# Dihedral Galois covers of algebraic varieties and the simple cases ☆

Fabrizio Catanese, [a], Fabio Perroni [b,*]

[a] *Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth, Germany*
[b] *Dipartimento di Matematica e Geoscienze, Sezione di Matematica e Informatica, Università di Trieste, Via Valerio 12/1, 34127 Trieste, Italy*

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ABSTRACT

In this article we investigate the algebra and geometry of dihedral covers of smooth algebraic varieties. To this aim we first describe the Weil divisors and the Picard group of divisorial sheaves on normal double covers. Then we provide a structure theorem for dihedral covers, that is, given a smooth variety $Y$, we describe the algebraic "building data" on $Y$ which are equivalent to the existence of such covers $\pi : X \to Y$. We introduce then two special very explicit classes of dihedral covers: the simple and the almost simple dihedral covers, and we determine their basic invariants. For the simple dihedral covers we also determine their natural deformations. In the last section we give an application to fundamental groups.

## 1. Introduction

A main issue in the classification theory of algebraic varieties is the construction of interesting and illuminating examples. For instance in the book of Enriques [1] one can see that a recurrent method is the one of considering the minimal resolution $S$ of double covers $f : X \to Y$; these are called 'piani doppi', double planes, when $Y = \mathbb{P}^2$.

Burniat [2] considered more general bidouble covers, i.e., Galois covers $f : X \to Y$ with group $G = (\mathbb{Z}/2)^2$, and some work of the first author [3,4] was focused on looking at the invariants and the deformations of bidouble covers, deriving basic results for the moduli spaces of surfaces.

Comessatti [5] was the first to study Galois coverings with abelian group $G$, and their relations to topology, while Pardini [6] described neatly the algebraic structure of such coverings, their invariants and deformations.

Just to give a flavour of the result: when $G$ is a cyclic group of order $n$ and $Y$ is factorial, normal $G$-coverings correspond to building data $(L, D_1, \dots, D_{n-1})$ consisting of reduced effective divisors $D_1, \dots, D_{n-1}$ without common components, and of the isomorphism class of a divisor $L$ such that $nL \equiv \sum_i iD_i$ ($\equiv$ is the classical notation for linear equivalence). The theorem is the scheme counterpart of the field theoretic description

$$\mathbb{C}(X) = \mathbb{C}(Y)[z]/(z^n - \Pi_i \delta_i^i).$$

Here $D_i = \{\delta_i = 0\}$ and the branch locus $\mathcal{B}_f$ is the union of the divisors $D_i$.

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* Corresponding author.
*E-mail addresses:* Fabrizio.Catanese@uni-bayreuth.de (F. Catanese,), fperroni@units.it (F. Perroni).

In Pardini's theorem one can replace the field $\mathbb{C}$ by any algebraically closed field of characteristic coprime to $n$, but over the complex numbers we have a more general result, namely, the extension of the Riemann existence theorem due to Grauert and Remmert: normal schemes with a finite covering $f : X \to Y$, and with branch locus contained in a divisor $\mathcal{B}$ correspond to conjugacy classes of monodromy homomorphisms $\mu : \pi_1(Y \setminus \mathcal{B}) \to \mathfrak{S}_n$. The scheme is a variety (i.e., irreducible) iff $\mathrm{Im}(\mu)$ is a transitive subgroup, and Galois if it is simply transitive (similarly for Abelian coverings in [7] the criterion for irreducibility of $X$ was explicitly given).

On the other hand, the goal of finding explicit algebraic equations for the case of Galois coverings with non Abelian group $G$ is like the quest for the Holy Graal, and in this paper, widely extending previous results of Tokunaga [8], we give a full characterization of Galois coverings $\pi : X \to Y$ with $Y$ smooth and with Galois group the dihedral group $D_n$ of order $2n$.

The underlying idea is of course the same and is easy to explain in general terms: we can factor $\pi : X \to Y$ as the composition of a cyclic covering of order $n$, $p : X \to Z := X/H$, (here $H \subset D_n$ is the group of rotations) and a (singular) double covering $q : Z \to Y$. A main new technical tool, developed in Sections 3 and 4, consists in describing Weil divisors and the Picard group of divisorial sheaves on a normal double cover $Z$. To make the results more appealing, we describe in detail the very special case of the Picard group of hyperelliptic curves, especially the algebraic determination of torsion line bundles; this is an extension of a previous partial result by Mumford [9], it boils down to determinantal equations for triples of polynomials in one variable, and bears interesting similarities with the resultant of two polynomials.

In Section 5 we describe the general theorem, whose application in concrete cases is however not so straightforward. For this reason we concentrate in the next sections on two special very explicit classes of dihedral covers of algebraic varieties: the simple and the almost simple dihedral covers.

The underlying idea of simple covers is the one of giving a schematic equation which looks exactly as the equation describing the field extension. The first well known instance is the one of a simple cyclic cover: $X$ is given by an equation

$$z^n = F,$$

where $z$ is a fibre variable on the line bundle $\mathbb{L}$ associated to a divisor $L$ on $Y$, and $F$ is a section in $H^0(\mathcal{O}_Y(nL))$.

In [10] the analysis of everywhere nonreduced moduli spaces was based also on the technical notion of an almost simple cyclic cover, given as the locus $X$ inside the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathbb{L}_0 \oplus \mathbb{L}_\infty)$, associated to a pair of two line bundles $\mathbb{L}_0, \mathbb{L}_\infty$, defined by the equation

$$z_0^n a_\infty = z_\infty^n a_0, \; a_0 \in H^0(\mathcal{O}_Y(nL_0)), \; a_\infty \in H^0(\mathcal{O}_Y(nL_\infty)),$$

and where the divisors $E_0 = \{a_0 = 0\}$ and $E_\infty = \{a_\infty = 0\}$ are disjoint (else the covering is not finite).

Let us now explicitly describe the equations of a simple dihedral covering: $X$ is contained in a rank two split vector bundle of the form $\mathbb{L} \oplus \mathbb{L}$ over $Y$, and is defined by equations

$$\begin{cases} u^n + v^n & = 2a, \quad a \in H^0(\mathcal{O}_Y(nL)), \\ uv & = F, \quad F \in H^0(\mathcal{O}_Y(2L)). \end{cases}$$

The dihedral action is generated by an element $\sigma$ of order $n$ such that

$$u \mapsto \zeta u, \; v \mapsto \zeta^{-1} v,$$

where $\zeta$ is a primitive $n$th root of 1, and by an involution $\tau$ exchanging $u$ with $v$.

The branch locus is here the divisor $\mathcal{B} = \{a^2 - F^n = 0\}$, $Z$ is defined by $z^2 = a^2 - F^n$, and the cyclic covering $X$ is defined by $u^n = z + a$: it is smooth if the two divisors $\{a = 0\}$ and $\{F = 0\}$ intersect transversally and $\mathcal{B}$ is smooth outside of $F = 0$.

In particular, as it is well known from the work of Zariski [11] and many other followers, the fundamental group $\pi_1(Y \setminus \mathcal{B})$ is non Abelian and admits a surjection to the group $D_n$ (irreducibility is granted if there are points where $\{a = 0\}$ and $\{F = 0\}$, since there the covering is totally ramified).

In the case of simple dihedral coverings, the covering $X \to Z$ is only ramified in the $A_{n-1}$ singularities of $Z$, the points where $z = a = F = 0$. In order to get a covering with more ramification we introduce then the almost simple dihedral covers, defined this time on the fibre product of two $\mathbb{P}^1$-bundles. $X$ is the subset of $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L})$ defined by the following two equations:

$$\begin{cases} u_1 v_1 - u_0 v_0 F & = 0 \\ a_\infty v_1^n u_0^n - 2a_0 v_0^n u_0^n + a_\infty v_0^n u_1^n & = 0, \end{cases}$$

where, $v_1, u_1$ are fibre coordinates on $\mathbb{L}$, $v_0, u_0$ are fibre coordinates on $\underline{\mathbb{C}}$ (so that $([u_0 : u_1], [v_0 : v_1])$ are projective fibre coordinates on $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L})$ ).

As we said earlier, we want to find the invariants and study the deformations of the varieties that we construct in this way. We first do this in full generality for any Galois covering in Section 2; later we write more explicit formulae for simple and almost simple dihedral coverings. We apply these in the case of easy examples, deferring the study of more complicated cases to the future.

One interesting application, to real forms of curves with cyclic symmetry, and related loci in the moduli space of real curves, shall be given in a forthcoming joint work with Michael Lönne [12].

Finally, in this paper we consider also the non Galois case: namely, coverings of degree $n$ whose Galois group is $D_n$. In doing so, we establish connections with the theory of triple coverings of algebraic varieties described by Miranda [13] (if the triple covering is not Galois, then its Galois group is $D_3$!).

## 2. Direct images of sheaves under Galois covers

We consider (algebraic) varieties which are defined over the field of complex numbers $\mathbb{C}$, though many results remain valid for complex manifolds and for algebraic varieties defined over an algebraically closed field of characteristic $p$ not dividing $2n$.

**Definition 2.1.** Let $Y$ be a variety and let $G$ be a finite group. A **Galois cover** of $Y$ with group $G$, shortly a $G$-**cover** of $Y$, is a finite morphism $\pi : X \to Y$, where $X$ is a variety with an effective action by $G$, such that $\pi$ is $G$-invariant and induces an isomorphism $X/G \cong Y$.

A **dihedral cover** of $Y$ is a $D_n$-cover of $Y$, where $D_n$ is the dihedral group of order $2n$.

If $\pi : X \to Y$ is a $G$-cover, the **ramification locus** of $\pi$ is the locus $R := R_\pi \subset X$ of points with non-trivial stabilizer; the **(reduced) branch locus** of $\pi$ is $\mathcal{B}_\pi := \pi(R) \subset Y$.

In the case where $X, Y$ are smooth, $R$ is the reduced divisor defined by the Jacobian determinant of $\pi$, and also $\mathcal{B}_\pi$ is a divisor (purity of the branch locus).

In the following, if there is no danger of confusion, we will denote $\mathcal{B}_\pi$ by $\mathcal{B}$.

Let $\pi : X \to Y$ be a $G$-cover. Since $\pi : X \to Y$ is finite, $\pi_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras. The $\mathcal{O}_Y$-algebra structure on $\pi_*\mathcal{O}_X$ corresponds to a morphism $m : \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X$ of $\mathcal{O}_Y$-modules that gives a commutative and associative product on $\pi_*\mathcal{O}_X$. These data determine $X$ as $\operatorname{Spec} \pi_*\mathcal{O}_X$. Furthermore, the $G$-action on $X$ corresponds to a $G$-action on $\pi_*\mathcal{O}_X$ which is the identity on $\mathcal{O}_Y$. The two structures on $\pi_*\mathcal{O}_X$, of an algebra over $\mathcal{O}_Y$ and of a $G$-module, impose restrictions that sometimes allow one to determine certain "building data" for the $G$-cover $\pi$. One of our aims here is the determination of such building data when $G = D_n$ (Theorem 5.6).

We will restrict ourselves to the case where $\pi$ is flat, or equivalently $\pi_*\mathcal{O}_X$ is locally free. We recall in the next proposition a useful characterization of flat $G$-covers.

**Proposition 2.2.** *Let $\pi : X \to Y$ be a $G$-cover, with $X$ irreducible and $Y$ smooth. Then $\pi$ is flat $\Leftrightarrow$ $X$ is Cohen–Macaulay.*

**Proof.** ($\Leftarrow$) This is a direct application of the Corollary to Theorem 23.1 in [14].

($\Rightarrow$) Let $x \in X$ and $y = \pi(x)$. Since $\pi$ is flat, for any regular sequence $a_1, \dots, a_r \in m_y$ for $\mathcal{O}_{Y,y}$, $\pi^*(a_1), \dots, \pi^*(a_r)$ is a regular sequence for $\mathcal{O}_{X,x}$. Hence, since $Y$ is smooth and $\pi$ is finite, $X$ is Cohen–Macaulay. □

In the rest of this section we assume that $X$ and $Y$ are smooth. The aim here is to relate the basic sheaves of $X$, namely $\Omega^i_X, \Theta_X, \dots$, with the corresponding ones of $Y$. Concretely, for example for the $\Omega^i_X$'s, there is a chain of inclusions as follows:

$$\Omega^i_Y \otimes \pi_*\mathcal{O}_X \hookrightarrow \pi_*\Omega^i_X \hookrightarrow \Omega^i_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X,$$

where $\Omega^i_Y(\log \mathcal{B})$ is the sheaf of $i$-forms with logarithmic poles along $\mathcal{B}$ and $\Omega^i_Y(\log \mathcal{B})^{**}$ is its double dual. These three sheaves coincide on $Y \setminus \mathcal{B}$ and it is possible to describe explicitly $\pi_*\Omega^i_X$ inside $\Omega^i_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$ at the generic points $\mathcal{B}_i$ of $\mathcal{B}$. Then, by Hartogs' theorem, one obtains a description of $\pi_*\Omega^i_X$. We work out in detail the steps of this procedure for $\Omega^1_X$ and $\Theta_X$.

