_{\mathcal{P}}$ the corresponding complex polytope norm. Moreover, let $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ and let $\|\cdot\|_{\mathcal{P}^*}$ the corresponding adjoint complex polytope norm. Then, for any complex $d \times d$-matrix A and its adjoint A^*, it holds that*

$$\|A\|_{\mathcal{P}^*} = \|A^*\|_{\mathcal{P}}. \tag{3.3}$$

As a consequence $\|\mathcal{F}\|_{\mathcal{P}^*} = \|\mathcal{F}^*\|_{\mathcal{P}}$.

In $\mathbb{R}^d$ the geometry of symmetric polytopes and adjoint symmetric polytopes is the same, but this fact is not inherited by balanced complex polytopes in $\mathbb{C}^d$. For a detailed discussion we refer the reader to [GZ1].

3.1. An algorithm to compute a polytope extremal norm. In [GWZ] and [GP] (under the name of Algorithm (**C**)) two slightly different algorithms to compute a complex polytope extremal norm (when it exists) for a finite family $\mathcal{F}$ are presented.

The method presented in [GP], which slightly improves the one given in [GWZ], is summarized by Algorithm 1.

In order to illustrate Algorithm 1 we need to introduce the following definitions.

Definition 3.4 (Spectrum maximizing product). *If $\mathcal{F}$ is a bounded family of $d \times d$-matrices, any matrix $P \in \Sigma_k(\mathcal{F})$ satisfying $\rho(P)^{1/k} = \rho(\mathcal{F})$ for some $k \geq 1$ is called a spectrum maximizing product (s.m.p.) for $\mathcal{F}$.*

We remark that, as is well-known, s.m.p.'s are not always guaranteed to exist (see, e.g., [BTV]).

Definition 3.5 (Dominant s.m.p.). *An s.m.p. P of a bounded family of $d \times d$-matrices $\mathcal{F}$ is said to be dominant if there exists a constant $q < 1$ such that, for any matrix $Q \in \Sigma(\mathcal{F})$ - other than P and the powers of it and of its cyclic permutations - it holds that $\rho(Q)^{1/k} \leq q \cdot \rho(\mathcal{F})$, where k is the degree of Q (i.e., $Q \in \Sigma_k(\mathcal{F})$).*

Algorithm 1: Basic polytope algorithm for finding extremal norms

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Data: $\mathcal{F}$
Result: $\mathcal{P}$
begin
- Preprocessing: find a product P of length $k \geq 1$ such that $\rho(P)^{1/k}$ is maximal among $\Sigma_k(\mathcal{F})$ (P is a candidate s.m.p.);
- Set $R := \rho(P)^{1/k}$ and $\widehat{\mathcal{F}} := R^{-1}\mathcal{F}$;
- Compute $v_1, \ldots, v_k$ leading eigenvectors of P and its cyclic permutations;
- Set $V_0 := \{v_1, \ldots, v_k\}$;
- Set $i = 0$;
- **while** $\mathrm{span}(V_i) \neq \mathbb{C}^d$ **do**
  - $V_{i+1} := V_i \cup \widehat{\mathcal{F}}\, V_i$;
  - Set $i = i + 1$;
- Set $\mathcal{P}^{(i)} = \mathrm{absco}(V_i)$;
- **while** $\widehat{\mathcal{F}}\, V_i \not\subseteq \mathcal{P}^{(i)}$ **do**
  - Set $i = i + 1$;
  - Determine an essential system of vertices V_i of $\mathrm{absco}(V_{i-1} \cup \widehat{\mathcal{F}}\, V_{i-1})$;
  - Set $\mathcal{P}^{(i)} = \mathrm{absco}(V_i)$;
- Return $\mathcal{P} := \mathcal{P}^{(i)}$ (polytope extremal unit ball);

The main idea behind Algorithm 1 is that of constructing iteratively a complex polytope which is finally mapped into itself by the family $\widehat{\mathcal{F}} := \mathcal{F}/\rho(\mathcal{F})$. If the procedure concludes successfully, it proves that the chosen product P is indeed an s.m.p. and, at the same time, it provides a polytope extremal norm.

The main difference between Algorithm 1 and the one presented in [GWZ] is that, in presence of a unique s.m.p. P, the former considers as an initial set of vectors the leading eigenvector of P along with those of all its permutations, whereas the latter starts from the sole leading eigenvector of P.

Algorithm 1 includes stopping criteria to detect whether the given product P is actually an s.m.p. or not. The two algorithms which we refer to have indeed different stopping criteria. A more sophisticated one is implemented in the algorithm presented by Guglielmi and Protasov and is based on [GP, Proposition 2]. The algorithm halts when the polytope $\mathcal{P}$ is mapped into itself by the family $\widehat{\mathcal{F}}$.

The most time-consuming part of Algorithm 1 is usually the preprocessing phase, whose cost is exponential in k. The use of suitable criteria to exclude certain products (see, e.g., [G] and [GZ2, Section 6]) may help to reduce this cost, but the optimization of this part remains an interesting research topic.

A sufficient condition for the successful termination of Algorithm 1 within finitely many iterations is that the starting product P be a dominant s.m.p. of the family $\mathcal{F}$ with a unique and simple leading eigenvector (see [GP, Theorem 4]).

3.2. Monotone norms on cones. We recall that a subset $K \subset \mathbb{R}^d$ is said to be a cone if and only if $x, y \in K$, $\alpha, \beta \geq 0$ implies that $\alpha x + \beta y \in K$. A cone K is said to be pointed if and only if $K \cap -K = \{0\}$.

In this paper, by *cone* we mean a convex, closed, pointed cone with nonempty interior. Any such cone defines a partial order in $\mathbb{R}^d$: we write $x \succeq_K y$ ($x \succ_K y$) for $x - y \in K$ ($x - y \in \text{int } K$) [BV, Section 2.4.1]. The cone K is an invariant cone for a matrix A if $A\,K \subseteq K$. If K is invariant for all matrices of some set $\mathcal{F}$, then it is said to be an invariant cone for that set.

In this subsection we consider families of matrices $\mathcal{F}$ that share a common invariant *cone* $K \subset \mathbb{R}^d$ like, for example, families of *nonnegative matrices*, in which case $K = \mathbb{R}^d_+ = \big\{x \in \mathbb{R}^d \;\big|\; x_i \geq 0,\ i = 1, \dots, d\big\}$.

For a discussion of the properties of matrices with an invariant cone we refer the reader to Vandergraft [V].

Here our interest is the construction of an *extremal monotone norm.* Recall that a function $g(\cdot)$ is *monotone* with respect to a cone K if $g(x) \geq g(y)$ whenever $x - y \in K$.

If $\|\cdot\|$ is a monotone norm defined on a cone K, then it is extended onto $\mathbb{R}^d$ in a standard way: the unit ball of the extended norm is given by

$$\mathrm{co}_s\left(\big\{\, x \in K \;\big|\; \|x\| \leq 1 \,\big\}\right), \tag{3.4}$$

where, given a set X, $\mathrm{co}(X)$ denotes the *convex hull* of X and

$$\mathrm{co}_s\,(X) = \mathrm{co}\big(X \cup -X\big)$$

is the *symmetric convex hull* of X.

All extreme points of the ball defined by (3.4) are in the cones K and $-K$. Consider a norm which is an extension to $\mathbb{R}^d$ of a monotone norm in K, via (3.4). Since for any matrix A, such norm is attained at an extreme point of the unit ball, we see that, if A leaves K invariant, then it attains its norm in the cone K. Thus,

$$\|A\| = \max_{x \in K \,:\, \|x\| \leq 1} \|Ax\|.$$

In particular, if $\|\cdot\|$ is an extremal norm for a family $\mathcal{F}$, then its extension defined by (3.4) is extremal as well. Thus, for families with a common invariant cone it suffices to construct an extremal monotone norm $\|\cdot\|$ on that cone.

Recall that a face F of a cone K is invariant for a matrix A if $Ax \in F$ for all $x \in F$. The following result is proved in [GP].