Let $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ be the fields of rational functions on $X$ and $Y$, respectively. The space of differentials of $\mathbb{C}(X)$, $\Omega^1_{\mathbb{C}(X)}$, is a vector space of dimension

$$d = \dim_{\mathbb{C}(X)} \Omega^1_{\mathbb{C}(X)} = \dim X$$

and a basis is given by the differentials of a transcendence basis of the field extension $\mathbb{C} \subset \mathbb{C}(X)$. The same holds true for $\Omega^1_{\mathbb{C}(Y)}$ and, since $\dim X = \dim Y$, the pull-back morphism $\pi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ induces an isomorphism of $\mathbb{C}(X)$-vector spaces:

$$\Omega^1_{\mathbb{C}(Y)} \otimes_{\mathbb{C}(Y)} \mathbb{C}(X) \to \Omega^1_{\mathbb{C}(X)}. \tag{1}$$

In other words, the rational differentials of $X$ can be written as linear combinations

$$\sum_{i=1}^{d} f_i dy_i,$$

where $f_1, \dots, f_d \in \mathbb{C}(X)$, $y_1, \dots, y_d \in \mathbb{C}(Y)$ form a transcendence basis for the extension $\mathbb{C} \subset \mathbb{C}(Y)$. Since a regular differential 1-form on $X$ is a rational differential 1-form which is regular at each point, the isomorphism (1) induces an inclusion

$$\varphi : \pi_*\Omega^1_X \hookrightarrow \Omega^1_{\mathbb{C}(Y)} \otimes_{\mathbb{C}(Y)} \mathbb{C}(X). \tag{2}$$

Since $X$ and $Y$ are smooth, by the theorem of purity of the branch locus [15] the ramification locus $R$ and the branch locus $\mathcal{B}$ of $\pi$ are divisors in $X$ and $Y$ respectively. Let $\mathcal{B} = \sum_{i=1}^r \mathcal{B}_i$, where $\mathcal{B}_i \subset Y$ are prime divisors. We recall, following [16], the following definition:

$$\Omega^1_Y(\log \mathcal{B}) := \operatorname{Im}\left(\Omega^1_Y \oplus \mathcal{O}_Y^{\oplus r} \to \Omega^1_Y(\mathcal{B})\right),$$

where $\Omega^1_Y \hookrightarrow \Omega^1_Y(\mathcal{B})$ is the natural inclusion, $\mathcal{O}_Y^{\oplus r} \to \Omega^1_Y(\mathcal{B})$ is given by

$$e_i \mapsto \frac{db_i}{b_i}, \quad i = 1, \ldots, r,$$

where, for any $i$, $e_i$ is the $i$th element of the standard $\mathcal{O}_Y$-basis of $\mathcal{O}_Y^{\oplus r}$, and $b_i$ is a global section of $\mathcal{O}_Y(\mathcal{B}_i)$ with $\mathcal{B}_i = \{b_i = 0\}$.

Let $\Omega^1_Y(\log \mathcal{B})^{**}$ be the double dual of $\Omega^1_Y(\log \mathcal{B})$. It coincides with the sheaf of germs of logarithmic 1-forms with poles along $\mathcal{B}$ defined in [17]. Indeed both sheaves are reflexive and they coincide on the complement $Y \setminus \mathrm{Sing}(\mathcal{B})$ of the singular locus of $\mathcal{B}$.

**Proposition 2.3.** *Let $\pi : X \to Y$ be a $G$-cover with $X$ and $Y$ smooth. Then there are $G$-equivariant inclusions of sheaves of $\mathcal{O}_Y$-modules as follows:*

$$\Omega^1_Y \otimes \pi_*\mathcal{O}_X \hookrightarrow \pi_*\Omega^1_X \hookrightarrow \Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X,$$

*where the morphisms are isomorphisms on the complement $Y \setminus \mathcal{B}$ of the branch divisor.*

*Moreover, let $\xi_i \in \pi_*\mathcal{O}_X$ be the generator of the ideal of the reduced divisor $R_i := \pi^{-1}(\mathcal{B}_i)$ and a maximal local root of $b_i$ ($b_i = \xi_i^{m_i}$), for any $i = 1, \ldots, r$. Then $\pi_*\Omega^1_X$ is characterized as the subsheaf of $\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$ such that at the smooth points of $\mathcal{B}_i = \{b_i = 0\}$ it coincides with the subsheaf of $\mathcal{O}_Y$-modules generated by $\Omega^1_Y \otimes \pi_*\mathcal{O}_X$ and by the elements $\xi_i^k d\log(b_i)$, $k = 1, \ldots, m_i - 1$.*

**Proof.** The first arrow to the left is the push-forward under $\pi_*$ of the natural inclusion

$$(T\pi)^* : \pi^*\Omega^1_Y \hookrightarrow \Omega^1_X,$$

where $T\pi$ is the tangent map of $\pi$.

Now consider the morphism $\varphi$ in (2). We first show that $(\mathrm{Im}\varphi)_q \subset (\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X)_q$ for any $q \in Y' := Y \setminus \mathrm{Sing}(\mathcal{B})$. On the complement of $\mathcal{B}$, $Y \setminus \mathcal{B}$, this follows from the projection formula, since $\pi$ is not ramified on $\pi^{-1}(Y \setminus \mathcal{B})$ and hence

$$\pi_*\Omega^1_X \cong \Omega^1_Y \otimes \pi_*\mathcal{O}_X \cong \Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X \quad \text{on } Y \setminus \mathcal{B}.$$

Let now $q \in \mathcal{B}' := \mathcal{B} \setminus \mathrm{Sing}(\mathcal{B})$ and $p \in \pi^{-1}(q)$. Choose local coordinates $y_1, \ldots, y_d$ at $q$ and $x_1, \ldots, x_d$ at $p$, such that $\pi$ has the following local expression:

$$y_1 = x_1^m, y_2 = x_2, \ldots, y_d = x_d.$$

Hence $\mathcal{O}_{X,p} = \mathcal{O}_{Y,q}[x_1]/(x_1^m - y_1)$, and the stalk $(\Omega^1_Y)_q$ is generated over $\mathcal{O}_{Y,q}$ by

$$dy_1, dy_2, \ldots, dy_d,$$

while the stalk $(\Omega^1_X)_p$ is generated over $\mathcal{O}_{X,p}$ by

$$dx_1, dx_2, \ldots, dx_d,$$

with the obvious relations

$$m \cdot d\log(x_1) = d\log(y_1), \quad \text{equivalently} \quad dx_1 = \frac{x_1}{my_1}dy_1,$$
$$dx_i = dy_i, \quad i \geq 2.$$

From this it follows that $(\pi_*\Omega^1_X)_q$ is generated as $\mathcal{O}_{Y,q}$-module by the elements of the $G$-orbit of

$$\begin{cases} x_1^i dy_j & 0 \leq i \leq m-1, 2 \leq j \leq d, \\ x_1^k \dfrac{dy_1}{y_1}, & 1 \leq k \leq m-1 \\ dy_1 \end{cases}$$

which implies the claim at the points $q \in \mathcal{B}'$, since $y_1 = 0$ is a defining equation for $\mathcal{B}$ near $q$. Moreover, we see that $(\pi_*\Omega^1_X)_q$ is generated, as an $\mathcal{O}_Y$-module, by $(\Omega^1_Y)_q \otimes (\pi_*\mathcal{O}_X)_q$ and by the elements $x_1^k \frac{dy_1}{y_1}$, $1 \leq k \leq m-1$.

Notice that this argument also shows that the inclusion $(\pi_*\Omega^1_X)_q \subset (\Omega_{Y,q}(\log \mathcal{B})^{**} \otimes (\pi_*\mathcal{O}_X))_q$ is $G$-equivariant, if $q \in Y \setminus \mathrm{Sing}(\mathcal{B})$.

To conclude the proof, observe that $\pi_*\Omega^1_X$ is locally free and that $\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$ is reflexive, so that any section of $\pi_*\Omega^1_X$ on $Y \setminus \mathrm{Sing}(\mathcal{B})$ has a unique extension in $\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$. □

Let now $\Theta_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ and $\Theta_Y := \mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$ be the tangent sheaves of $X$ and $Y$, respectively. In the following proposition we relate the sheaf $\pi_*\Theta_X$ with $\Theta_Y$.

Define as in [17] or [18, Def. 9.15] the sheaf $\Theta_Y(-\log \mathcal{B})$ of logarithmic vector fields on $Y$ with respect to $\mathcal{B} = \{b = 0\}$ as

$$\Theta_Y(-\log \mathcal{B}) := \{v | v \cdot \log(b) \in \mathcal{O}_Y\} = \{v | v(b) \in b\mathcal{O}_Y\} = (\Omega^1_Y(\log \mathcal{B}))^*.$$

Observe that the quotient sheaf $\Theta_Y/\Theta_Y(-\log \mathcal{B})$ equals the equisingular normal sheaf $N'_{\mathcal{B}|Y}$ of $\mathcal{B}$ in $Y$, which coincides with the normal sheaf at the points where $\mathcal{B}$ is smooth ([18], rem. 9.16).

Let us recall that the usual pairing $\Theta_Y \times \Omega^1_Y \to \mathcal{O}_Y$ extends to a perfect pairing

$$\Theta_Y(-\log \mathcal{B}) \times \Omega^1_Y(\log \mathcal{B})^{**} \to \mathcal{O}_Y.$$

**Proposition 2.4.** *Let $\pi : X \to Y$ be a G-cover with X and Y smooth. Let $\mathcal{B} \subset Y$ be the branch divisor of $\pi$. Then the tangent map of $\pi$ identifies $\pi_*\Theta_X$ with a subsheaf of $\Theta_Y \otimes \pi_*\mathcal{O}_X$. Under this identification we have the following inclusions of sheaves*

$$\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X \subset \pi_*\Theta_X \subset \Theta_Y \otimes \pi_*\mathcal{O}_X$$

*which are compatible with the G-actions. Moreover the three sheaves coincide on $Y \setminus \mathcal{B}$ and $\pi_*\Theta_X$ is characterized as the subsheaf of $\Theta_Y \otimes \pi_*\mathcal{O}_X$ sending $\pi_*\Omega^1_X$ to $\pi_*\mathcal{O}_X$.*

*More concretely, it is the subsheaf such that at the smooth points of $\mathcal{B}_i = \{b_i = 0\}$ coincides with the subsheaf of $\mathcal{O}_Y$-modules generated by $\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X$ and by $\xi_i^{-1} b_i \frac{\partial}{\partial b_i}$, where $\xi_i \in \pi_*\mathcal{O}_X$ is the generator of the ideal of the reduced divisor $R_i := \pi^{-1}(\mathcal{B}_i)$ (and a maximal local root of $b_i$, $b_i = \xi_i^{m_i}$), while $b_i \frac{\partial}{\partial b_i}$ is a local generator of $N'_{\mathcal{B}|Y}$.*

**Proof.** The tangent morphism $T\pi : \Theta_X \to \pi^*\Theta_Y$ gives a $G$-equivariant morphism of sheaves

$$\pi_*\Theta_X \to \Theta_Y \otimes \pi_*\mathcal{O}_X. \tag{3}$$

We first prove that (3) is injective and that its image contains $\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X$. By Hartogs' theorem and the definition of $\Theta_Y(-\log \mathcal{B})$ it suffices to prove this on the complement of $\mathrm{Sing}(\mathcal{B})$.

On $Y \setminus \mathcal{B}$, $\pi$ is not ramified, hence (3) is an isomorphism. Let now $q \in \mathcal{B} \setminus \mathrm{Sing}(\mathcal{B})$ and let $p \in \pi^{-1}(q)$. Choose as before coordinates $y_1, \ldots, y_d$ at $q$ and $x_1, \ldots, x_d$ at $p$, such that $\pi$ has the following local expression:

$$y_1 = x_1^m, y_2 = x_2, \ldots, y_d = x_d.$$

Then $(\pi_*\Theta_X)_q$ is generated by the $G$-orbit of

$$x_1^k \left( \frac{\partial}{\partial x_1} + m x_1^{m-1} \frac{\partial}{\partial y_1} \right), x_1^k \frac{\partial}{\partial y_2}, \ldots, x_1^k \frac{\partial}{\partial y_d}, 0 \leq k \leq m-1,$$

as $\mathcal{O}_{Y,q}$-module. The image of these generators under (3) is the $G$-orbit of

$$m \frac{\partial}{\partial y_1} \otimes x_1^{m-1+k}, \frac{\partial}{\partial y_2} \otimes x_1^k, \ldots, \frac{\partial}{\partial y_d} \otimes x_1^k,\ 0 \leq k \leq m-1.$$

From this it follows that (3) is injective and that its image contains the sheaf $\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X$, which is generated by the $G$-orbit of

$$y_1 \frac{\partial}{\partial y_1} \otimes x_1^k = \frac{\partial}{\partial y_1} \otimes x_1^{m+k}, \frac{\partial}{\partial y_2} \otimes x_1^k, \ldots, \frac{\partial}{\partial y_d} \otimes x_1^k, \quad 0 \leq k \leq m-1,$$

as $\mathcal{O}_{Y,q}$-module. The only missing term to get from $\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X$ the full direct image is then $y_1 \frac{\partial}{\partial y_1} \otimes x_1^{-1}$. We observe that $y_1 \frac{\partial}{\partial y_1} \otimes x_1^{-1} = \frac{1}{x_1} y_1 \frac{\partial}{\partial y_1}$ and the vector field $y_1 \frac{\partial}{\partial y_1}$ is the local generator of $N'_{\mathcal{B}|Y} = \Theta_Y/\Theta_Y(-\log \mathcal{B})$. □

**Remark 2.5.** Notice that the previous results are valid for more general Galois covers of complex manifolds.

### *2.1. The non-Galois case*

We extend Propositions 2.3 and 2.4 to branched covers $\pi : X \to Y$ which are not necessarily Galois. Notice that also in this case we have an inclusion of the sheaf $\pi_*\Omega^1_X$ in the constant sheaf $\Omega^1_{\mathbb{C}(Y)} \otimes_{\mathbb{C}(Y)} \mathbb{C}(X)$ as in (2).

**Proposition 2.6.** *Let $\pi : X \to Y$ be a branched cover with X and Y smooth. Let $\mathcal{B} \subset Y$ be the branch divisor of $\pi$. Then we have the following inclusions of $\mathcal{O}_Y$-modules:*

$$\Omega^1_Y \otimes \pi_*\mathcal{O}_X \hookrightarrow \pi_*\Omega^1_X \hookrightarrow \Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X,$$

*where the morphism on the left is induced by the pull-back morphism $(T\pi)^* : \pi^*\Omega^1_Y \to \Omega^1_X$ and the one on the right is induced by (2).*

**Proof.** Since $\pi : X \setminus R \to Y \setminus \mathcal{B}$ is non-ramified, $(T\pi)^* : \pi^*\Omega^1_Y \to \Omega^1_X$ is injective. Applying the push-forward functor $\pi_*$ we obtain an injective homomorphism $\Omega^1_Y \otimes \pi_*\mathcal{O}_X \hookrightarrow \pi_*\Omega^1_X$.