PROPOSITION 3.6. *If a finite family of matrices $\mathcal{F} = \{A_i\}_{i=1}^m$ has a common invariant cone K and does not have any common invariant face of that cone, then $\mathcal{F}$ has an extremal monotone norm $\|\cdot\|$ on K.*

If $K = \mathbb{R}^d_+$, then the faces of K are included in coordinate planes, i.e., they are sets of the type $F_i = \{x \in \mathbb{R}^d_+ \mid x_i = 0\}$, $i = 1, \ldots, d$.

DEFINITION 3.7. *A family of nonnegative matrices $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ ($\mathcal{I}$ set of indeces) is called positively irreducible if it does not have any common invariant face of the cone $K = \mathbb{R}^d_+$.*

By applying Proposition 3.6 to the case $K = \mathbb{R}^d_+$, we obtain the following result.

COROLLARY 3.8. *A finite positively irreducible family $\mathcal{F} = \{A_i\}_{i=1}^m$ of nonnegative matrices has an extremal monotone norm $\|\cdot\|$ on $\mathbb{R}^d_+$.*

In [GP] a polytope monotone norm was defined, which has been used in the construction of an extremal monotone norm for a nonnegative family of matrices.

To this aim, we consider the cone $K = \mathbb{R}^d_+$ and, for a given set $Q \subset K$, we define

$$\mathrm{co}_-(Q) \;=\; \big(\mathrm{co}(Q) - K\big) \cap K \;=\; \big\{ x \in K \;\big|\; x = y - t\,,\; y \in \mathrm{co}\,(Q)\,,\; t \in K \big\}\,.$$

Then, given a set of vertices $V = \{v_j\}_{j=1}^n$, $v_j \in K$, $j = 1, \ldots, n$, we define the *positive polytope*

$$\mathcal{P} = \mathrm{co}_-(V)$$

to which a monotone norm is associated.

In fact, it turns out that, if $x \in \mathcal{P}$ and $y \le x$, then $y \in \mathcal{P}$ too.

A variant of Algorithm 1, able to compute, under suitable assumptions, a polytope monotone norm, is given in [GP]. It is still based on the guess of a candidate s.m.p. and, as an initial vector, its Perron-Frobenius (leading) eigenvector.

4. Barabanov (complex) polytope norms. Without loss of generality, let $\rho(\mathcal{F}) = 1$. Moreover, assume that $\mathcal{F}$ admits a complex polytope extremal norm, whose unit ball is given by a balanced complex polytope $\mathcal{P}$.

More precisely, assume that $\mathcal{P}$ has been constructed by using Algorithm 1, which means that every vertex v_p, $p = 1, \ldots, n$, of the polytope $\mathcal{P}$ has been generated in such a way that

$$v_p = A_{i_p} v_{j_p} \qquad \text{for some } \; j_p \in \{1, \ldots, n\} \text{ and } i_p \in \{1, \ldots, m\}. \tag{4.1}$$

Then let $\|\cdot\|_{\mathcal{P}}$ be the extremal norm for $\mathcal{F}$ determined by $\mathcal{P}$ and $\|\cdot\|_{\mathcal{P}^*}$ be its dual norm, that is the norm determined by $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$, which is extremal for $\mathcal{F}^*$ as well (see Guglielmi and Zennaro [GZ1]).

We have the following main result.

THEOREM 4.1. *Let $\mathcal{P}$ be a balanced complex polytope defining an extremal norm $\|\cdot\|_{\mathcal{P}}$ for $\mathcal{F}$ and assume that* (4.1) *holds. Then the norm $\|\cdot\|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}^*$.*

Proof. Since $\rho(\mathcal{F}) = 1$, by extremality we have that $\|A\|_{\mathcal{P}} \le 1$ for all $A \in \mathcal{F}$.

Now let $x \in \partial \mathcal{P}^*$. Then, by definition of dual norm,

$$1 = \|x\|_{\mathcal{P}^*} = \max_{1 \le j \le n} |\langle x, v_j \rangle| = |\langle x, v_\ell \rangle|$$

for some $\ell \in \{1, \ldots, n\}$. Now, by (4.1) there exist $v_q \in V$ and $A_p \in \mathcal{F}$ such that

$$\begin{aligned} |\langle x, v_\ell \rangle| &= |\langle x, A_p v_q \rangle| = |\langle A_p^* x, v_q \rangle| \le \max_{1 \le j \le n} |\langle A_p^* x, v_j \rangle| = \|A_p^* x\|_{\mathcal{P}^*} \\ &\le \|A_p^*\|_{\mathcal{P}^*} \|x\|_{\mathcal{P}^*} = \|A_p^*\|_{\mathcal{P}^*} = \|A_p\|_{\mathcal{P}} \le 1 \end{aligned}$$

(see [GZ1, Corollary 5.6]). Therefore, we get

$$\|A_p^* x\|_{\mathcal{P}^*} = 1,$$

which means that $\| \cdot \|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}^*$. $\square$

As a consequence, in order to construct a polytope Barabanov norm for $\mathcal{F}$ it is sufficient to apply the procedure described in [GWZ] to $\mathcal{F}^*$ and then consider the corresponding dual norm which, by Theorem 4.1, is a Barabanov norm for $\mathcal{F}$.

Remark 4.1. *If $\mathcal{F} = \mathcal{F}^*$ and $\| \cdot \|_{\mathcal{P}}$ is an extremal norm for $\mathcal{F}$ for which* (4.1) *holds, then the norm $\| \cdot \|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}$.*

We also have the following result, which is the converse of Theorem 4.1.

Theorem 4.2. *Let $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$ be an adjoint balanced complex polytope defining a Barabanov norm $\| \cdot \|_{\mathcal{P}^*}$ for $\mathcal{F}^*$. Then the dual norm $\| \cdot \|_{\mathcal{P}}$ is an extremal norm for $\mathcal{F}$ which satisfies* (4.1).

Proof. Let $v_\ell \in V$ be a vertex of $\mathcal{P}$. By [GZ1, Proposition 2.12], there exists $\widehat{x} \in \partial \mathcal{P}^*$ such that

$$1 = \|\widehat{x}\|_{\mathcal{P}^*} = |\langle \widehat{x}, v_\ell \rangle| \quad \text{and} \quad |\langle \widehat{x}, v_j \rangle| < 1 \qquad \forall j \ne \ell. \tag{4.2}$$

Therefore, since $\| \cdot \|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}^*$, there exists $A_p^* \in \mathcal{F}^*$ such that

$$\begin{aligned} 1 &= |\langle \widehat{x}, v_\ell \rangle| = \|A_p^* \widehat{x}\|_{\mathcal{P}^*} = \max_{1 \le j \le n} |\langle A_p^* \widehat{x}, v_j \rangle| \\ &= \max_{1 \le j \le n} |\langle \widehat{x}, A_p v_j \rangle| = |\langle \widehat{x}, A_p v_r \rangle| \qquad \text{for some } v_r \in V. \end{aligned}$$

Since $\mathcal{P} = \mathcal{P}^*$, $\mathcal{F}^{**} = \mathcal{F}$ and $\| \cdot \|_{\mathcal{P}^*}$ is extremal for $\mathcal{F}^*$, the norm $\| \cdot \|_{\mathcal{P}}$ is extremal for $\mathcal{F}$ (see again [GZ1]). Thus $A_p v_r \in \mathcal{P}$ and consequently, by (4.2) it follows that

$$v_\ell = A_p v_r,$$

which concludes the proof. $\square$

Remark 4.2. *It is immediate to realize that property* (4.1) *implies that $\| \cdot \|_{\mathcal{P}}$ is a Protasov norm, i.e., its unit ball $\mathcal{P}$ satisfies* (2.10).

Therefore, the foregoing Theorems 4.1 *and* 4.2 *might be obtained just as corollaries of the duality results proved in* [PW].

However, we preferred to prove them in an autonomous way since, in the next sections, we will extend the theory to the case of polytope monotone norms and polytope antinorms defined on cones.

Example 4.1. Consider the following example by Teichner and Margaliot [TM].

Let $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 14 & 6 \\ 1 & 13 \end{pmatrix}.$$

We consider the adjoint family $\mathcal{F}^* = \{A_1^{\mathrm{T}}, A_2^{\mathrm{T}}\}$ and apply the procedure described in [GWZ]. A real polytope extremal norm for $\mathcal{F}^*$ is $\mathcal{P} = \mathrm{co}\,(v_1, v_2, -v_1, -v_2)$, where

$$v_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \frac{5}{32} \\ \frac{1}{2} \end{pmatrix}$$

are the computed vectors.

The corresponding dual norm (the Barabanov one) is given by

$$\mathcal{P}^* = \left\{ \begin{array}{rcl} \frac{1}{2}x_1 + x_2 & \leq & 1 \\ \frac{1}{2}x_1 + x_2 & \geq & -1 \\ \frac{5}{32}x_1 + \frac{1}{2}x_2 & \leq & 1 \\ \frac{5}{32}x_1 + \frac{1}{2}x_2 & \geq & -1 \end{array} \right. = \mathrm{co}\left(\begin{pmatrix} -\frac{16}{3} \\ \frac{11}{3} \end{pmatrix}, \begin{pmatrix} \frac{16}{3} \\ -\frac{11}{3} \end{pmatrix}, \begin{pmatrix} -16 \\ 7 \end{pmatrix}, \begin{pmatrix} 16 \\ -7 \end{pmatrix} \right)$$

and is plotted in Figure 4.1. $\diamond$

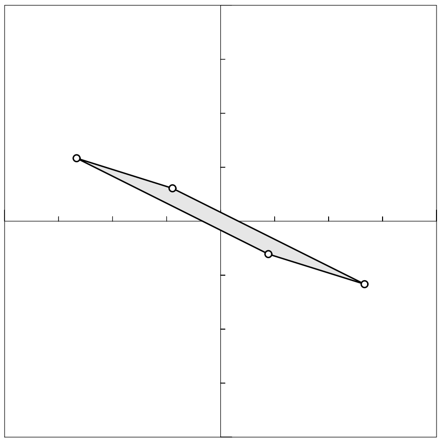

FIG. 4.1. *The polytope Barabanov norm for Example* 4.1.

EXAMPLE 4.2. We consider the following example by Cicone et al. [CGSZ]. Let $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we consider the adjoint family $\mathcal{F}^*$. It can be proved that $P = A_1^{\mathrm{T}}\, A_2^{\mathrm{T}}\, (A_1^{\mathrm{T}})^2\, A_2^{\mathrm{T}}$ is an s.m.p., $\rho(\mathcal{F}^*) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{5}}{2}\right)^{1/5}$ and a real polytope extremal norm is given by $\mathcal{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_1 is the leading eigenvector of P,

$$v_2 = B_2^{\mathrm{T}} v_1,\ v_3 = B_1^{\mathrm{T}} v_2,\ v_4 = B_1^{\mathrm{T}} v_3,\ v_5 = B_2^{\mathrm{T}} v_3,\ v_6 = B_2^{\mathrm{T}} v_4,$$

with $B_1 = A_1/R, B_2 = A_2/R$ and $R = \rho(P)^{1/5} \approx 1.212258$.

The polytope extremal norm for $\mathcal{F}^*$ (left) and the polytope Barabanov norm for $\mathcal{F}$ (right) are shown in Figure 4.2.

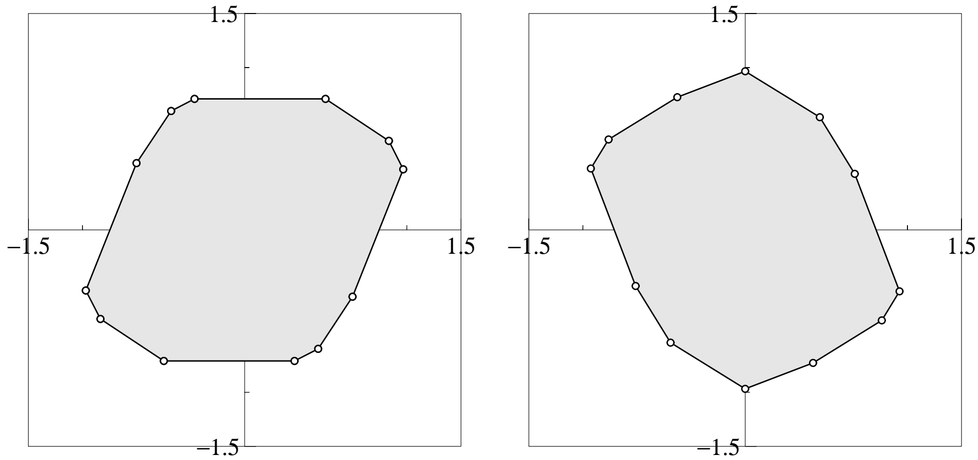

FIG. 4.2. *Polytope extremal norm (left) for the family $\mathcal{F}^*$ and polytope Barabanov norm (right) for the family $\mathcal{F}$ of Example* 4.2.

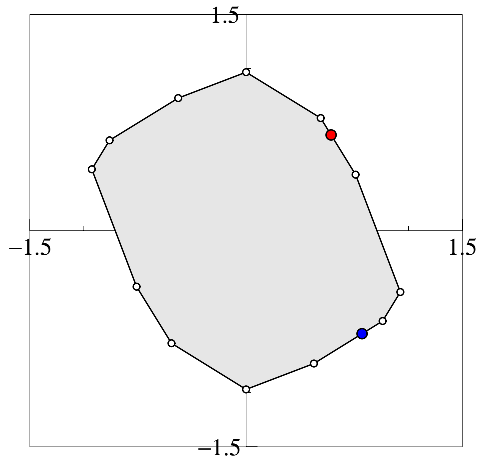

FIG. 4.3. *A random initial vector x (in red) on the boundary $\partial\mathcal{P}^*$ of the unit ball of the polytope Barabanov norm for $\mathcal{F}$ and the vector B_1x (in blue) which also lies on the boundary $\partial\mathcal{P}^*$ (see Example* 4.2*).*

In Figure 4.3 we choose a random initial vector x on the boundary $\partial\mathcal{P}^*$ of the polytope Barabanov norm and obtain, as we expect, that at least one of the two vectors B_1x, B_2x lies on $\partial\mathcal{P}^*$ as well. $\diamond$

Next we consider the following example (see e.g. [BTV, K2]), which has recently received particular attention in the literature[1].

EXAMPLE 4.3.

Let $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[1]We thank an anonymous referee for addressing us to the recent results concerning the uniqueness of the Barabanov norm of this example [M3].

The family is self-dual so we do not have to introduce $\mathcal{F}^*$. It is easy to observe that the 2-norm is an extremal norm. In fact

$$\rho\,(A_1A_2)^{\frac{1}{2}} = \|A_1\|_2 = \|A_2\|_2 = \frac{1}{2}\left(1+\sqrt{5}\right).$$

An interesting question is whether there exists an ellipsoid Barabanov norm. In [M3] it has been proven that the Barabanov norm is unique for this example and an iterative construction of such a norm is also considered in [K2]. We are going to show that there exists a polytope Barabanov norm, which implies that no ellipsoid norm can be a Barabanov norm.