We now prove that the morphism (2) has image in $\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$. Notice that by the same argument as in the proof of Proposition 2.3, it suffices to prove this on the complement of the singular locus of $\mathcal{B}$. So we assume that $\mathcal{B}$ is smooth. The assertion is true on $Y \setminus \mathcal{B}$, because here $\pi$ is unramified. Let now $q \in \mathcal{B}$ and let $V \subset Y$ be an open neighbourhood (in the complex topology) of $q$, such that:

$$\pi^{-1}(V) = U_1 \sqcup \cdots \sqcup U_s,$$

with $U_1, \ldots, U_s \subset X$ disjoint subsets; for any $i = 1, \ldots, s$ there are coordinates $(x_{i,1}, \ldots, x_{i,d})$ on $U_i$ and $(y_1, \ldots, y_d)$ on $V$, such that the restriction of $\pi$ on $U_i$ has the form

$$(x_{i,1}, \ldots, x_{i,d}) \mapsto (y_1 = x_{i,1}^{e_i}, y_2 = x_{i,2}, \ldots, y_d = x_{i,d}),$$

$e_1, \ldots, e_s$ are integers $\geq 1$ (see e.g. [19]). Then $(\pi_*\Omega^1_X)_{|V}$ is generated as $\mathcal{O}_Y(V)$-module by:

$$\begin{cases} x_{i,1}^{k_i} dx_{i,1} = \dfrac{x_{i,1}^{k_i+1}}{e_i} \dfrac{dy_1}{y_1}, & k_i = 0, \ldots, e_i - 1, i = 1, \ldots, s \\ x_{i,1}^{k_i} dx_{i,j} = x_{i,1}^{k_i} dy_j, & j = 2, \ldots, d, i = 1, \ldots, s, k_i = 0, \ldots, e_1 - 1. \end{cases}$$

Since $y_1 = 0$ is a local equation for $\mathcal{B}$, it follows that the image of $\pi_*\Omega^1_X$ under (2) is contained in $\Omega^1_Y(\log \mathcal{B})^{**} \otimes \pi_*\mathcal{O}_X$. □

Analogously, for the tangent sheaves we have the following.

**Proposition 2.7.** *Let $\pi : X \to Y$ be a branched cover, with $X$ and $Y$ smooth. Let $\mathcal{B} \subset Y$ be the branch locus. Then the tangent map of $\pi$ induces an injective morphism of sheaves*

$$\pi_*\Theta_X \hookrightarrow \Theta_Y \otimes \pi_*\mathcal{O}_X$$

*whose image contains the sheaf $\Theta_Y(-\log \mathcal{B}) \otimes \pi_*\mathcal{O}_X$.*

**Remark 2.8.** We can obtain more precise results by taking the Galois closure $p : W \to Y$, which is a normal variety, and writing $X = W/H$ where $H$ is a suitable subgroup of the Galois group $G$. The only problem is that $W$ may not be smooth, and we denote then by $W'$ its smooth locus. Then we can write $\pi_*\Omega^1_X$ as essentially the submodule of $H$-invariants inside $p_*\Omega^1_W$: for instance we have

$$\pi_*\Omega^1_X = ((p_*\Omega^1_W)^H)^{**},$$

since the regular 1-forms on $X' = W'/H$ are just the $H$-invariant 1-forms on $W'$.

## 3. Line bundles and divisorial sheaves on flat double covers

In this section we consider the following general situation. We have a flat finite double cover, where $Y$ is smooth:

$$q : Z \to Y, \ Z = \mathrm{Spec}(q_*\mathcal{O}_Z),$$

$$\mathcal{R} := q_*\mathcal{O}_Z = (\mathcal{O}_Y \oplus z\mathcal{O}_Y(-L)/(z^2 - F)), \ F \in H^0(\mathcal{O}_Y(2L)).$$

Our goal is to have a description of divisorial sheaves, that is, rank 1 reflexive sheaves $\mathcal{L}$ on the Gorenstein variety $Z$ in terms of their direct image $q_*(\mathcal{L}) =: \mathcal{N}$.

By flatness of $\mathcal{L}$ over $Y$, which we assume throughout this section, $\mathcal{N}$ is a rank two vector bundle on $Y$ (a locally free sheaf of rank 2), and its $\mathcal{R}$- module structure is fully encoded in an endomorphism

$$N : \mathcal{N}(-L) \to \mathcal{N}$$

such that

$$N^2 = F \cdot \mathrm{Id}_{\mathcal{N}} : \mathcal{N}(-2L) \to \mathcal{N}.$$

In particular, observe that $\mathrm{Tr}(N) = 0$, $\det(N) = -F$, so that, on any open set where $\mathcal{N}$ and the divisor $L$ are trivialized, $N$ is given by a matrix of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \ a^2 + bc = F.$$

Conversely, any such pair $(\mathcal{N}, N)$ as above determines a divisorial sheaf $\mathcal{L}$ on $Z$.

**Lemma 3.1.** *Given a pair $(\mathcal{N}, N)$ where $N : \mathcal{N}(-L) \to \mathcal{N}$ satisfies $N^2 = F \cdot \mathrm{Id}_{\mathcal{N}} : \mathcal{N}(-2L) \to \mathcal{N}$, it determines a saturated rank 1 torsion free $\mathcal{R}$-module which is locally free exactly in the points of $Y$ where the endomorphism N does not vanish. In particular, a divisorial sheaf $\mathcal{L}$ on $Z$.*

**Proof.** View locally $\mathcal{N}$ as $\mathcal{O}_Y^2$: then a local section $(x, y)$ is a local $\mathcal{R}$ generator at a point $P$ if and only if the row vectors $(x, y)$ and $N(x, y)$ yield a local basis. This means, in terms of the matrix $N$, that $(x, y)$ and $(ax+by, cx-ay)$ are linearly independent; equivalently, the determinant

$$q(x, y) = cx^2 - 2axy - by^2 \neq 0.$$

Hence $\mathcal{N}$ is an invertible $\mathcal{R}$-module if and only if the quadratic form obtained by evaluating $a, b, c$ at $P$ is not identically zero, which amounts to the condition that $a, b, c$ do not vanish simultaneously. □

**Remark 3.2.** Of course, the points where $N$ vanishes yield singular points of the branch locus $\mathcal{B} = \{F = 0\}$, and singular points of $Z$.

In the case where $Z$ is normal, then $\mathcal{L}$ is determined by its restriction on the smooth locus $Z^0$ of $Z$ since $\mathcal{L} = i_*(\mathcal{L}|_{Z^0})$ (we are denoting by $i : Z^0 \to Z$ the inclusion map), hence we are just dealing with the Picard group $\mathrm{Pic}(Z^0)$. We want now to spell out in detail the group structure of $\mathrm{Pic}(Z^0)$ in terms of the direct image rank 2 vector bundles.

**Proposition 3.3.** *Let $Z$ be normal, let $\mathcal{L}_1^0, \mathcal{L}_2^0$ be invertible sheaves on $Z^0$, and let $\mathcal{N}_j = (q \circ i)_*(\mathcal{L}_j^0)$, $j = 1, 2$. Then the tensor product operation in* $\mathrm{Pic}(Z^0)$*, associating to $\mathcal{L}_1^0, \mathcal{L}_2^0$ their tensor product $\mathcal{L}_1^0 \otimes_{\mathcal{O}_{Z^0}} \mathcal{L}_2^0$ gives the following pair:*

(1) *the vector bundle $(q \circ i)_*(\mathcal{L}_1^0 \otimes_{\mathcal{O}_{Z^0}} \mathcal{L}_2^0)$ equals the $\mathcal{O}_Y$-double dual of $\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2$, which is the cokernel of the following exact sequence:*

$$\psi : (\mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2)(-L) \to \mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2 \to \mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2 \to 0,$$

*where*

$$\psi = N_1 \otimes_{\mathcal{O}_Y} \mathrm{Id}_{\mathcal{N}_2} - \mathrm{Id}_{\mathcal{N}_1} \otimes_{\mathcal{O}_Y} N_2.$$

(2) *To the inverse invertible sheaf ${\mathcal{L}^0}^{-1}$ corresponds the $\mathcal{R}$-module associated to the pair $(\mathcal{N}^*(-L), {}^tN(-2L))$, where $\mathcal{N}^* := \mathcal{H}om(\mathcal{N}, \mathcal{O}_Y)$.*

(3) *Two invertible sheaves on $Z^0$, $\mathcal{L}_1^0, \mathcal{L}_2^0$, are isomorphic if and only if there is an isomorphism $\Psi : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\Psi \circ N_1 = N_2 \circ \Psi(-L)$.*

(4) *Effective Weil divisors $D$ on $Z$ correspond to points $[\delta]$ of some projective space $\mathbb{P}(H^0(\mathcal{N}))$, for some isomorphism class of a pair $(\mathcal{N}, N)$ as above. The sum $D_1 + D_2$ corresponds to the image of $\delta_1 \otimes_{\mathcal{R}} \delta_2$ in the double dual $(\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2)^{**}$.*

**Proof.** (1) First of all we have that $(q \circ i)_*(\mathcal{L}_1^0 \otimes_{\mathcal{O}_{Z^0}} \mathcal{L}_2^0)$ is the saturation of the sheaf equal to $\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2$ on the open set $Y^0 = Y \setminus \mathrm{Sing}(\mathcal{B})$ (letting $j : Y^0 \to Y$ be the inclusion, the saturation of $\mathcal{F}$ is here for us $j_*(\mathcal{F}|_{Y^0})$).

Moreover, by definition, $\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2$ is the quotient of $\mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2$ by the submodule generated by the elements $(z \cdot n_1) \otimes_{\mathcal{O}_Y} n_2 - n_1 \otimes_{\mathcal{O}_Y} (z \cdot n_2)$, i.e., the submodule image of

$$N_1 \otimes_{\mathcal{O}_Y} \mathrm{Id}_{\mathcal{N}_2} - \mathrm{Id}_{\mathcal{N}_1} \otimes_{\mathcal{O}_Y} N_2 : \mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2(-L) \to \mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2.$$

Since the above is an antisymmetric map of vector bundles, its rank at one point is either 2, or 0: but the latter happens, as it is easy to verify, exactly when both $N_1$ and $N_2$ vanish, i.e., in a locus contained in the singular locus of the branch locus $\mathcal{B} = \{F = 0\}$, which has codimension 2 since we assume $Z$ to be normal.

Hence we have that $\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2$ is a rank 2 bundle at $y \in Y$ if either $N_1$ or $N_2$ do not vanish at $y$; in the contrary case, the sheaf $\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2$ has rank 4 at the point $y$, and one needs to take $(\mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2)^{**}$, where we set $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$.

We observe moreover that we have an infinite complex

$$\ldots \to (\mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2)(-2L) \to (\mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2)(-L) \to \mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2 \to \mathcal{N}_1 \otimes_{\mathcal{R}} \mathcal{N}_2 \to 0,$$

where the maps are given by

$$\psi' = N_1 \otimes_{\mathcal{O}_Y} \mathrm{Id}_{\mathcal{N}_2} + \mathrm{Id}_{\mathcal{N}_1} \otimes_{\mathcal{O}_Y} N_2,$$

respectively by the above $\psi$. The complex is exact on the right and also at the points where either $N_1$ or $N_2$ are not vanishing.

(2) Given the module $\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{R})$, its elements consist of elements $\phi = \phi_1 + z\phi_2$, where $\phi_1 \in \mathcal{N}^*$, and $\phi_2 : \mathcal{N} \to \mathcal{O}_Y(-L)$. $\mathcal{R}$-linearity is equivalent to $\phi(Nn) = z\phi(n)$, hence to

$$\phi_2(Nn) = \phi_1(n), \phi_1(Nn) = F\phi_2(n).$$

The first equation implies the second, hence $\phi_2$ determines $\phi$; moreover, given any $\phi_2 : \mathcal{N} \to \mathcal{O}_Y(-L)$, and defining $\phi_1(n) := \phi_2(Nn)$, we get an $\mathcal{R}$-linear homomorphism by the above formulae.

Clearly, multiplication by $z$ acts on $\phi$ by sending $\phi = \phi_1 + z\phi_2 \mapsto F\phi_2 + z\phi_1$, hence it sends $\phi_2 : \mathcal{N} \to \mathcal{O}_Y(-L)$ to $\phi_2 \circ N$, and it is given by ${}^tN(-2L) : \mathcal{N}^*(-L)(-L) \to \mathcal{N}^*(-L)$.

(3) Is easy to show, applying $(q \circ i)_*$ to the given isomorphism.

(4) A section of $\mathcal{L}$ determines, again by applying $(q \circ i)_*$, a section $\delta \in H^0(\mathcal{N})$, which represents the image of $1 \in \mathcal{O}_Y$ and which in turn determines the image of an element $\alpha + z\beta \in \mathcal{R}$: since we must have $\alpha + z\beta \mapsto \delta(\alpha) + N\delta(\beta)$. □

## 4. Line bundles on hyperelliptic curves

We apply the theory developed in the previous section to discuss the particular, but very interesting case, of line bundles on hyperelliptic curves. Here $Y = \mathbb{P}^1$, and $Z = C$ is a hyperelliptic curve of genus $g \geq 1$ defined by the equation

$$z^2 = F(x_0, x_1),$$

where $F$ is a homogeneous polynomial of degree $2g + 2$ without multiple roots.

The results of this section are also very relevant in order to show the complexity of the equations defining dihedral covers. In fact, a $D_n$-covering of $\mathbb{P}^1$, $X \to \mathbb{P}^1$, factors through a hyperelliptic curve $C$. In the case where $X \to C$ is étale, $X$ is determined by the choice of a line bundle $L \in \mathrm{Pic}^0(C)$ which is an element of $n$-torsion. This is the reason why we dedicate special attention to the description of $n$-torsion line bundles on hyperelliptic curves.

For shorthand notation we write $\mathcal{O}(d) := \mathcal{O}_{\mathbb{P}^1}(d)$ and we observe that every locally free sheaf on $\mathbb{P}^1$ is a direct sum of some invertible sheaves $\mathcal{O}(d_j)$.

Our first goal here is to parametrize the Picard group of $C$. The first preliminary remark is that it suffices to parametrize the subsets $\mathrm{Pic}^0(C)$, $\mathrm{Pic}^{-1}(C)$, since for any line bundle $\mathcal{L}$ we can choose an integer $d$ such that

$$\deg(\mathcal{L} \otimes_{\mathcal{O}_C} q^*(\mathcal{O}(d))) \in \{0, -1\}.$$

Let us then consider a line bundle $\mathcal{L}$ of degree zero or $-1$. In the case $\mathcal{L} = \mathcal{O}_C$ we know that the corresponding pair is the vector bundle

$$\mathcal{O} \oplus z\mathcal{O}(-g-1), N(\alpha + z\beta) = F\beta + z\alpha,$$

and the corresponding matrix is

$$N = \begin{pmatrix} 0 & F \\ 1 & 0 \end{pmatrix}$$

whose determinant yields a trivial factorization $F = 1 \cdot F$.