The construction of the polytope Barabanov norm follows. Using the s.m.p. $P = A_1A_2$, we apply Algorithm 1 and obtain the real polytope extremal norm for $\mathcal{F}$ with unit ball $\mathcal{P} = \mathrm{co}\,(v_1, v_2, v_3, v_4, -v_1, -v_2, -v_3, -v_4)$ (see Figure 4.4, left picture) where

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{pmatrix}, & v_2 &= B_1 v_1 = \begin{pmatrix} 1 \\ \frac{2}{3+\sqrt{5}} \end{pmatrix}, \\ v_3 &= B_2 v_1 = \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{pmatrix}, & v_4 &= B_2 v_3 = \begin{pmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{pmatrix}, \end{aligned}$$

with $B_1 = A_1/R, B_2 = A_2/R$ and $R = \rho(A_1A_2)^{1/2}$. The corresponding dual norm (the Barabanov one) turns out to be given by the real polytope norm with unit ball $\mathcal{P}^* = \mathrm{co}\,(u_1, u_2, u_3, u_4, -u_1, -u_2, -u_3, -u_4)$ (see Figure 4.4, right picture) where

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_4 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

We denote by $\overline{u_i u_j}$ the segment connecting the vectors u_i and u_j. It is direct to check that

$$\begin{aligned} & B_1 u_1 = u_2 \in \partial\mathcal{P}^*, & & B_2 u_1 = \begin{pmatrix} 0 \\ \frac{2}{1+\sqrt{5}} \end{pmatrix} \in \mathring{\mathcal{P}}^*, \\ & B_1 u_2 = \begin{pmatrix} \frac{2\left(-1+\sqrt{5}\right)}{1+\sqrt{5}} \\ \frac{-1+\sqrt{5}}{1+\sqrt{5}} \end{pmatrix} \in \overline{u_2 u_3} \in \partial\mathcal{P}^*, & & B_2 u_2 = \begin{pmatrix} \frac{-1+\sqrt{5}}{1+\sqrt{5}} \\ \frac{2\left(-1+\sqrt{5}\right)}{1+\sqrt{5}} \end{pmatrix} \in \overline{u_1 u_2} \in \partial\mathcal{P}^*, \\ & B_1 u_3 = \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ 0 \end{pmatrix} \in \mathring{\mathcal{P}}^*, & & B_2 u_3 = u_2 \in \partial\mathcal{P}^*, \\ & B_1 u_4 = -u_1 \in \partial\mathcal{P}^*, & & B_2 u_4 = u_3 \in \partial\mathcal{P}^*. \end{aligned}$$

As a consequence, exploiting central symmetry, we can limit ourselves to consider the boundary of $\mathcal{P}^*$ for $x_1 > 0$ and state that

(i) for all $x \in \overline{u_1 u_2}$ it holds $B_1 x \in \overline{u_2 u_3}$;

(ii) for all $x \in \overline{u_2 u_3}$ it holds $B_2 x \in \overline{u_1 u_2}$;

(iii) for all $x \in \overline{u_3 u_4}$ it holds $B_2 x \in \overline{u_2 u_3}$;

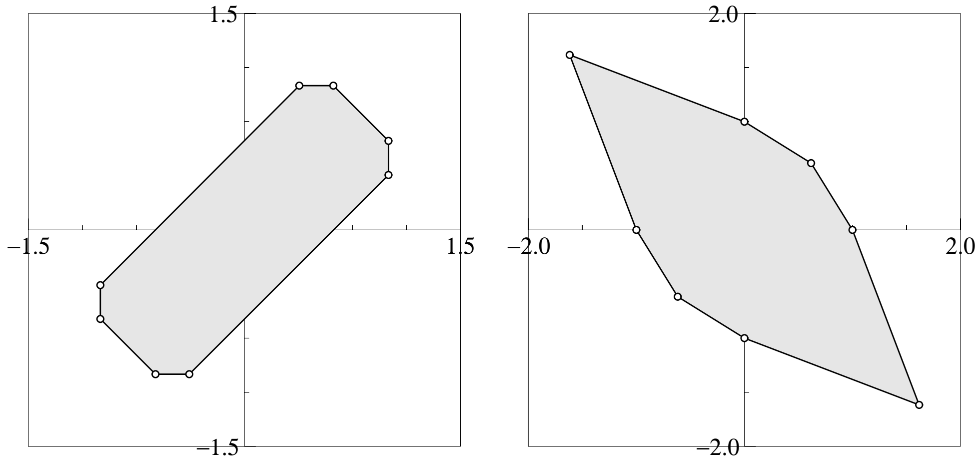

FIG. 4.4. *Polytope extremal norm (left) for the family $\mathcal{F}^*$ and polytope Barabanov norm (right) for the family $\mathcal{F}$ of Example* 4.3.

(iv) for all $x \in \overline{u_4(-u_1)}$ it holds $B_1 x \in \overline{(-u_1)(-u_2)}$.

The properties (i)–(iv) above imply that $\|\cdot\|_{\mathcal{P}^*}$ is a polytope Barabanov norm for $\mathcal{F}$, which is the unique Barabanov norm [M3].

Note that $B_1\mathcal{P}^* \cup B_2\mathcal{P}^* \subset \mathcal{P}^*$, which means that the norm associated to $\mathcal{P}^*$ is not a Protasov norm for $\mathcal{F}$. This is consistent with the fact that its dual norm, associated to $\mathcal{P}$, is not a Barabanov norm. $\diamond$

EXAMPLE 4.4. Consider the family of transition matrices $\mathcal{F} = \{A_1, A_2\}$ arising in the construction of the well-known Daubechies wavelet D_4 (see, e.g., [D, C]), where

$$A_1 = \begin{pmatrix} 5.212854848820774 & 0 & 0 \\ 1.703224934278843 & -4.676287953813834 & 5.212854848820774 \\ 0 & -0.239791829285782 & 1.703224934278843 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -4.676287953813834 & 5.212854848820774 & 0 \\ -0.239791829285782 & 1.703224934278843 & -4.676287953813834 \\ 0 & 0 & -0.239791829285782 \end{pmatrix}.$$

Then we consider the adjoint family $\mathcal{F}^*$. It can be proved that $P = A_1^{\mathrm{T}}$ is an s.m.p., $\rho(\mathcal{F}^*) = \rho(P) = 5.212854848820774$ and a real polytope extremal norm is given by $\mathcal{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where $v_1 = (1\ 0\ 0)^{\mathrm{T}}$ is the leading eigenvector of P,

$$v_2 = B_2^{\mathrm{T}} v_1,\ v_3 = B_1^{\mathrm{T}} v_2,\ v_4 = B_2^{\mathrm{T}} v_2,\ v_5 = B_1^{\mathrm{T}} v_3,\ v_6 = B_2^{\mathrm{T}} v_3,\ v_7 = B_1^{\mathrm{T}} v_5,$$

with $B_1 = A_1/\rho(P), B_2 = A_2/\rho(P)$.

The centrally symmetric polytope which is the unit ball of the Barabanov norm turns out to have 44 vertices and is plotted in Figure 4.5. $\diamond$

4.1. Nonnegative matrices: polytope Barabanov monotone norms.

Now assume here that $\mathcal{F}$ is nonnegative. Then, according to [GP] (see Algorithm (**P**)), the procedure for the construction of a polytope extremal norm can be refined to give a real polytope monotone norm.

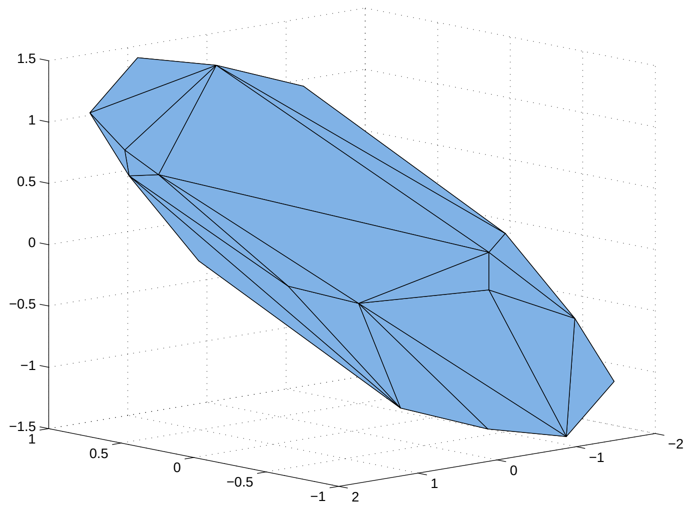

FIG. 4.5. *Polytope Barabanov norm for the family $\mathcal{F}$ of Example* 4.4.

If a cone $K \subset \mathbb{R}^d_+$ is fixed, then for a given set $V \subset K$ we have that

$$\mathcal{P} = \mathrm{co}_-(V)$$

is a positive polytope to which a monotone norm is associated.

Here we describe the construction of a polytope Barabanov monotone norm starting from a polytope extremal monotone norm.

DEFINITION 4.3 (Dual monotone norm). *Given a monotone norm $\|\cdot\|$ on the cone K we define the dual norm as*

$$\|x\|_* = \max_{u \in K\ :\ \|u\|=1} \langle x, u \rangle, \qquad x \in K. \tag{4.3}$$

It is direct to check that the definition (4.3) gives indeed a monotone norm. In fact, if $x - y \in K$, then $\|x\|_* \geq \|y\|_*$. Moreover, $\|\cdot\|_*$ is nonnegative, continuous and convex, since the maximum of $n \geq 1$ linear functionals is a convex function.