If instead we assume that $\mathcal{L}$ is nontrivial, i.e. it has no sections, we can write

$$\mathcal{N} := q_*(\mathcal{L}) = \mathcal{O}(-a) \oplus \mathcal{O}(-b), \ 0 < a \leq b, \ a + b = g + 1 - d, \ d = \deg(\mathcal{L}).$$

The integer $a$ is equal to

$$\min\{m | H^0(\mathcal{L} \otimes q^*\mathcal{O}(m)) \neq 0\},$$

and it determines a Zariski locally closed stratification of the Picard group. We fix $a$ and $d \in \{0, 1\}$ for the time being, and observe that then the vector bundle $\mathcal{N}$ is uniquely determined, whereas the matrix $N$ (determining the $\mathcal{R}$-structure) has the form

$$N = \begin{pmatrix} P & f \\ q & -P \end{pmatrix}, -\det(N) = (P^2 + qf) = F.$$

Here,

$$\deg(P) = g + 1, \deg(f) = g + 1 - a + b, \deg(q) = g + 1 + a - b.$$

Recall in fact that, in terms of fibre variables $(u, v)$, we have

$$zu = Pu + qv, zv = fu - Pv.$$

We see the matrix $N$ as providing a factorization

$$F - P^2 = qf.$$

It follows that $N$ is fully determined by the degree $g + 1$ polynomial $P$ and by a partial factorization of the polynomial $F - P^2 = \Pi_1^{2g+2} l_i(x_0, x_1)$ (here the $l_i$ are linear forms), as the product $fq$ of two polynomials of respective degrees $\deg(f) = g + 1 - a + b \geq g + 1$, $\deg(q) = g + 1 + a - b \leq g + 1$. Observe that the choice for the polynomial $q$ (hence of $f$) is only unique up to a constant $\lambda \in \mathbb{C}^*$, once the partial factorization is fixed.

Since for $P$ general the linear forms $l_i$ are distinct, we have in this case exactly $\frac{(2g+2)!}{(g+1-a+b)!(g+1+a-b)!}$ such possible factorizations.

Denote by $\mathcal{V}(a, b)$ the variety of such matrices: $\mathcal{V}(a, b) = \{(P, f, q) | P^2 + qf = F\}$. The preceding discussion shows that the variety $\mathcal{V}(a, b)$ parametrizing such matrices $N$ is a $\mathbb{C}^*$ bundle over a finite covering of degree $\frac{(2g+2)!}{(g+1-a+b)!(g+1+a-b)!}$ of the affine space $\mathbb{C}^{g+2}$ parametrizing our polynomials $P$. Hence $\mathcal{V}(a, b)$ is an affine variety of dimension $g + 3$. However, in order to get isomorphism classes of line bundles (elements of the Picard group) we must divide by the adjoint action of the group $\mathcal{G}$ of the automorphisms of the vector bundle $\mathcal{N} := q_*(\mathcal{L}) = \mathcal{O}(-a) \oplus \mathcal{O}(-b)$ which have determinant 1.

For $a = b$ we get $\mathcal{G} = SL(2, \mathbb{C})$, whereas for $a < b$ we get a triangular group of matrices

$$B = \begin{pmatrix} \lambda & \beta \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\beta$ is any homogeneous polynomial of degree $b - a$.

We also observe that the stabilizer of a matrix $N$ corresponds to an isomorphism of $\mathcal{L}$, hence to a scalar $\mu \in \mathbb{C}^*$ whose square equals $\det(B) = 1$, hence $\mu = \pm 1$.

We do not investigate here the GIT stability of the orbits, but just observe that the $\mathcal{V}(a, b)$ yields an open set of dimension $g$ in the case where $a = b$, else it gives a stratum of dimension $g + 1 - (b - a)$.

Recalling that $a + b = g + 1 - d$, for $d = 0$ the open set corresponds to: the case $a = b = \frac{g+1}{2}$ for $g$ odd, and the case $a = \frac{g}{2}$, $b = \frac{g}{2} + 1$ for $g$ even; similarly for $d = -1$.

Next we investigate the explicit description of tensor powers of an invertible sheaf $\mathcal{L}$ on the hyperelliptic curve $C$, with two motivations: the first one is to try to get useful results towards the description of dihedral coverings of the projective line, the second in order to describe torsion line bundles on hyperelliptic curves.

**Proposition 4.1.** *There are exact sequences*

$$0 \to \mathcal{K}_2 := \mathcal{O}(-a - b - (g + 1)) \to S^2(q_*\mathcal{L}) \to q_*(\mathcal{L}^2) \to 0,$$

$$0 \to \mathcal{K}_3 := \mathcal{K}_2 \otimes q_*\mathcal{L} \to S^3(q_*\mathcal{L}) \to q_*(\mathcal{L}^3) \to 0,$$

$$0 \to \mathcal{K}_n := \mathcal{K}_2 \otimes S^{n-2}(q_*\mathcal{L}) \to S^n(q_*\mathcal{L}) \to q_*(\mathcal{L}^n) \to 0.$$

**Proof.** The local sections of $\mathcal{N} = q_*\mathcal{L}$ can be written as pairs $(u, v)$. Hence $q_*(\mathcal{L}^2)$ is generated by $u^2, uv, v^2$, subject to the relation

$$u(fu - Pv) = u(zv) = z(uv) = v(zu) = v(Pu + qv)$$

$$\Leftrightarrow \Xi := fu^2 - 2Puv - qv^2 = 0.$$

That this is the only relation follows since the kernel $\mathcal{K}_2$ has rank 1 and first Chern class equal to $-a - b - (g + 1)$.

Similarly, $q_*(\mathcal{L}^3)$ is generated by the cubic monomials $u^3, u^2v, uv^2, v^3$, and we can simply multiply the relation $\Xi$ by $u$, respectively $v$.

In general, we simply observe that there is a natural morphism

$$q_{\mathcal{L}} : C \to \operatorname{Proj}(q_*\mathcal{L})$$

given by evaluation, and whose image is the relative quadric $\Gamma := \{(u, v) | \Xi(u, v) = 0\}$.

Recall that, since $\mathcal{L}$ is invertible, the polynomials $f, P, q$ cannot vanish simultaneously, in particular $\Gamma := \{\Xi(u, v) = 0\}$ is irreducible. Moreover the branch locus of $\Gamma \to \mathbb{P}^1$ is $\{(x_0, x_1) | fq + P^2 = F = 0\}$, therefore $\Gamma$ is isomorphic to $C$.

Hence $C$ is the hypersurface in $\mathbb{P}' := \operatorname{Proj}(q_*\mathcal{L})$ defined by $\Xi = 0$, where $\Xi$ is a section of $\mathcal{O}_{\mathbb{P}'}(2) \otimes q^*(\mathcal{K}_2)^{-1}$; hence we obtain the general exact sequence via the pushforward of

$$0 \to \mathcal{O}_{\mathbb{P}'}(n)(-C) = \mathcal{O}_{\mathbb{P}'}(n-2) \otimes q^*(\mathcal{K}_2) \to \mathcal{O}_{\mathbb{P}'}(n) \to \mathcal{O}_C(n) \to 0. \quad \square$$

**Corollary 4.2.** *A line bundle $\mathcal{L}$ of $n$-torsion on the hyperelliptic curve $C$ corresponds to a pair $(\mathcal{N}, N)$,*

$$\mathcal{N} = \mathcal{O}(-a) \oplus \mathcal{O}(-b),\ 0 < a \le b,\ a + b = g + 1,$$

$$N = \begin{pmatrix} P & f \\ q & -P \end{pmatrix}, \det(N) = -(P^2 + qf) = -F$$

*where* $\deg(P) = g + 1$, $\deg(f) = 2b$, $\deg(q) = 2a$, *such that the following linear map*

$$H^0(S^n(\mathcal{O}(a) \oplus \mathcal{O}(b))(-2)) \to H^0(S^{n-2}(\mathcal{O}(a) \oplus \mathcal{O}(b))(2g)),$$

*equal to the dual of $\mathcal{K}_2 \otimes S^{n-2}(q_*\mathcal{L}) \to S^n(q_*\mathcal{L})$ twisted by $(-2)$, is not surjective.*

**Proof.** Let $\mathcal{L}$ be a degree zero line bundle on $C$, so that we have $a + b = g + 1$. The condition that $\mathcal{L}^n$ is trivial is clearly equivalent to the condition $H^0(q_*(\mathcal{L}^n)) \neq 0$. In view of the exact cohomology sequence (here $H^0(S^n(q_*\mathcal{L})) = H^0(S^n(\mathcal{N})) = 0$ since $a, b > 0$):

$$0 \to H^0(\mathcal{K}_n) = 0 \to H^0(S^n(q_*\mathcal{L})) = 0 \to H^0(q_*(\mathcal{L}^n))$$

$$\to H^1(\mathcal{K}_n) \to H^1(S^n(q_*\mathcal{L})) \to H^1(q_*(\mathcal{L}^n)) \to 0,$$

the condition $H^0(q_*(\mathcal{L}^n)) \neq 0$ is equivalent to the non injectivity of $H^1(\mathcal{K}_n) \to H^1(S^n(q_*\mathcal{L}))$, equivalently, to the non surjectivity of the homomorphism of Serre dual vector spaces:

$$\begin{aligned}&\operatorname{coker}(H^0(S^n(\mathcal{N}^*)(-2)) \to H^0(S^{n-2}(\mathcal{N}^*)(2g))) \neq 0\\ &\quad\Leftrightarrow \operatorname{coker}(H^0(S^n(\mathcal{O}(a)\oplus\mathcal{O}(b))(-2)) \to H^0(S^{n-2}(\mathcal{O}(a)\oplus\mathcal{O}(b))(2g))) \neq 0.\end{aligned}$$

For $n = 2$ this means that, denoting by $\mathcal{A}[m] = H^0(\mathcal{O}(m))$ the space of homogeneous polynomials of degree $m$, the linear map

$$\mathcal{A}[2a-2] \oplus \mathcal{A}[a+b-2] \oplus \mathcal{A}[2b-2] \to \mathcal{A}[2a+2b-2]$$

given by the matrix $(f, -2P, q)$ is not surjective (equivalently, the linear map given by the matrix $(f, P, q)$ is not surjective).

Writing in non homogeneous coordinates

$$f = \sum_i^{2b} f_i x^i,\ P = \sum_i^{a+b} P_i x^i,\ q = \sum_i^{2a} q_i x^i,$$

this means that the matrix

$$A(f,P,q) = \begin{pmatrix} f_0 & f_1 & \dots & f_{2b-1} & f_{2b} & 0 & 0 & \dots & 0\\ 0 & f_0 & f_1 & \dots & f_{2b-1} & f_{2b} & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & f_0 & f_1 & \dots & f_{2b-1} & f_{2b}\\ P_0 & P_1 & \dots & P_{a+b} & 0 & 0 & 0 & \dots & 0\\ 0 & P_0 & P_1 & \dots & P_{a+b} & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & P_0 & P_1 & \dots & P_{a+b}\\ q_0 & q_1 & \dots & q_{2a} & 0 & 0 & 0 & \dots & 0\\ 0 & q_0 & q_1 & \dots & q_{2a} & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & q_0 & q_1 & \dots & q_{2a} \end{pmatrix},$$

does not have maximal rank. $\square$

**Remark 4.3.** The reader may notice the similarity of the matrix $A(f, P, q)$ with the matrix giving the resultant of two homogeneous polynomials in two variables. Moreover, notice that $A(f, P, q)$ is a $(3a+3b-3)\times(2a+2b-1)$ matrix, hence the condition that its rank be at most $2a+2b-2$ amounts to a codimension $g = a+b-1$ condition, which is the expected codimension of the set of n-torsion points in $\mathrm{Pic}^0(C)$.

### *4.1. Powers of divisorial sheaves on double covers*

We would now like to show how the result of Proposition 4.1 generalizes to any flat double cover $q : Z \to Y$. Let $\mathcal{L}$ be a divisorial sheaf on $Z$, and $(\mathcal{N}, N)$ the associated pair, where $\mathcal{N} = q_*(\mathcal{L})$. There is a natural map $q_{\mathcal{L}} : Z \dashrightarrow \mathrm{Proj}(\mathcal{N})$ given by evaluation, and which is a morphism on the smooth locus of $Z$.

Set $\mathbb{P}' := \mathrm{Proj}(\mathcal{N})$, and let $\Gamma$ be the image of $q_{\mathcal{L}}$. Clearly $Z$ is birational to $\Gamma$, which is a Gorenstein variety, since it is a divisor in $\mathbb{P}'$, and $q_{\mathcal{L}}$ is bijective outside of the inverse image of the branch locus $\mathcal{B}_q$. Moreover $q$ factors through $q_{\mathcal{L}}$ and the natural projection $\pi : \mathbb{P}' \to Y$.

At the points where $\mathcal{L}$ is invertible, then $\mathcal{L}$ is isomorphic with $\mathcal{O}_Z$, with an isomorphism compatible with the projection $q$, hence we conclude that $q_{\mathcal{L}}$ is an embedding on the smooth locus of $Z$.

At a point $P$ where $\mathcal{L}$ is not invertible, we use the local description of the pair $(\mathcal{N}, N)$ as $(\mathcal{O}_Y^2, N)$, where $N$ is the matrix sending two local generators $x, y$ to $ax+cy$, respectively to $bx-ay$. Since $\mathcal{L}$ is not invertible, by Lemma 3.1 we get that $a, b, c$ vanish at $P$. We use now the relations

$$xz = ax + cy \Leftrightarrow x(z-a) = cy,\ \ yz = bx - ay \Leftrightarrow y(z+a) = bx$$

to obtain

$$(x : y) = (c : z - a) = (z + a : b).$$

These formulae (observe that $(c : z - a) = (z + a : b)$ since $z^2 - a^2 = bc$) show that, at the points $P$ where $a, b, c$ (hence also $z$) vanish, the rational map $q_{\mathcal{L}}$ blows up the point $q^{-1}(P)$ to the whole fibre $\mathbb{P}^1$ lying over $P$. Hence $\Gamma$ is a small resolution of $Z$, and we have that the inverse of $q_{\mathcal{L}}$ is obtained blowing down these $\mathbb{P}^1$'s lying over such points $P$.