LEMMA 4.4. *Assume that $\|\cdot\|$ is an extremal monotone norm for $\mathcal{F}$. Then $\|\cdot\|_*$ is an extremal monotone norm for $\mathcal{F}^*$.*

Proof.

Assume that $\rho(\mathcal{F}) = 1$ and let $x \in K$ be such that $\|x\|_* = 1$. We write

$$\begin{aligned} \|A_i^{\mathrm{T}} x\|_* x &= \max_{y \in K\ :\ \|y\|=1} \langle A_i^{\mathrm{T}} x, y \rangle = \max_{y \in K\ :\ \|y\|=1} \langle x, A_i y \rangle \\ &= \max_{y \in K\ :\ \|y\|=1} \|A_i y\| \left\langle x, \frac{A_i y}{\|A_i y\|} \right\rangle \leq \max_{y \in K\ :\ \|y\|=1} \left\langle x, \frac{A_i y}{\|A_i y\|} \right\rangle, \end{aligned} \tag{4.4}$$

where we use the fact that $\|A_i y\| \leq 1$ by extremality of $\|\cdot\|$.

Therefore, since

$$\left\| \frac{A_i y}{\|A_i y\|} \right\| = 1,$$

we obtain

$$\|A_i^{\mathrm{T}} x\|_* \le \max_{u \in K \,:\, \|u\|=1} \langle x, u \rangle = \|x\|_* = 1.$$

This completes the proof. □

Definition 4.5. *Let a family $\mathcal{F}$ have a common invariant cone K. A norm $\|\cdot\|$ is called invariant if there is a constant $\lambda > 0$ such that*

$$\max_{i=1,\ldots,m} \|A_i x\| \;=\; \lambda \, \|x\| \qquad \forall x \in K.$$

Moreover, if $\lambda = \rho(\mathcal{F})$, then $\|\cdot\|$ is called a Barabanov norm.

Assume that $\rho(\mathcal{F}) = 1$ and V has been computed by Algorithm (**P**) presented in [GP] applied to $\mathcal{F}^*$, which means that every vertex v_p, $p = 1, \ldots, n$, of the polytope $\mathcal{P} = \mathrm{co}_-(V)$, which determines an extremal monotone polytope norm, has been generated in such a way that

$$v_p = A_{i_p}^{\mathrm{T}} v_{j_p} \qquad \text{for some} \;\; j_p \in \{1, \ldots, n\} \text{ and } i_p \in \{1, \ldots, m\}. \tag{4.5}$$

Then we have the following (constructive) result which shows the existence of a related polytope Barabanov monotone norm.

Theorem 4.6. *Assume that $\rho(\mathcal{F}) = 1$ and, for a finite set $V = \{v_i\}_{i=1}^n \subset K$, let $\mathcal{P} = \mathrm{co}_-(V)$ be a positive polytope which determines a polytope extremal monotone norm $\|\cdot\|$ for the family $\mathcal{F}^* = \{A_i^{\mathrm{T}}\}_{i=1}^m$. Moreover, assume that (4.5) is fulfilled for $p = 1, \ldots, n$. Then the dual norm $\|\cdot\|_*$ is a polytope Barabanov monotone norm for $\mathcal{F}$.*

Proof. Definition (4.3) implies that

$$\|x\|_* = \max_{1 \le j \le n} \langle x, v_j \rangle.$$

Therefore, its unit ball $\{x \in K \;\big|\; \|x\|_* \le 1\}$ is just $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$.

Since $\mathcal{F}^{**} = \mathcal{F}$, Lemma 4.4 implies that $\|\cdot\|_*$ is an extremal monotone norm for $\mathcal{F}$. Thus

$$\|A_i x\|_* \le 1 \qquad \forall x \in \partial \mathcal{P}^*, \quad i = 1, \ldots, m. \tag{4.6}$$

In order to prove the theorem we have to show that for every $x \in \partial \mathcal{P}^*$ there exists $i \in \{1, \ldots, m\}$ such that $\|A_i x\|_* = 1$.

To this aim observe that

$$\max_{1 \le i \le m} \|A_i x\|_* \;=\; \max_{1 \le i \le m} \max_{1 \le j \le n} \langle A_i x, v_j \rangle \;=\; \max_{1 \le i \le m} \max_{1 \le j \le n} \langle x, A_i^{\mathrm{T}} v_j \rangle. \tag{4.7}$$

Now let v_k be such that

$$\langle x, v_k \rangle = \max_{1 \le j \le n} \langle x, v_j \rangle = \|x\|_* = 1. \tag{4.8}$$

By (4.5), there esist r, s such that $v_k = A_r^{\mathrm{T}} v_s$. Therefore, (4.7) yields

$$\max_{1 \le i \le m} \|A_i x\|_* \ge \langle x, v_k \rangle,$$

which, together with (4.6) and (4.8), concludes the proof. □

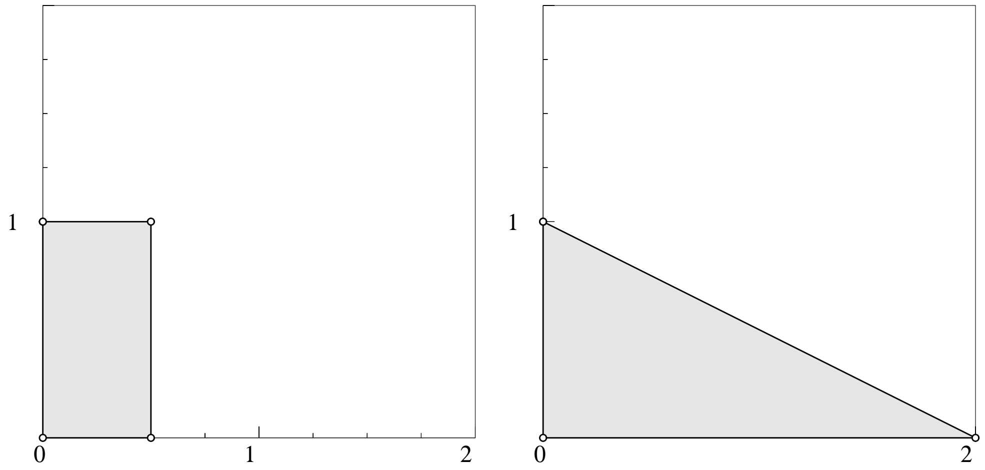

FIG. 4.6. *The polytope extremal monotone norm for $\mathcal{F}^*$ (left) and the polytope Barabanov monotone norm for $\mathcal{F}$ (right) of Example 4.1.*

EXAMPLE 4.5. Let us consider again Example 4.1, where the matrices A_1 and A_2 are nonnegative. It is immediate to construct a polytope extremal monotone norm for $\mathcal{F}^*$, which is given by

$$\mathcal{P} = \text{co}_-(v_1), \qquad v_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

This implies that the corresponding polytope Barabanov monotone norm for $\mathcal{F}$ (see Figure 4.6) has the set

$$\mathcal{P}^* = \begin{cases} x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ \frac{1}{2}x_1 + x_2 & \leq & 1 \end{cases} \tag{4.9}$$

as unit ball. Note that $\mathcal{P}^*$ intersects the principal axes in

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Using the fact that $\rho(\mathcal{F}) = 16$, we can conclude that (4.9) provides a polytope Barabanov monotone norm by noticing that the vectors

$$\frac{1}{16}A_2u_1 = \begin{pmatrix} \frac{3}{8} \\ \frac{13}{16} \end{pmatrix} \quad \text{and} \quad \frac{1}{16}A_2u_2 = \begin{pmatrix} \frac{7}{4} \\ \frac{1}{8} \end{pmatrix}$$

fulfil the equality $\frac{1}{2}x_1 + x_2 = 1$ and, hence, belong to the boundary of $\mathcal{P}^*$.