Since we have an isomorphism $Z^0 \cong \Gamma^0$, we see that the line bundle $\mathcal{O}_\Gamma(1)$ restricts to $\mathcal{L}$ on $Z^0$, hence the sections of $\mathcal{L}^n$ on $Z^0$ correspond to sections of $\mathcal{O}_\Gamma(n)$ on $\Gamma^0$.

Notice that Pic($\mathbb{P}'$) is generated by Pic($Y$) and by $\mathcal{O}_{\mathbb{P}'}(1)$, hence there is an invertible sheaf $\mathcal{K}_2$ on $Y$ such that $\Gamma$ is the zero set of a section of $\mathcal{O}_{\mathbb{P}'}(1) \otimes \pi^*(\mathcal{K}_2)^{-1}$.

Consider now the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}'}(n)(-\Gamma) = \mathcal{O}_{\mathbb{P}'}(n-2) \otimes \pi^*(\mathcal{K}_2) \to \mathcal{O}_{\mathbb{P}'}(n) \to \mathcal{O}_\Gamma(n) \to 0.$$

Taking the direct image, and observing that $\pi_* \mathcal{O}_\Gamma(n) = q_*(\mathcal{L}^n)$, we obtain the following.

**Proposition 4.4.** *For each divisorial sheaf $\mathcal{L}$ on $Z$ there is an exact sequence*

$$0 \to \mathcal{K}_n := \mathcal{K}_2 \otimes S^{n-2}(q_*\mathcal{L}) \to S^n(q_*\mathcal{L}) \to q_*(\mathcal{L}^n) \to 0.$$

## 5. Dihedral field extensions and generalities on dihedral covers

In this section we describe dihedral field extensions $\mathbb{C}(Y) \subset \mathbb{C}(X)$ in the case where $Y$ is factorial and $X$ is normal, thus we obtain a birational classification of $D_n$-covers. Recall that for any $G$-cover $\pi : X \to Y$ of normal varieties the field extension $\mathbb{C}(Y) \subset \mathbb{C}(X)$ is Galois with group $G$ (a $G$-extension); on the other hand any such field extension determines $\pi$ as the normalization of $Y$ in $\mathbb{C}(X)$.

In the second part of this section we determine the geometric building data that make the above normalization process explicit. This is important, for instance to calculate invariants of $X$, to determine the direct images of basic sheaves on $X$ (e.g. $\pi_*\mathcal{O}_X$, $\pi_*\Omega_X$, $\pi_*\Theta_X$, etc.) and to provide explicitly families of such covers.

### 5.1. Dihedral field extensions

Let us fix the following presentation for the dihedral groups:

$$D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

The split exact sequence

$$0 \to \langle \sigma \rangle = \mathbb{Z}/n\mathbb{Z} \to D_n \to \mathbb{Z}/2\mathbb{Z} \to 0$$

gives a factorization of the $D_n$-cover $\pi : X \to Y$ as the composition of two intermediate cyclic covers: $\pi = q \circ p$, where $p : X \to Z$ is a $\mathbb{Z}/n\mathbb{Z}$-cover, $q : Z \to Y$ is a $\mathbb{Z}/2\mathbb{Z}$-cover, $Z := X/\langle \sigma \rangle$. If $X$ is normal and irreducible, then also $Z$ and $Y$ are so, and we have the following chain of field extensions:

$$\mathbb{C}(Y) \subset \mathbb{C}(Z) \subset \mathbb{C}(X),$$

where the field of rational functions on $Z$ is the field $\mathbb{C}(X)^{\langle \sigma \rangle}$ of invariant functions under $\sigma$.

Let us recall, following [20], the structure of cyclic extensions $\mathbb{C}(W) \subset \mathbb{C}(V)$, where $W, V$ are normal varieties. Here the Galois group $G$ is cyclic of order $m$, $G \cong \mathbb{Z}/m\mathbb{Z}$; later we use this description with $m = 2$ and with $m = n$ to study $D_n$-field extensions. Let $\sigma \in G$ be a generator and let $\zeta \in \mathbb{C}$ be a primitive $m$th root of the unity. Then there exists $v \in \mathbb{C}(V)$ and $f \in \mathbb{C}(W)$, such that

$$\begin{cases} \mathbb{C}(V) & = \mathbb{C}(W)(v), \quad v^m = f, \\ \sigma(v) & = \zeta v. \end{cases} \tag{4}$$

Concretely, $v$ can be chosen to be any non-zero element of the form

$$v = \sum_{i=0}^{m-1} \zeta^{m-i} \sigma^i(\tilde{v}), \quad \tilde{v} \in \mathbb{C}(V).$$

Furthermore, it is possible to choose $v$ and $f$, such that (4) holds and

$$f = \prod_{i=1}^{m-1} (\delta_i)^i, \tag{5}$$

where $\delta_1, \dots, \delta_{m-1}$ are regular sections of invertible sheaves on $W \setminus \mathrm{Sing}(W)$. To see this, let $\hat{v}$ and $\hat{f}$ satisfy (4), and consider the Weil divisor associated to $\hat{f}$, $(\hat{f}) = \sum_U v_U(\hat{f})U$, where $U \subset W$ varies among the prime divisors of $W$ and $v_U$ is the valuation of $U$. On the non-singular locus $W \setminus \mathrm{Sing}(W)$, $U \cap (W \setminus \mathrm{Sing}(W))$ is an effective Cartier divisor, hence $U \cap (W \setminus \mathrm{Sing}(W)) = \{\delta_U = 0\}$, where $\delta_U$ is a regular section of an invertible sheaf on $W \setminus \mathrm{Sing}(W)$, hence

$$\hat{f} = \prod_U \delta_U^{v_U(\hat{f})}.$$

Let now $v_U(\hat{f}) = mq_U(\hat{f}) + r_U(\hat{f})$, where $q_U(\hat{f}), r_U(\hat{f}) \in \mathbb{Z}$, $0 \le r_U(\hat{f}) < m$, and define

$$\delta_i = \prod_{r_U(\hat{f})=i} \delta_U, \quad i = 1, \dots, m-1\,;$$

then the claim holds true with $v := \hat{v}\prod_U(\delta_U)^{-q_U(\hat{f})}$.

**Remark 5.1.** If $W$ is factorial, the group of Weil divisors coincides with that of Cartier divisors, hence $\delta_i$ corresponds to a Weil divisor $D_i$ which is reduced but not necessarily irreducible. Geometrically $D_i$ is the divisorial part of the branch locus $\mathcal{B} = \sum_{i=1}^{m-1} D_i$ where the local monodromy is $\sigma^i$ and $v$ is a rational section of a line bundle $L$ on $W$ which satisfies the linear equation

$$mL \equiv \sum_{i=1}^{m-1} iD_i. \tag{6}$$

Conversely, one can construct in a natural way a $\mathbb{Z}/m\mathbb{Z}$-cover starting from a line bundle $L$ and effective reduced divisors without common components $D_1, \dots, D_{m-1}$, such that (6) holds [6,20].

The following proposition describes dihedral field extensions (see also [8]).

**Proposition 5.2.** *Let $\mathbb{C}(Y) \subset \mathbb{C}(X)$ be a $D_n$-extension. Then there exist $a, F \in \mathbb{C}(Y)$ and $x \in \mathbb{C}(X)$, such that*

$$\mathbb{C}(X) = \mathbb{C}(Y)(x), \qquad x^{2n} - 2ax^n + F^n = 0,$$

*with $D_n$-action given as follows: $\sigma(x) = \zeta x$, $\tau(x) = F/x$, where $\zeta \in \mathbb{C}$ is a primitive nth root of 1.*

*Conversely, given $a, F \in \mathbb{C}(Y)$, such that $x^{2n} - 2ax^n + F^n \in \mathbb{C}(Y)[x]$ is irreducible, then $\frac{\mathbb{C}(Y)[x]}{(x^{2n}-2ax^n+F^n)}$ is a $D_n$-field extension of $\mathbb{C}(Y)$ with $D_n$-action given as before. Hence the normalization of $Y$ in $\frac{\mathbb{C}(Y)[x]}{(x^{2n}-2ax^n+F^n)}$ is a $D_n$-covering of $Y$.*

**Proof.** Consider the quotient $Z := X/\langle\sigma\rangle$ of $X$ by the cyclic subgroup $\langle\sigma\rangle \cong \mathbb{Z}/n\mathbb{Z}$ and let $q : Z \to Y$ be the induced double cover. From the previous description of cyclic field extensions, we have that

$$\begin{cases} \mathbb{C}(Z) & = \mathbb{C}(Y)(z), \quad z^2 = f \in \mathbb{C}(Y) \\ \bar{\tau}(z) & = -z \end{cases}$$

where $\bar{\tau} \in D_n/\langle\sigma\rangle$ is the class of $\tau$. Since $Y$ is factorial we can assume that $f$ is a regular section of an invertible sheaf on $Y$.

Now consider the extension $\mathbb{C}(Z) \subset \mathbb{C}(X)$. Let $x \in \mathbb{C}(X)$, $g \in \mathbb{C}(Z)$, such that

$$\begin{cases} \mathbb{C}(X) & = \mathbb{C}(Z)(x), \quad x^n = g, \\ \sigma(x) & = \zeta x, \end{cases}$$

where $\zeta \in \mathbb{C}$ is a primitive $n$th root of 1, and notice that $g \in \mathbb{C}(Z)$ can be written as

$$g = a + zb, \qquad \text{with} \quad a, b \in \mathbb{C}(Y).$$

Without loss of generality $b = 1$ in the previous formula, otherwise we replace $f$ with $b^2f$ and $z$ with $bz$. Moreover, $x\tau(x)$ is invariant under the action of $D_n$, therefore

$$x\tau(x) = F \in \mathbb{C}(Y), \quad \text{equivalently} \quad \tau(x) = F/x.$$

Finally, $F^n = x^n\tau(x^n) = g\bar{\tau}(g) = (a+z)(a-z) = a^2 - f$, hence

$$f = a^2 - F^n.$$

To conclude, we observe that $z \in \mathbb{C}(Y)(x)$ because $x^n = a + z$, so $\mathbb{C}(X) = \mathbb{C}(Y)(x)$ and by construction it follows that

$$x^{2n} - 2ax^n + F^n = 0.$$

For the last statement, notice that the field extension

$$\mathbb{C}(Y) \subset \frac{\mathbb{C}(Y)[x]}{(x^{2n} - 2ax^n + F^n)}$$

is Galois with group $D_n$, indeed the conjugates of $x$, namely $\zeta^i x$ and $\zeta^i F/x$, for $i = 0, \dots, n-1$, belong to $\frac{\mathbb{C}(Y)[x]}{(x^{2n}-2ax^n+F^n)}$. $\square$

Let $\pi : X \to Y$ be a flat $D_n$-cover with $Y$ smooth. Then $\pi_*\mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_Y$-modules, i.e. a vector bundle on $Y$. Recall that the sheaf $\pi_*\mathcal{O}_X$ carries a natural structure of $\mathcal{O}_Y$-algebras, which is given by the product of regular functions on $X$,

$$m : \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X \,; \tag{7}$$

the $D_n$-action on $X$ gives to $\pi_*\mathcal{O}_X$ the structure of a $D_n$-sheaf (see below). On the other hand, the variety $X$ is completely determined by $\pi_*\mathcal{O}_X$ and $m$, since $X = \mathrm{Spec}(\pi_*\mathcal{O}_X)$ ([21, II, 5.17]); the $D_n$-action on $X$ is given by the structure of $D_n$-sheaf on $\pi_*\mathcal{O}_X$.

Recall that, for any finite group $G$ the *regular representation* of $G$, $\mathbb{C}[G]$, is the vector space with a basis $\{e_g\}_{g\in G}$ indexed by the elements of $G$; for any $h \in G$, an endomorphism of $\mathbb{C}[G]$ is defined by $e_g \mapsto e_{hg}$, for all $g \in G$. In a similar way one defines a sheaf of $\mathcal{O}_Y$-algebras $\mathcal{O}_Y[G]$, for any variety $Y$. A sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules is a $G$-**sheaf**, if it has a structure of sheaf of $\mathcal{O}_Y[G]$-modules. If moreover $\mathcal{F}$ is a vector bundle, then its fibres carry a linear $G$-action and so we can see $\mathcal{F}$ as a family of representations of $G$ parametrized by $Y$.

For any representation $\rho : G \to \mathrm{GL}(V)$ of $G$, its *canonical decomposition* (see [22, §2.6]) is the unique decomposition

$$V = V_1 \oplus \cdots \oplus V_N$$

defined as follows. Let $W_1, \ldots, W_N$ be the different irreducible representations of $G$. Then each $V_i$ is the direct sum of all the irreducible representations of $G$ in $V$ that are isomorphic to $W_i$. If $\chi_1, \ldots, \chi_N$ are the characters of $W_1, \ldots, W_N$, and $n_i = \dim W_i$, then

$$p_i = \frac{n_i}{|G|} \sum_{g\in G} \overline{\chi_i(g)}\rho(g) \in \mathrm{End}(V) \tag{8}$$

is the projection of $V$ onto $V_i$, for any $i = 1, \ldots, N$, where $\overline{\chi_i(g)}$ is the complex-conjugate of $\chi_i(g)$.

Let now $\mathcal{F}$ be a locally free $G$-sheaf on $Y$ with action $\rho : G \to \mathrm{GL}_{\mathcal{O}_Y}(\mathcal{F})$. Via (8) we define an endomorphism $p_i \in \mathrm{End}_{\mathcal{O}_Y}(\mathcal{F})$, for any $i = 1, \ldots, N$. Setting $\mathcal{F}_i := \mathrm{Im}(p_i)$, we have the following decomposition:

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_N.$$

For any $i = 1, \ldots, N$, $\mathcal{F}_i$ is the *eigensheaf* of $\mathcal{F}$ corresponding to the (irreducible) representation with character $\chi_i$. Notice that $\mathcal{F}_i$ is a vector sub-bundle of $\mathcal{F}$, for all $i$. In particular, when $\mathcal{F} = \pi_*\mathcal{O}_X$, $\pi : X \to Y$ is a flat $G$-cover, we have that $\pi_*\mathcal{O}_X$ is a locally free sheaf of $\mathcal{O}_Y[G]$-modules of rank one, i.e. the fibres of $\pi_*\mathcal{O}_X$ are isomorphic to $\mathbb{C}[G]$ as $G$-representations. Indeed, the previous procedure gives the decomposition $\pi_*\mathcal{O}_X = (\pi_*\mathcal{O}_X)_1 \oplus \cdots \oplus (\pi_*\mathcal{O}_X)_N$, with $(\pi_*\mathcal{O}_X)_i \subset \pi_*\mathcal{O}_X$ a sub-bundle, for any $i$. By construction, the fibres of $(\pi_*\mathcal{O}_X)_i$ are isomorphic to each other as $G$-representations. So, it is enough to consider the restriction of $\pi_*\mathcal{O}_X$ on the complement $Y \setminus \mathcal{B}$ of the branch divisor $\mathcal{B}$, where the assertion follows easily.

In order to describe the sheaves $(\pi_*\mathcal{O}_X)_i$, $i = 1, \ldots, N$, when $\pi : X \to Y$ is a flat $D_n$-cover, let us briefly recall the representation theory of the dihedral groups. They depend on the parity of $n$.

<u>$n$ odd</u>. There are two irreducible representations of degree 1 with characters $\chi_1$ and $\chi_2$,

| | $\sigma^k$ | $\sigma^k\tau$ |
|---|---|---|
| $\chi_1$ | 1 | 1 |
| $\chi_2$ | 1 | $-1$, |

and $\frac{n-1}{2}$ irreducible representations of degree 2,

$$\rho^\ell(\sigma^k) = \begin{pmatrix} \zeta^{k\ell} & 0 \\ 0 & \zeta^{-k\ell} \end{pmatrix}, \qquad \rho^\ell(\sigma^k\tau) = \begin{pmatrix} 0 & \zeta^{k\ell} \\ \zeta^{-k\ell} & 0 \end{pmatrix}, \tag{9}$$

where $\zeta \in \mathbb{C}^*$ is a primitive $n$th root of 1, $1 \le \ell \le \frac{n-1}{2}$, $k = 0, \ldots, n-1$.

<u>$n$ even</u>. In this case there are 4 representations of degree 1 with characters $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$,

| | $\sigma^k$ | $\sigma^k\tau$ |
|---|---|---|
| $\chi_1$ | 1 | 1 |
| $\chi_2$ | 1 | $-1$ |
| $\chi_3$ | $(-1)^k$ | $(-1)^k$ |
| $\chi_4$ | $(-1)^k$ | $(-1)^{k+1}$, |

and, for any $1 \le \ell \le \frac{n}{2} - 1$, the irreducible representation $\rho^\ell$ defined by (9).

As a consequence we have that

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \bigoplus_{\ell=1}^{\frac{n-1}{2}} (\pi_*\mathcal{O}_X)_\ell, \qquad \text{if } n \text{ is odd},$$

and

$$\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{M} \oplus \mathcal{N} \bigoplus_{\ell=1}^{\frac{n}{2}-1} (\pi_*\mathcal{O}_X)_\ell, \qquad \text{if } n \text{ is even,}$$

where $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are the line bundles corresponding to the 1-dimensional representations with characters respectively $\chi_2, \chi_3, \chi_4$.

Notice that the sections of $\mathcal{L}$ are invariant under the action of the subgroup $\langle\sigma\rangle \cong \mathbb{Z}/n\mathbb{Z} \le D_n$, hence they are regular functions on $Z$. Moreover we have:

$$q_*\mathcal{O}_Z = \mathcal{O}_Y \oplus \mathcal{L}, \quad \mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(-\mathcal{B}_q), \tag{10}$$

where $\mathcal{B}_q \subset Y$ is the branch divisor of $q$.

If $n$ is even, the line bundles $\mathcal{M}$ and $\mathcal{N}$ have a similar interpretation, they arise from the $(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$-cover $X/\langle\sigma^2, \tau\rangle \to Y$.

In order to get further information on the rank 4 vector bundles $(\pi_*\mathcal{O}_X)_\ell$, we assume that $X$ is normal and use the factorization $\pi = q \circ p$ and the theory of cyclic covers. Let us denote with $Z^0$ the smooth locus of $Z$, $X^0 = p^{-1}(Z^0)$ and $p^0 : X^0 \to Z^0$ be the restriction of $p$. From the structure theorem for cyclic covers [6,7,20] it follows that $p^0$ is determined by divisor classes $L_1, \dots, L_{n-1}$ and reduced effective divisors $D_1^0, \dots, D_{n-1}^0 \subset Z^0$ without common components, such that

$$L_i + L_j \equiv L_{\overline{i+j}} - \sum_{k=1}^{n-1} \varepsilon_{i,j}^k D_k^0, \tag{11}$$

where $\overline{i+j} \in \{0, \dots, n-1\}$, $\overline{i+j} = i + j (\bmod n)$, $L_0 := \mathcal{O}_{Z^0}$.

Let us briefly recall the geometric interpretation of the previous data. For any $i = 0, \dots, n-1$, the line bundle $\mathcal{O}(L_i)$ is the subsheaf of $(p^0)_*\mathcal{O}_{X^0}$ consisting of the regular functions $f$ on $X^0$ such that $\sigma^* f = \exp\left(\frac{2\pi\sqrt{-1}}{n} i\right) f$. For any $k = 1, \dots, n-1$, the divisor $D_k^0 \subset Z^0$ is the union of the components $\Delta$ of the branch divisor $\mathcal{B}_{p^0}$ of $p^0$ such that, for any component $T \subset (p^0)^{-1}(\Delta)$, the stabilizer of the generic point of $T$ is the cyclic subgroup of $\langle\sigma\rangle$ generated by $\sigma^k$ and there is a uniformizing parameter $x \in \mathcal{O}_{X^0,T}$ such that $(\sigma^k)^* x = \exp\left(\frac{2\pi\sqrt{-1}}{|\langle\sigma^k\rangle|}\right) x$, where $|\langle\sigma^k\rangle|$ is the order of $\sigma^k$. For every $i, j = 0, \dots, n-1$, $\varepsilon_{i,j}^k$ is defined as follows: let $\imath_i(k), \imath_j(k) \in \{0, \dots, |\langle\sigma^k\rangle| - 1\}$ be such that

$$\exp\left(\frac{2\pi\sqrt{-1}}{n} ik\right) = \exp\left(\frac{2\pi\sqrt{-1}}{|\langle\sigma^k\rangle|}\right)^{\imath_i(k)},$$

$$\exp\left(\frac{2\pi\sqrt{-1}}{n} jk\right) = \exp\left(\frac{2\pi\sqrt{-1}}{|\langle\sigma^k\rangle|}\right)^{\imath_j(k)}$$

respectively, then

$$\varepsilon_{i,j}^k = \begin{cases} 1, & \text{if } \quad \imath_i(k) + \imath_j(k) \ge |\langle\sigma^k\rangle| \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

**Proposition 5.3.** *Let $Y$ be a smooth variety and let $\pi : X \to Y$ be a flat $D_n$-cover with $X$ normal. Let $p : X \to Z$ and $q : Z \to Y$ be the intermediate cyclic covers defined previously. Then the following holds true.*

(i) *$p_*\mathcal{O}_X = \bigoplus_{i=0}^{n-1} \mathcal{F}_i$, where $\mathcal{F}_0 = \mathcal{O}_Z$ and, for any $i = 1, \dots, n-1$, $\mathcal{F}_i$ is a divisorial sheaf on $Z$. For any $i, j = 1, \dots, n-1$ the product (7) induces an isomorphism as follows:*

$$(\mathcal{F}_i \otimes_{\mathcal{O}_Z} \mathcal{F}_j)^{**} \cong \left(\mathcal{F}_{\overline{i+j}} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z\left(-\sum_{k=1}^{n-1} \varepsilon_{i,j}^k D_k\right)\right)^{**}, \tag{13}$$

*where $\overline{i+j} \in \{0, \dots, n-1\}$, $\overline{i+j} = i + j(\bmod n)$, for any $k = 1, \dots, n-1$, $D_k = \overline{D_k^0}$ is the closure of the divisor $D_k^0$ in (11), and $()^{**}$ is the double dual in the category of $\mathcal{O}_Z$-modules.*

(ii) *For any $i = 1, \dots, n-1$, $U_i := q_*(\mathcal{F}_i)$ is a vector bundle of rank 2 on $Y$, in particular $\mathcal{F}_i$ is flat over $Y$. Furthermore,*

$$(\pi_*\mathcal{O}_X)_\ell = U_\ell \oplus U_{n-\ell},$$

*where $\ell = 1, \dots, \frac{n-1}{2}$, if $n$ is odd, and $\ell = 1, \dots, \frac{n}{2} - 1$ if $n$ is even; in the case where $n$ is even, $U_{\frac{n}{2}} = \mathcal{M} \oplus \mathcal{N}$.*

**Proof.** Consider the following Cartesian diagram:

$$\begin{array}{ccc} X^0 & \xrightarrow{\imath} & X \\ {\scriptstyle p^0}\downarrow & & \downarrow{\scriptstyle p} \\ Z^0 & \xrightarrow{\jmath} & Z \end{array} \tag{14}$$

where $Z^0$ is the smooth locus of $Z$, $\jmath : Z^0 \to Z$ is the inclusion, $X^0 = p^{-1}(Z^0)$, $\imath : X^0 \to X$ is the inclusion, and $p^0 = p_{|X^0}$. Notice that under our hypotheses $Z$ is normal.

Since $\mathcal{O}_X = \imath_* \mathcal{O}_{X^0}$ and $\mathcal{O}_Z = \jmath_* \mathcal{O}_{Z^0}$ (see, e.g. [23]),

$$\begin{aligned} p_* \mathcal{O}_X &= p_* \imath_* \mathcal{O}_{X^0} = (p \circ \imath)_* \mathcal{O}_{X^0} \\ &= \jmath_* p^0_* \mathcal{O}_{X^0} = \jmath_* \left( \mathcal{O}_{Z^0} \bigoplus_{i=1}^{n-1} \mathcal{O}_{Z^0}(L_i) \right) \\ &= \mathcal{O}_Z \bigoplus_{i=1}^{n-1} \jmath_* \mathcal{O}_{Z^0}(L_i). \end{aligned} \tag{15}$$

Let us define $\mathcal{F}_i := \jmath_* \mathcal{O}_{Z^0}(L_i)$, for any $i = 1, \ldots, n-1$. The product (7) gives morphisms $m_{ij} : \mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{F}_{\overline{i+j}}$. Since $\mathcal{F}_{\overline{i+j}}$ is reflexive, $m_{ij}$ is determined by its restriction on the smooth locus $Z^0$. On $Z^0$, by (11), $m_{ij}$ gives an isomorphism $\mathcal{O}_{Z^0}(L_i + L_j) \cong \mathcal{O}_{Z^0}(L_{\overline{i+j}} - \sum_{k=1}^{n-1} D_k^0)$. This implies (13).

To prove (ii), apply $q_*$ to (15) and use the following equalities: $\pi = q \circ p$, $q_* \mathcal{O}_Z = \mathcal{O}_Y \oplus \mathcal{L}$. Notice that, $q_* \mathcal{F}_i$ is locally free for any $i = 0, \ldots, n-1$, since $\pi_* \mathcal{O}_X$ is locally free and $\pi_* \mathcal{O}_X = \oplus_{i=0}^{n-1} q_* \mathcal{F}_i$. The equality $(\pi_* \mathcal{O}_X)_\ell = U_\ell \oplus U_{n-\ell}$ follows directly from the definition of $(\pi_* \mathcal{O}_X)_\ell \subset \pi_* \mathcal{O}_X$ as the eigensheaf corresponding to the representation $\rho^\ell$. Finally, in the case where $n$ is even, $\tau^*$ acts on $U_{\frac{n}{2}}$; $\mathcal{M}$ is the $\tau^*$-invariant subsheaf, while $\mathcal{N}$ is the $\tau^*$-anti-invariant one. □

Notice that (13) implies the following relation between $\mathcal{F}_1$ and the Weil divisors $D_1, \ldots, D_{n-1}$,

$$\left(\mathcal{F}_1^{\otimes n}\right)^{**} \cong \mathcal{O}_Z \left( -\sum_{k=1}^{n-1} k D_k \right),$$

which is the analogous of Remark 5.1 and Eq. (6) when the base of the cover is normal.

For later use we observe that the branch divisor $\mathcal{B}_{p^0} = \sum_{k=1}^{n-1} D_k^0$ of $p^0 : X^0 \to Z^0$ is invariant under the involution $\overline{\tau} : Z^0 \to Z^0$ induced by $\tau$, that is $\overline{\tau}^* \mathcal{B}_{p^0} = \mathcal{B}_{p^0}$, since $\sigma\tau = \tau\sigma^{-1}$. So, there exists an effective Cartier divisor $\Delta_{p^0} \subset Y^0$, such that $(q^0)^* \Delta_{p^0} = \mathcal{B}_{p^0}$, where $q^0 := q_{|Z^0}$. Let us define

$$\Delta_p := \overline{\Delta_{p^0}} \in \mathrm{Div}(Y). \tag{16}$$

**Remark 5.4.** Notice that, for any $i = 1, \ldots, n-1$, $\overline{\tau}(D_i) = D_{n-i}$, so $\overline{\tau}(D_i)$ does not have any common component with $D_i$ for any $i < n/2$, while, if $n$ is even, $\overline{\tau}(D_{\frac{n}{2}}) = D_{\frac{n}{2}}$.

In the remaining of the section we determine building data for $D_n$-covers of $Y$. To this aim we first derive some properties of the vector bundles $U_1, \ldots, U_{n-1}$ (**1**–**3** below).

**1.** For any $i = 1, \ldots, n-1$, the involution $\tau : X \to X$ induces an isomorphism $\tau^* : U_i \to U_{n-i}$, such that $\tau^* \circ \tau^*$ is the identity.