This means that for every $x \in \partial\mathcal{P}^*$ there exists a trajectory for the scaled family $\{B_1, B_2\} = \{A_1/16, A_2/16\}$ which belongs to $\partial\mathcal{P}^*$. If $x = (0 \quad \beta)^{\mathrm{T}}$ or $x = (2\beta \quad 0)^{\mathrm{T}}$, $\beta \leq 1$, then $B_1^n x \in \partial\mathcal{P}^*$ for all $n \geq 1$. If x is such that $\frac{1}{2}x_1 + x_2 = 1$, then $B_2^n x \in \mathcal{P}^*$ for all $n \geq 1$. $\diamond$

REMARK 4.3. *The results of this section generalize and improve the theory developed by Teichner and Margaliot* [TM]*, yielding a more general approach.*

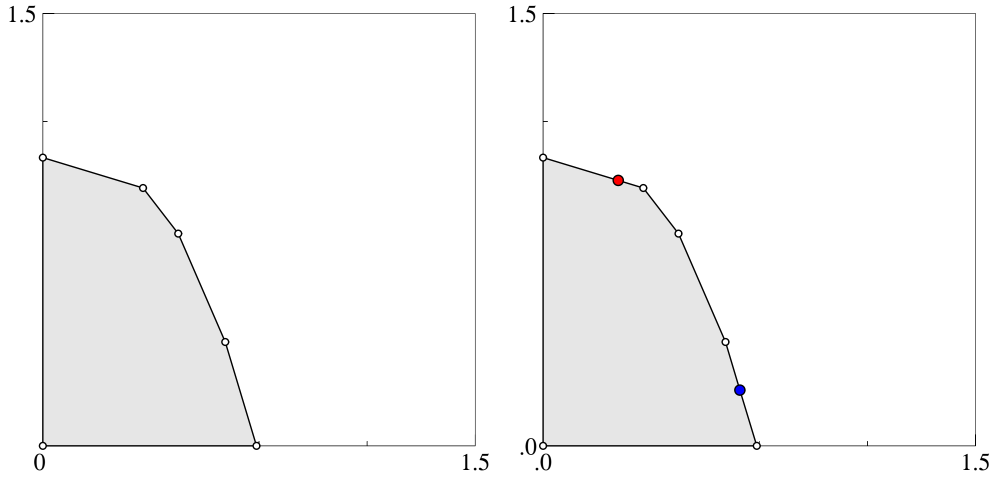

Fig. 4.7. *The polytope Barabanov monotone norm $\mathcal{P}^*$ for Example 4.6 (left) with a random initial vector x (in red) on the boundary $\partial\mathcal{P}^*$ and the vector B_1x (in blue) which also lies on the boundary $\partial\mathcal{P}^*$ (right).*

Example 4.6. Consider the following example by Jungers and Blondel [JB].

Let $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We consider the adjoint family $\mathcal{F}^* = \{A_1^{\mathrm{T}}, A_2^{\mathrm{T}}\}$ and apply the procedure described in [GP].

It turns out that an s.m.p. for $\mathcal{F}^*$ is given by $P = A_1^{\mathrm{T}} \left(A_2^{\mathrm{T}}\right)^3$ and that the unit ball of a polytope extremal monotone norm for $\mathcal{F}^*$ is $\mathrm{co}_-\,(v_1, v_2, v_3, v_4)$, where v_1 is the leading eigenvector of P, $v_2 = B_2^{\mathrm{T}}\, v_1$, $v_3 = B_2^{\mathrm{T}}\, v_2$ and $v_4 = B_2^{\mathrm{T}}\, v_3$, where $B_i = A_i/\rho(P)^{1/4}$, $i = 1, 2$.

The corresponding polytope Barabanov monotone norm for $\mathcal{F}$ is given in Figure 4.7. $\diamond$

Example 4.7. Let us consider again Example 4.3.

Since $\mathcal{F}$ is nonnegative it also admits a polytope Barabanov monotone norm.

A polytope extremal monotone norm for $\mathcal{F}$ turns out to have unit ball given by $\mathrm{co}_-\,(\{v_1, v_3\})$; correspondingly, the polytope Barabanov monotone norm has unit ball $\mathrm{co}_-\,(\{u_1, u_2, u_3\})$, where the vectors $\{v_i\}$ and $\{u_j\}$ are those defined in Example 4.3.

Interestingly, if we extend the Barabanov monotone norm to the whole space $\mathbb{R}^2$ we do not get a Barabanov norm for $\mathcal{F}$, but only an extremal norm. If this were not true, then we would contradict the uniqueness of the Barabanov norm proved in [M3] (see also [M2]). The polytope monotone norms are illustrated in Figure 4.8. $\diamond$

5. Lower spectral radius of families with an invariant cone. Consider a family $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ (non necessarily possessing an invariant cone) and let

$$\check{\rho}_k(\mathcal{F}) = \min_{P \in \Sigma_k(\mathcal{F})} \|P\|^{1/k}. \tag{5.1}$$

Definition 5.1 (lower spectral radius [G]). *The number*

$$\check{\rho}(\mathcal{F}) = \liminf_{k \to \infty} \check{\rho}_k(\mathcal{F}) \tag{5.2}$$

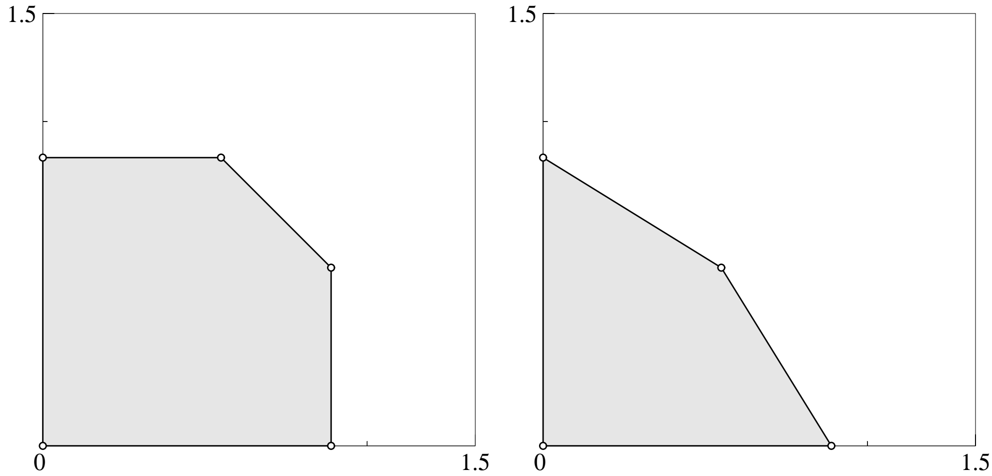

FIG. 4.8. *Polytope extremal monotone norm (left) and polytope Barabanov monotone norm (right) for the family $\mathcal{F}$ of Example* 4.3.

is said to be the lower spectral radius (l.s.r.) of the family $\mathcal{F}$.

Thus the l.s.r. is the exponent of *minimal* asymptotic growth of the products of operators from the family $\mathcal{F}$. The limit in (5.2) always exists and does not depend on the norm. A simple observation is that the l.s.r. can be estimated by means of the standard spectral radius as follows:

$$\check{\rho}(\mathcal{F}) \le \min_{P\in\Sigma_k(\mathcal{F})} \rho(P)^{\,1/k} \le \min_{P\in\Sigma_k(\mathcal{F})} \|P\|^{\,1/k}. \tag{5.3}$$

In contrast to inequality (2.6) for the j.s.r., estimation (5.3) only gives upper bounds.

Now assume that the family $\mathcal{F}$ has an invariant cone K. Here the role of the norms is taken by the *antinorms* (see [P2]).

DEFINITION 5.2. *Given a cone $K \subset \mathbb{R}^d$, an antinorm $a(\cdot)$ is a continuous nonnegative nontrivial (not identically zero) concave positively homogeneous function defined on K.*

Moreover, if $a(x) \ge a(y)$ whenever $x - y \in K$, then the antinorm is said to be monotone.

DEFINITION 5.3. *Let $a(\cdot)$ be an antinorm on a cone K. Then the set*

$$\mathcal{A} = \{x \in K \;\big|\; a(x) \ge 1\}$$

is the corresponding unit antiball.