**Proof.** For any open $V \subset Y$, $U_i(V) := \mathcal{F}_i(q^{-1}(V))$ consists of the regular functions $f \in \mathcal{O}_X(\pi^{-1}(V))$ such that $\sigma^* f = \exp(\frac{2\pi\sqrt{-1}}{n} i) f$. From the relation $\sigma\tau = \tau\sigma^{-1}$ it follows that $f \mapsto \tau^* f$ gives a morphism $\tau^* : U_i(V) \to U_{n-i}(V)$ of $\mathcal{O}_Y$-modules. Since $\tau^2 = 1$, $\tau^* \circ \tau^* = \mathrm{Id}$ and $\tau^*$ is an isomorphism. □

**2.** For any $i = 1, \ldots, n-1$, $U_i$ has a structure of $\mathcal{R}$-module given by the restriction $m_i : \mathcal{L} \otimes U_i \to U_i$ of the product (7), where $\mathcal{R} = \mathcal{O}_Y \oplus \mathcal{L} = q_* \mathcal{O}_Z$ (see Section 3.). In particular $m_i^2 = \delta_q \cdot \mathrm{Id}_{U_i}$, where $\delta_q \in H^0(Y, \mathcal{O}_Y(\mathcal{B}_q))$ is such that $\mathcal{B}_q = \{\delta_q = 0\}$ ($\delta_q = F$ of Section 3). Furthermore, the isomorphism $\tau^* : U_i \to U_{n-i}$ allows us to identify $m_{n-i}$ with $-m_i$, as it follows from the commutativity of the following diagram and the fact that $m_i \circ (\tau^* \otimes \mathrm{Id}_{U_i}) = -m_i$

$$\begin{array}{ccc} \mathcal{L} \otimes U_i & \xrightarrow{m_i \circ (\tau^* \otimes \mathrm{Id}_{U_i})} & U_i \\ {\scriptstyle \mathrm{Id}_{\mathcal{L}} \otimes \tau^*} \downarrow & & \downarrow {\scriptstyle \tau^*} \\ \mathcal{L} \otimes U_{n-i} & \xrightarrow{m_{n-i}} & U_{n-i} \end{array} \tag{17}$$

**3.** For any $i, j = 1, \ldots, n-1$, the product (7) induces, by restriction, a morphism

$$m_{ij} : U_i \otimes U_j \to U_{\overline{i+j}},$$

where $\overline{i+j} \in \{0, \ldots, n-1\}$, $\overline{i+j} = i + j \pmod n$, and by definition $U_0 = q_* \mathcal{O}_Z$. In particular, for any $i = 1, \ldots, n-1$, there is a morphism

$$m_{i,n-i} : U_i \otimes U_{n-i} \to \mathcal{O}_Y \oplus \mathcal{L}.$$

In the following, we will denote again with $m_{i,n-i}$ the previous morphism under the identification $\tau^* : U_{n-i} \to U_i$, hence

$$m_{i,n-i} : U_i \otimes U_i \to \mathcal{O}_Y \oplus \mathcal{L}.$$

Let $m^+_{i,n-i} : U_i \otimes U_i \to \mathcal{O}_Y$ and $m^-_{i,n-i} : U_i \otimes U_i \to \mathcal{L}$ be the compositions of $m_{i,n-i}$ with the projections onto $\mathcal{O}_Y$ and $\mathcal{L}$ respectively. Notice that, with respect to the involution on $U_i \otimes U_i$ that exchanges the factors, $m^+_{i,n-i}$ is symmetric, while $m^-_{i,n-i}$ is antisymmetric, hence they can be seen as morphisms

$$m^+_{i,n-i} : \mathrm{Sym}^2(U_i) \to \mathcal{O}_Y, \qquad m^-_{i,n-i} : \wedge^2(U_i) \to \mathcal{L}, \tag{18}$$

or equivalently as sections

$$m^+_{i,n-i} \in H^0(Y, \mathrm{Sym}^2(U_i^\vee)),\ m^-_{i,n-i} \in H^0(Y, \wedge^2(U_i^\vee) \otimes \mathcal{L}). \tag{19}$$

**Proposition 5.5.** *For any $i = 1, \ldots, n-1$ the following statements hold true.*

(i) *$m^+_{i,n-i}$ is determined by $m_i$ and $m^-_{i,n-i}$.*
(ii) *The divisor of zeros of $m^-_{i,n-i}$ coincides with the divisor $\Delta_p$ defined in (16). In particular $m^-_{i,n-i}$ yields an isomorphism between $\wedge^2 U_i$ and $\mathcal{L} \otimes \mathcal{O}_Y(-\Delta_p)$.*

**Proof.** (i) Let $y \in Y$, let $(U_i)_y$ be the stalk of $U_i$ over $y$ and let $s_1, s_2 \in (U_i)_y$. Then

$$m_{i,n-i}(s_1 \otimes s_2) = s_1 \tau^*(s_2),$$

hence

$$m^+_{i,n-i}(s_1 \otimes s_2) = \frac{1}{2}\left(s_1\tau^*(s_2) + \tau^*(s_1)s_2\right),$$
$$m^-_{i,n-i}(s_1 \otimes s_2) = \frac{1}{2}\left(s_1\tau^*(s_2) - \tau^*(s_1)s_2\right);$$

where the product $s_1\tau^*(s_2)$ (respectively $\tau^*(s_1)s_2$) is the usual one between stalks of regular functions defined in some neighbourhood of $\pi^{-1}(y)$. For any $r \in (\mathcal{L})_y$, the associativity of the multiplication implies that

$$m_{i,n-i}\left(m_i(r \otimes s_1) \otimes s_2\right) = m\left(r \otimes m_{i,n-i}(s_1 \otimes s_2)\right)$$

and hence

$$m^\pm_{i,n-i}\left(m_i(r \otimes s_1) \otimes s_2\right) = m\left(r \otimes m^\mp_{i,n-i}(s_1 \otimes s_2)\right).$$

This implies that, under the natural identification $\mathcal{O}_{Y,y} \cong \mathrm{End}((\mathcal{L})_y)$, $m^+_{i,n-i}(s_1 \otimes s_2) = m^-_{i,n-i}(m_i((\_) \otimes s_1) \otimes s_2)$, and hence the claim follows.

(ii) Without loss of generality we assume that $Z$ is smooth. Indeed, since $Z$ is normal, its singular locus $\mathrm{Sing}(Z)$ has codimension $\geq 2$, so, for $Z^0 := Z \setminus \mathrm{Sing}(Z)$ and $Y^0 := q(Z^0)$, the divisor of zeros of $m^-_{i,n-i}$ and $\Delta_p$ are determined by their restrictions to $Y^0$.

For any $y \in Y$, let us consider the stalk $(m^-_{i,n-i})_y$ of $m^-_{i,n-i}$ at $y$. Let $u_i$, $v_i$ be a basis of $(U_i)_y$ as $\mathcal{O}_{Y,y}$-module, then $u_i \otimes v_i - v_i \otimes u_i$ is a basis of $(\wedge^2 U_i)_y$, viewed as the submodule of $U_i \otimes U_i$. We have:

$$\begin{aligned} m^-_{i,n-i}(u_i \otimes v_i - v_i \otimes u_i) &= u_i\tau^*(v_i) - v_i\tau^*(u_i) \\ &= u_i\tau^*(v_i) - \tau^*(u_i\tau^*(v_i)), \end{aligned} \tag{20}$$

where we consider $u_i$ and $v_i$ as stalks of regular functions defined on some open neighbourhood of $\pi^{-1}(y)$ in $X$, such that $\sigma^*(u_i) = \exp(\frac{2\pi\sqrt{-1}}{n}i)u_i$ and $\sigma^*(v_i) = \exp(\frac{2\pi\sqrt{-1}}{n}i)v_i$.

Let us choose local analytic coordinates $(y_1, \ldots, y_d)$ for $Y$ at $y$, and $w$ on $Z$, such that $Z$ is given locally by the equation $w^2 = y_1$. Furthermore, for any $i = 1, \ldots, n-1$, let $e_i$ be a basis of $\mathcal{F}_i$ as $\left(\frac{\mathbb{C}[y_1,\ldots,y_d,w]}{(w^2-y_1)}\right)$-module. Then let

$$u_i := 1 \cdot e_i, \quad v_i := w \cdot e_i. \tag{21}$$

Notice that we can choose the $e_i$'s in such a way that $\tau^*(e_i) = e_{n-i}$, for any $i = 1, \ldots, n-1$, since $\tau^*$ identifies $\mathcal{F}_i$ with $\mathcal{F}_{n-i}$. So, substituting (21) in (20) and using the equations $\tau^*(w) = -w$ and $\tau^*(e_i) = e_{n-i}$, we obtain:

$$\begin{aligned} m^-_{i,n-i}(u_i \otimes v_i - v_i \otimes u_i) &= u_i\tau^*(v_i) - \tau^*(u_i\tau^*(v_i)) \\ &= e_i\tau^*(we_i) - \tau^*(e_i\tau^*(we_i)) \\ &= e_i(-we_{n-i}) - \tau^*(e_i(-we_{n-i})) \\ &= -we_ie_{n-i} - we_ie_{n-i} \\ &= -2we_ie_{n-i} = -2wb_p, \end{aligned} \tag{22}$$

where we have used the fact that $e_i e_{n-i} = b_p$, $b_p$ is a local equation for $\mathcal{B}_p$ (this follows from (11)). Finally, let $\delta_p$ be a local equation for $\Delta_p$ such that $q^*(\delta_p) = b_p$, then the previous computation shows that

$$m^-_{i,n-i}(u_i \otimes v_i - v_i \otimes u_i) = -2wq^*(\delta_p),$$

from which the claim follows. □

The following result is a converse to Proposition 5.3.

**Theorem 5.6** (*Structure of $D_n$-covers*). *Let $Y$ be a smooth variety and $n$ be a positive integer. Then, to the following data* (a), (b) *and* (c), *we can associate a $D_n$-cover $\pi : X \to Y$ in a natural way.*

(a) *A line bundle $\mathcal{L}$ and an effective reduced divisor $\mathcal{B}_q$ on $Y$, such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(-\mathcal{B}_q)$.*
(b) *Reduced effective Weil divisors $D_1, \dots, D_{\lfloor \frac{n}{2} \rfloor}$ on $Z := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$ without common components, such that $\overline{\tau}(D_1 \cup \dots \cup D_{\lfloor \frac{n-1}{2} \rfloor})$ doesn't have common components with $D_1 \cup \dots \cup D_{\lfloor \frac{n-1}{2} \rfloor}$, and, in the case where $n$ is even, $\overline{\tau}(D_{\frac{n}{2}}) = D_{\frac{n}{2}}$; where $\lfloor a \rfloor$ denotes the integer part of a number $a \in \mathbb{Q}$, and $\overline{\tau}$ is the involution of the double cover $q : Z \to Y$.*
(c) *Divisorial sheaves $\mathcal{F}_1, \dots, \mathcal{F}_{\lfloor \frac{n}{2} \rfloor}$ on $Z$ flat over $\mathcal{O}_Y$, such that, for any $i, j = 1, \dots, n-1$,* (13) *holds, and if $n$ is even $\mathcal{F}_{\frac{n}{2}} = \overline{\tau}^*(\mathcal{F}_{\frac{n}{2}})$; where for $\lfloor \frac{n}{2} \rfloor < \ell, k \leq n-1$, $\mathcal{F}_\ell := \overline{\tau}^*(\mathcal{F}_{n-\ell})$ and $D_k := \overline{\tau}(D_{n-k})$, the coefficients $\varepsilon^k_{ij}$ are defined in* (12).

*The variety $X$ so constructed is normal if and only if, setting $\varkappa := \gcd\{k = 1, \dots, n-1 \,|\, D_k \neq 0\}$, then either $\varkappa = 1$, or $\frac{n}{\varkappa}\mathcal{F}_1 - \sum_{k=1}^{n-1} \frac{k}{\varkappa} D_k$ has order precisely $\varkappa$ in the group of divisorial sheaves of $Z$. In this case $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus_{i=1}^{\lfloor n/2 \rfloor} q_* \mathcal{F}_i$, in particular $\pi$ is flat.*

Before giving the proof of the previous theorem, two remarks are in order.

**Remark 5.7.** **1.** Using the results of Section 3, the divisorial sheaves in (c) correspond to pairs $(U_1, m_1), \dots, (U_{\lfloor \frac{n}{2} \rfloor}, m_{\lfloor \frac{n}{2} \rfloor})$ consisting of rank 2 vector bundles on $Y$, $U_1, \dots, U_{\lfloor \frac{n}{2} \rfloor}$, and morphisms $m_i : U_i \otimes \mathcal{L} \to U_i$ such that $m_i^2 = \delta_q \mathrm{Id}_{U_i}$, where $\mathcal{B}_q = \{\delta_q = 0\}$. With this notation, $\overline{\tau}^* \mathcal{F}_i$ corresponds to $(U_i, -m_i)$, for any $i = 1, \dots, n-1$. Furthermore, using Proposition 3.3, the relation (13) can be written in terms of the $(U_i, m_i)$'s and the $D_i$'s.
**2.** A similar criterion for the existence of $D_n$-covers $\pi : X \to Y$ has been given in [8], where $X$ is constructed as the normalization of $Y$ in a certain dihedral field extension of $\mathbb{C}(Y)$. Here, using the structure theorem for cyclic covers, we construct $\pi : X \to Y$ explicitly, we hope that this procedure could be useful for further investigations of Galois covers.

**Proof.** The data (a) determines a flat double cover $q : Z \to Y$ in the usual way, $Z := \mathrm{Spec}\,(\mathcal{O}_Y \oplus \mathcal{L})$ and $q$ is given by the inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y \oplus \mathcal{L}$ as the first summand. Notice that $Z$ is normal since $\mathcal{B}_q$ is reduced.