The following result is proved in [GP].

PROPOSITION 5.4. *If for an antinorm $a(\cdot)$ and for a constant $\lambda > 0$ it holds that*

$$a(A_i x) \ge \lambda\, a(x) \quad \forall x \in K \text{ and } \forall A_i \in \mathcal{F},$$

then $\check{\rho}(\mathcal{F}) \ge \lambda$.

DEFINITION 5.5. *An antinorm $a(\cdot)$ is called extremal for the family $\mathcal{F}$ if*

$$a(A_i x) \;\ge\; \check{\rho}(\mathcal{F})\, a(x) \quad \forall x \in K\,,\ \forall A_i \in \mathcal{F}.$$

We also define a Barabanov antinorm.

Definition 5.6. *Let a family $\mathcal{F}$ have a common invariant cone K. An antinorm $a(\cdot)$ is called invariant if there exists a constant $\lambda > 0$ such that*

$$\min_{i=1,\ldots,m} a(A_i x) \;=\; \lambda\, a(x)\,, \quad \forall x \in K.$$

Moreover, if $\lambda = \check{\rho}(\mathcal{F})$, then $a(\cdot)$ is called a Barabanov antinorm.

5.1. Polytope extremal antinorms. Here we discuss general antinorms and, in particular, antinorms of polytope type and their use in the computation of the lower spectral radius of a family $\mathcal{F}$ of matrices having an invariant cone K.

To this aim, we shall make use of sets of the type

$$\mathrm{co}_+(X) \;=\; \mathrm{co}(X) \,+\, K \;=\; \big\{\, x + z \;\big|\; x \in \mathrm{co}\,(X)\,,\; z \,\in\, K \,\big\}, \tag{5.4}$$

where X is a subset of K.

Definition 5.7. *Given a finite set $V = \{v_i\}_{i=1}^n \subset K$, the set $\mathcal{P} = \mathrm{co}_+(V)$ is said to be a positive infinite polytope.*

If the unit antiball $\mathcal{A}$ of an antinorm $a(\cdot)$ is a positive infinite polytope $\mathcal{P}$, then it is said to be a *polytope antinorm* and it turns out to be monotone.

In fact, it turns out that, if $x \in \mathcal{P}$ and $y \geq x$, then $y \in \mathcal{P}$ too.

Theorem 5.8 (see [GP]). *For every family of matrices $\mathcal{F}$ with a common invariant cone K there exists an extremal monotone antinorm $a(\cdot)$ on K.*

From now on we assume that $\mathcal{F}$ is nonnegative and

$$K \subseteq \mathbb{R}^d_+. \tag{5.5}$$

5.2. An algorithm to compute a polytope extremal monotone antinorm. In [GP] (under the name of Algorithm (**L**)) a method to compute a polytope extremal monotone antinorm (when it exists) for a finite family $\mathcal{F}$ is presented, which is summarized by Algorithm 2.

In order to illustrate Algorithm 2 we need to introduce the following definitions.

Definition 5.9 (Spectrum minimizing product). *If $\mathcal{F}$ is a bounded family of $d \times d$-matrices, any matrix $P \in \Sigma_k(\mathcal{F})$ satisfying $\rho(P)^{1/k} = \check{\rho}(\mathcal{F})$ for some $k \geq 1$ is called a spectrum minimizing product (s.l.p.) for $\mathcal{F}$.*

To avoid confusion with s.m.p., we denote such a product as s.l.p., acronym of "spectral lowest product".

Definition 5.10 (Under-dominant s.l.p.). *An s.l.p. P of a bounded family of $d \times d$-matrices $\mathcal{F}$ is said to be under-dominant if there exists a constant $p > 1$ such that, for any matrix $Q \in \Sigma(\mathcal{F})$ - other than P and the powers of it and of its cyclic permutations - it holds that $\rho(Q)^{1/k} \geq p \cdot \check{\rho}(\mathcal{F})$, where k is the degree of Q (i.e., $Q \in \Sigma_k(\mathcal{F})$).*

The idea behind Algorithm 2 is to compute a candidate s.l.p. $P \in \Sigma_k(\mathcal{F})$, to define the family $\check{\mathcal{F}} := (1/\rho(P)^{1/k})\mathcal{F}$ and to construct a positive infinite polytope $\mathcal{P}$, which is mapped into itself by the family $\check{\mathcal{F}}$.

A sufficient condition for the successful termination of Algorithm 2 within finitely many iterations is that the starting product P be an under-dominant s.l.p. of the family $\mathcal{F}$ (see [GP, Theorem 7]).

6. Polytope Barabanov antinorms. Here we describe polytope Barabanov antinorms and a canonical procedure for the determination of a polytope Barabanov antinorm strating from the knowledge of a polytope extremal monotone antinorm.

Algorithm 2: Basic polytope algorithm for finding extremal monotone antinorms

Data: $\mathcal{F}$
Result: $\mathcal{P}$
begin
- Preprocessing: find a product P of length $k \geq 1$ such that $\rho(P)^{1/k}$ is minimal among $\Sigma_k(\mathcal{F})$ (P is a candidate s.l.p.);
- Set $R := \rho(P)^{1/k}$ and $\check{\mathcal{F}} := R^{-1}\mathcal{F}$;
- Compute $v_1, \ldots, v_k$ leading eigenvectors of P and its cyclic permutations;
- Set $V_0 := \{v_1, \ldots, v_k\}$;
- Set $i = 0$;
- Set $\mathcal{P}^{(0)} = \mathrm{co}_+(V_0)$;
- **while** $\check{\mathcal{F}}\, V_i \not\subseteq \mathcal{P}^{(i)}$ **do**
  - Set $i = i + 1$;
  - Determine an essential system of vertices V_i of $\mathrm{co}_+(V_{i-1} \cup \check{\mathcal{F}}\, V_{i-1})$;
  - Set $\mathcal{P}^{(i)} = \mathrm{co}_+(V_i)$;
- Return $\mathcal{P} := \mathcal{P}^{(i)}$ (extremal positive infinite polytope unit antiball);

Definition 6.1. *Given an antinorm $a(\cdot)$ on a cone $K \subseteq \mathbb{R}^d_+$, we define the dual antinorm as*

$$a^*(x) = \min_{u \in K \,:\, a(u)=1} \langle x, u \rangle, \qquad x \in K. \tag{6.1}$$

The continuity and the nonnegativity of $a^*(\cdot)$ are immediate to verify. Therefore, since the minimum of $n \geq 1$ linear functionals is a concave function, it turns out that (6.1) defines an antinorm.

Lemma 6.2. *Assume that $a(\cdot)$ is an extremal antinorm for $\mathcal{F}$. Then $a^*(\cdot)$ is an extremal antinorm for $\mathcal{F}^*$.*

Proof. Assume that $\check{\rho}(\mathcal{F}) = 1$ and let $x \in K$ be such that $a^*(x) = 1$. We write

$$\begin{aligned} a^*(A_i^* x) &= \min_{y \in K \,:\, a(y)=1} \langle A_i^* x, y \rangle = \min_{y \in K \,:\, a(y)=1} \langle x, A_i y \rangle \\ &= \min_{y \in K \,:\, a(y)=1} a(A_i y) \left\langle x, \frac{A_i y}{a(A_i y)} \right\rangle \geq \min_{y \in K \,:\, a(y)=1} \left\langle x, \frac{A_i y}{a(A_i y)} \right\rangle, \end{aligned} \tag{6.2}$$

where we use the fact that $a(A_i y) \geq 1$ by extremality of f. Therefore, since

$$a\left(\frac{A_i y}{a(A_i y)}\right) = 1,$$

we obtain

$$a^*(A_i^* x) \geq \min_{u \in K \,:\, a(u)=1} \langle x, u \rangle = a^*(x) = 1.$$

This completes the proof. $\square$

Assume that $\check{\rho}(\mathcal{F}) = 1$ and V has been computed by Algorithm 2 applied to $\mathcal{F}^*$, which means that every vertex v_p, $p = 1, \ldots, n$, of the positive infinite polytope $\mathcal{P} = \mathrm{co}_+(V)$, which determines an extremal monotone antinorm, is such that

$$v_p = A_{i_p}^{\mathrm{T}} v_{j_p} \qquad \text{for some} \;\; j_p \in \{1, \ldots, n\} \text{ and } i_p \in \{1, \ldots, m\}. \tag{6.3}$$

Then we have the following (constructive) result, which shows the existence of a related Barabanov antinorm.

THEOREM 6.3. *Assume that* $\check{\rho}(\mathcal{F}) = 1$ *and, for a finite set* $V = \{v_i\}_{i=1}^n \subset K \subseteq \mathbb{R}_+^d$, *let* $\mathcal{P} = \mathrm{co}_+(V)$ *be a positive infinite polytope which determines an extremal monotone antinorm* $a(\cdot)$ *for the family* $\mathcal{F}^* = \{A_i^{\mathrm{T}}\}_{i=1}^m$. *Moreover, assume that* (6.3) *is fulfilled for* $p = 1, \ldots, n$. *Then the dual antinorm* (6.1) *is a Barabanov antinorm for* $\mathcal{F}$.

Proof. Definition (6.1) implies that $a^*(x) = \min_{1 \le j \le n} \langle x, v_j \rangle$. Therefore, its unit antiball $\{x \in K \mid a^*(x) \ge 1\}$ is just $\mathcal{P}^* = \mathrm{adj}(\mathcal{P})$. Since $\mathcal{F}^{**} = \mathcal{F}$, Lemma 6.2 implies that $a^*(\cdot)$ is an extremal monotone antinorm for $\mathcal{F}$. Thus

$$a^*(A_i x) \ge 1 \qquad \forall x \in \partial \mathcal{P}^*, \quad i = 1, \ldots, m. \tag{6.4}$$

In order to prove the theorem we have to show that for every $x \in \partial \mathcal{P}^*$ there exists $i \in \{1, \ldots, m\}$ such that $a^*(A_i x) = 1$. To this aim observe that

$$\min_{1 \le i \le m} a^*(A_i x) \;=\; \min_{1 \le i \le m} \min_{1 \le j \le n} \langle A_i x, v_j \rangle \;=\; \min_{1 \le i \le m} \min_{1 \le j \le n} \langle x, A_i^{\mathrm{T}} v_j \rangle. \tag{6.5}$$

Now let v_k be such that

$$\langle x, v_k \rangle = \min_{1 \le j \le n} \langle x, v_j \rangle = a^*(x) = 1. \tag{6.6}$$

By (6.3), there esist r, s such that $v_k = A_r^{\mathrm{T}} v_s$. Therefore, (6.5) yields

$$\min_{1 \le i \le m} a^*(A_i x) \le \langle x, v_k \rangle,$$

which, together with (6.4) and (6.6), concludes the proof. $\square$

EXAMPLE 6.1. Let $\mathcal{F} = \{A_1, A_2\}$, where

$$A_1 \;=\; \begin{pmatrix} 7 & 2 \\ 0 & 3 \end{pmatrix}, \quad A_2 \;=\; \begin{pmatrix} 2 & 0 \\ 4 & 8 \end{pmatrix},$$

and consider the adjoint family $\mathcal{F}^*$.

In [GP] it is proved that the product $P = A_1^{\mathrm{T}} A_2^{\mathrm{T}} (A_1^{\mathrm{T}} A_1^{\mathrm{T}} A_2^{\mathrm{T}})^2$ is spectrum minimizing and that $\check{\rho}(\mathcal{F}^*) = \rho(P)^{1/8}$. Running Algorithm 2 yields a positive infinite polytope $\mathrm{co}_+(V)$ with $V = \{v_i\}_{i=1}^9$. By setting $B_{1,2}^{\mathrm{T}} = A_{1,2}^{\mathrm{T}} / \check{\rho}(\mathcal{F}^*)$ and denoting the leading eigenvector of P by v_1 one gets:

$$\begin{array}{llll} v_2 = B_2^{\mathrm{T}} v_1, & v_3 = B_1^{\mathrm{T}} v_2, & v_4 = B_1^{\mathrm{T}} v_3, & v_5 = B_2^{\mathrm{T}} v_4, \\ v_6 = B_1^{\mathrm{T}} v_5, & v_7 = B_1^{\mathrm{T}} v_6, & v_8 = B_2^{\mathrm{T}} v_7, & v_9 = B_2^{\mathrm{T}} v_6. \end{array}$$

The corresponding infinite polytope $\mathcal{P}$, which determines the extremal antinorm for $\mathcal{F}^*$, is shown in Figure 6.1 (left) along with the corresponding dual infinite polytope $\mathcal{P}^*$, which determines the Barabanov antinorm for $\mathcal{F}$ (right). $\diamond$

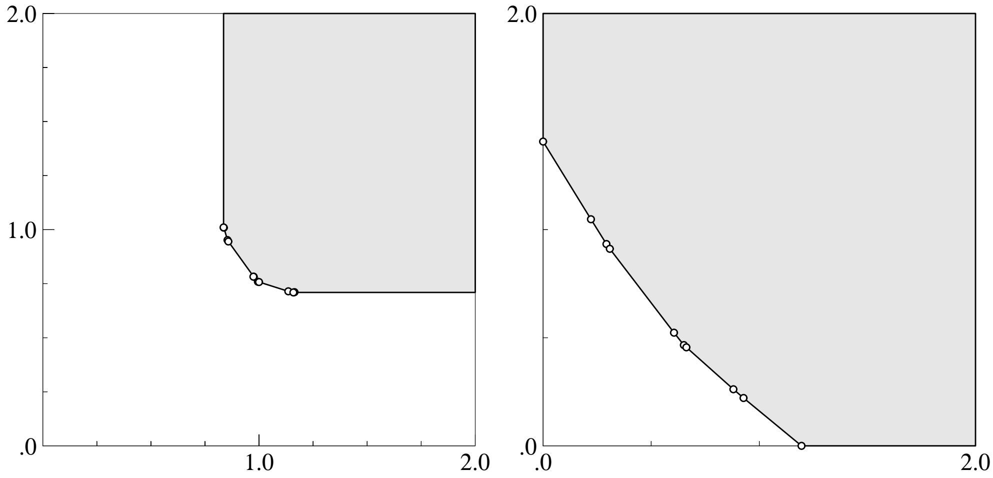

FIG. 6.1. *The polytope extremal antinorm (left) for $\mathcal{F}^*$ and the Barabanov antinorm for $\mathcal{F}$ (right) for Example 6.1.*

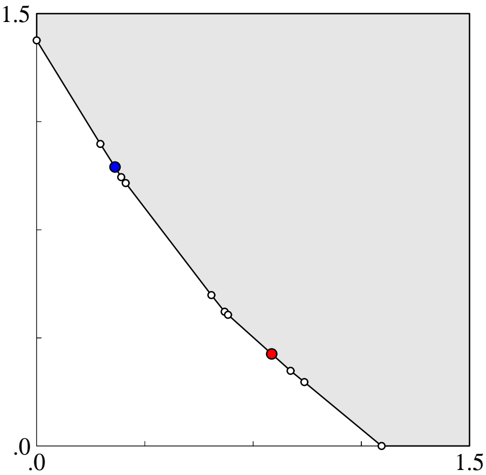

FIG. 6.2. *A random initial vector x (in red) on the boundary $\partial\mathcal{P}^*$ of the antiball of the Barabanov antinorm and the vector B_2x (in blue) which also lies on the boundary $\partial\mathcal{P}^*$ (see Example* 6.1).

Conclusions. In this article we have proposed a novel methodology which allows us to construct - under suitable assumptions - a polytope Barabanov norm for a finite family of matrices. Such a norm is associated to the joint spectral radius of the considered family. The assumptions appear quite general which means that we expect the procedure is successful in most cases. Under the assumption that the family of matrices has an invariant cone, we are able to extend the methodology to construct a Barabanov monotone norm. Analogously we are able to construct a Barabanov antinorm, which is associated to the computation of the lower spectral radius of the family. Unless a family possesses an invariant cone, the construction of monotone norms and antinorms, both extremal and Barabanov, is not possible. Analyzing possible extensions of the obtained results to families with more sophisticated invariant sets is an interesting topic which is not yet explored.

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