We define $X := \mathrm{Spec}\,(\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1})$, where $\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1}$ is the sheaf of $\mathcal{O}_Z$-algebras with algebra structure given by the isomorphisms (13) in the following way. Since $\mathcal{F}_{\overline{i+j}}$ is a divisorial sheaf, for $i, j = 1, \dots, n-1$, a morphism $\mathcal{F}_i \otimes \mathcal{F}_j \to \mathcal{F}_{\overline{i+j}}$ is uniquely determined by its restriction on the smooth locus $Z^0$ of $Z$. Let us fix firstly sections $\delta_k \in H^0(Z, \mathcal{O}_Z(D_k))$, such that $D_k = \{\delta_k = 0\}$ on $Z^0$, for any $k = 1, \dots, n-1$. By Lemma 3.1, on $Z^0$ the sheaves $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ are locally free, so, locally where they are trivial, we choose generators $e_1, \dots, e_{n-1}$ such that $e_{n-i} = \overline{\tau}^*(e_i)$, for any $i$. The algebra structure on the restriction of $\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1}$ on such open subsets is defined as usual by the equations

$$e_i e_j = e_{\overline{i+j}} \prod_{k=1}^{n-1} \delta_k^{\varepsilon^k_{ij}}, \quad \text{for any } i, j = 1, \dots, n-1.$$

Notice that, if we choose different generators, $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ satisfying the same conditions $\tilde{e}_{n-i} = \overline{\tau}^*(\tilde{e}_i)$, then we obtain an algebra canonically isomorphic to the previous one. Hence the construction globalizes and $\mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1}$ becomes a sheaf of $\mathcal{O}_Z$-algebras. The morphism $\pi : X \to Y$ is defined as usual.

From the previous construction and from the fact that $D_{n-k} = \overline{\tau}(D_k)$, it follows that $\overline{\tau}^* : \mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1} \to \mathcal{O}_Z \oplus \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_{n-1}$ is a morphism of $\mathcal{O}_Z$-algebras. This defines an involution $\tau : X \to X$. By construction we have that $\sigma^* \circ \tau^* = \tau^* \circ (\sigma^{-1})^*$, hence $X$ carries an action of $D_n$ such that $\pi : X \to Y$ is a $D_n$-cover.

The condition for $X$ to be normal follows from Theorem 1.1 in [20]. □

**Remark 5.8.** Similarly as for cyclic covers, $X$ in Theorem 5.6 is determined by the data (a), (b) and $\mathcal{F}_1$ (or $(U_1, m_1)$), such that

$$(\mathcal{F}_1^{\otimes n})^{**} \cong \mathcal{O}_Z\left(-\sum_{k=1}^{n-1} kD_k\right).$$

Furthermore, by Proposition 4.4, $(\mathcal{F}_1^{\otimes n})^{**} = \left(\mathrm{Sym}^n(U_1)\right)/\mathcal{K}_n$.

### 5.3. $D_3$-covers and triple covers

In this section we restrict ourselves to $D_3$-covers and relate the results that we obtained so far with the structure theorem for flat finite morphisms of degree 3 of algebraic varieties [13], also called *triple covers*. This relation originates from the following fact: for every $D_3$-cover $\pi : X \to Y$, any quotient $W := X/\langle \sigma^i \tau \rangle$ has a natural structure of triple cover of $Y$ induced by $\pi$; conversely, if $W \to Y$ is a triple cover which is not Galois, then its Galois closure is a $D_3$-cover. Since the elements $\sigma^i \tau$ are pairwise conjugate, the corresponding triple covers are isomorphic, hence in the following we consider only $X/\langle \tau \rangle$. Notice that the structure of $D_3$-covers has been studied also in [24].

Let $\pi : X \to Y$ be a $D_3$-cover. We have seen in the previous section that $\pi$ is determined by the locally free sheaves $\mathcal{L}, U_1, U_2$ and a morphism of $\mathcal{O}_Y$-modules

$$m : (\mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_2)^{\otimes 2} \to \mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_2$$

which gives an associative commutative product. The involution $\tau$ yields an isomorphism $\tau^* : U_1 \to U_2$ as explained in the previous section. Under this identification one easily see that $\pi : X \to Y$ is determined by $\mathcal{O}_Y, \mathcal{L}, U_1, m_{12}, m_1$ and $m_{11}$ (see the previous section for their definitions). In particular,

$$\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_1$$

and the involution $\tau^*$ acts on the element $(a, b, c, d) \in \mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_1$ as follows:

$$\tau^*(a, b, c, d) = (a, -b, d, c).$$

The subsheaf of $\tau^*$-invariants, $(\mathcal{O}_X)^{\tau^*} \subset \mathcal{O}_X$, consists of the elements of the form $(a, 0, c, c) \in \mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_1$, so the sheaf of regular functions on $W := X/\langle \tau \rangle$ is

$$\mathcal{O}_W \cong \mathcal{O}_Y \oplus U_1 \tag{23}$$

with inclusion in $\mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_1$ given by $(a, c) \mapsto (a, 0, c, c)$. The morphism $\pi : X \to Y$ descends to a morphism $f : W \to Y$, which is a triple cover. We refer to [13] for definitions, notations and results concerning triple covers.

**Proposition 5.9.** *Under the identification (23), $U_1$ coincides with the Tschirnhausen module of $\mathcal{O}_W$ over $\mathcal{O}_Y$, that is $U_1$ consists of the elements of $\mathcal{O}_W$ whose minimal cubic polynomial has no square term ($U_1 = E$ in the notation of [13]). Furthermore, the tensor $\phi_2$ in [13] coincides with $m_{11} : U_1 \otimes U_1 \to U_1$, which has been defined in the previous section.*

**Proof.** For $(0, c) \in \mathcal{O}_Y \oplus U_1$, we calculate its cubic power $(0, c)^3$. To this aim, we consider its image $(0, 0, c, c) \in \mathcal{O}_Y \oplus \mathcal{L} \oplus U_1 \oplus U_1$ and use the product $m$ of $\mathcal{O}_X$. In order to simplify the notation, we write $(0, 0, c, c) = c + \tau^*(c) \in \mathcal{O}_X$ and denote the product $m$ simply by $\cdot$. Then we have the following expression:

$$(c + \tau^*(c))^3 = c^3 + \tau^*(c)^3 + 2c\tau^*(c)\left(c + \tau^*(c)\right).$$

Since $c^3 + \tau^*(c)^3, 2c\tau^*(c) \in \mathcal{O}_Y$, the minimal cubic polynomial of $c + \tau^*(c)$ has no square term and so it belongs to the Tschirnhausen module of $\mathcal{O}_W$ over $\mathcal{O}_Y$. This proves the first claim.

For the second claim, recall that by definition $\phi_2$ is the composition of the product in $\mathcal{O}_W$ followed by the projection onto $U_1$. For any $c, c' \in U_1$, the product between $c + \tau^*(c), c' + \tau^*(c') \in \mathcal{O}_W$ has the following expression:

$$\left(c + \tau^*(c)\right)\left(c' + \tau^*(c')\right) = cc' + \tau^*(cc') + c\tau^*(c') + \tau^*(c\tau^*(c')).$$

Notice that $c\tau^*(c') + \tau^*(c\tau^*(c')) \in \mathcal{O}_Y$ and that $cc' + \tau^*(cc')$ corresponds to $(0, \tau^*(cc')) \in U_1$ under the identification (23). So $\phi_2$ coincides with $m_{11}$ under the identification $\tau^* : U_1 \to U_2$. $\square$

Let $f : W \to Y$ be a flat finite map of degree 3, let $E$ be the Tschirnhausen module of $\mathcal{O}_W$ over $\mathcal{O}_Y$ and let $\phi_2 : \mathrm{Sym}^2(E) \to E$ be the associated triple cover homomorphism, as defined in [13]. Under the standard identification $E \cong E^\vee \otimes \wedge^2(E) = \mathrm{Hom}(E, \wedge^2(E))$, $u \mapsto (v \mapsto u \wedge v)$, we can view $\phi_2$ as a morphism $\phi_2 : \mathrm{Sym}^2(E) \otimes E \to \wedge^2(E)$. By [13, Prop. 3.5], $\phi_2$ is symmetric, hence it induces a morphism $\Phi : \mathrm{Sym}^3(E) \to \wedge^2(E)$. The structure theorem for triple covers states that conversely $f : W \to Y$ is determined by a rank 2 locally free sheaf $E$ of $\mathcal{O}_Y$-modules and an $\mathcal{O}_Y$-morphism $\Phi : \mathrm{Sym}^3(E) \to \wedge^2(E)$ ([13, Thm. 3.6]). The proof of the fact that $\phi_2$ is symmetric given in [13] is done by a direct calculation using local coordinates and is a consequence of the fact that $E$ is the Tschirnhausen module of $\mathcal{O}_W$ over $\mathcal{O}_Y$.

We extend part of these results to $D_n$-covers, $n \geq 2$, and give a different proof of [13, Prop. 3.5] quoted above. For any $D_n$-cover $\pi : X \to Y$, there is a $\mathcal{O}_Y$-morphism

$$\phi_{n-1} : \mathrm{Sym}^{n-1}(U_1) \to U_1$$

defined as follows: for any $u_1 \otimes \cdots \otimes u_{n-1} \in \mathrm{Sym}^{n-1}(U_1)$, their product $m(u_1 \otimes \cdots \otimes u_{n-1})$ is in $U_{n-1}$, so applying $\tau^* : U_{n-1} \to U_1$ we obtain an element in $U_1$, this is by definition $\phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-1})$. Under the identification $U_1 \cong U_1^\vee \otimes \wedge^2(U_1)$ as before, we can see $\phi_{n-1}$ as a morphism

$$\phi_{n-1} : \mathrm{Sym}^{n-1}(U_1) \otimes U_1 \to \wedge^2(U_1).$$

**Proposition 5.10.** *The morphism $\phi_{n-1}$ is symmetric, hence induces a morphism of $\mathcal{O}_Y$-modules $\Phi_{n-1} : \mathrm{Sym}^n(U_1) \to \wedge^2(U_1)$.*

**Proof.** By the commutativity of $m$, $\Phi_{n-1}$ is symmetric in the first $n-1$ entries. So, it is enough to prove that

$$\Phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-1} \otimes v) = \Phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-2} \otimes v \otimes u_{n-1}), \tag{24}$$

for any $u_1, \ldots, u_{n-1}, v \in U_1$. To this aim, let us consider the morphism

$$m^-_{1,n-1} : \wedge^2(U_1) \to \mathcal{L}$$

defined in (18). By Proposition 5.5, $m^-_{1,n-1}$ identifies $\wedge^2(U_1)$ with $\mathcal{L}(-\Delta_p)$, hence (24) is equivalent to

$$m^-_{1,n-1}\left(\Phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-1} \otimes v)\right) = m^-_{1,n-1}\left(\Phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-2} \otimes v \otimes u_{n-1})\right). \tag{25}$$

From the explicit form of the isomorphism $U_1 \cong U_1^\vee \otimes \wedge^2(U_1)$ recalled before, we have that

$$\Phi_{n-1}(u_1 \otimes \cdots \otimes u_{n-1} \otimes v) = \tau^*(m(u_1 \otimes \cdots \otimes u_{n-1})) \wedge v.$$

Furthermore, by definition, $m^-_{1,n-1}(\tau^*(m(u_1 \otimes \cdots \otimes u_{n-1})) \wedge v)$ is the projection onto $\mathcal{L}$ of $m(\tau^*(m(u_1 \otimes \cdots \otimes u_{n-1})) \otimes \tau^*(v)) \in \mathcal{O}_Y \oplus \mathcal{L}$. Since $\tau^*$ commutes with $m$,

$$m\left(\tau^*(m(u_1 \otimes \cdots \otimes u_{n-1})) \otimes \tau^*(v)\right) = \tau^* m\left((m(u_1 \otimes \ldots \otimes u_{n-1})) \otimes v\right).$$

Using the associativity and the commutativity of $m$, we deduce that

$$\tau^* m\left((m(u_1 \otimes \ldots \otimes u_{n-1})) \otimes v\right) = \tau^* m\left((m(u_1 \otimes \ldots \otimes u_{n-2} \otimes v)) \otimes u_{n-1}\right).$$

Projecting both sides of this equation onto $\mathcal{L}$ we obtain (25), from which the statement follows. □

**Remark 5.11.** From Theorem 5.6 it follows that, for a $D_n$-cover $\pi : X \to Y$, the intermediate degree $n$ cover $f : W := X/\langle\tau\rangle \to Y$ embeds naturally in $\mathcal{O}_Y \oplus U_1 \oplus \cdots \oplus U_{\frac{n-1}{2}}$ if $n$ is odd, respectively in $\mathcal{O}_Y \oplus U_1 \oplus \cdots \oplus U_{\frac{n}{2}-1} \oplus \mathcal{M}$ when $n$ is even. Then the symmetric morphism $\Phi_{n-1} : \mathrm{Sym}^n(U_1) \to \wedge^2(U_1)$ determines $f : W \to Y$ birationally.

## 6. Simple and almost simple dihedral covers and their invariants

In this section we define the *simple dihedral covers* and the *almost simple dihedral covers*, and we investigate some of their properties. Throughout the section we assume that $Y$ is a smooth variety.

Recall that, from Proposition 5.2, a $D_n$-field extension $\mathbb{C}(Y) \subset E$ is given by $E = \mathbb{C}(Y)(x)$, where $x$ satisfies an irreducible equation of the following form:

$$x^{2n} - 2ax^n + F^n = 0,$$

with $a, F \in \mathbb{C}(Y)$. In order to construct varieties having $E$ as field of rational functions, one could proceed in the following way. Consider a geometric line bundle $\mathbb{L} \to Y$, sections $a \in H^0(Y, \mathbb{L}^{\otimes n})$, $F \in H^0(Y, \mathbb{L}^{\otimes 2})$, and define

$$X' := \{v \in \mathbb{L} \mid v^{2n} - 2av^n + F^n = 0\} \subset \mathbb{L},$$

where $v$ is a fibre coordinate of $\mathbb{L}$. The singular locus of $X'$ contains the locus $v = F = 0$, so it has codimension 1 and $X'$ is not normal.

We slightly modify this construction in the following way. Let $\mathbb{L} \to Y$ be a geometric line bundle, $a \in H^0(Y, \mathbb{L}^{\otimes n})$ and $F \in H^0(Y, \mathbb{L}^{\otimes 2})$. Define $X \subset \mathbb{L} \oplus \mathbb{L}$ to be the set of $(u, v) \in \mathbb{L} \oplus \mathbb{L}$ such that the following equations are satisfied:

$$\begin{cases} uv & = F \\ u^n - 2a + v^n & = 0, \end{cases} \tag{26}$$

where $u$ (resp. $v$) is a fibre coordinate of the first (resp. second) copy of $\mathbb{L}$ in $\mathbb{L} \oplus \mathbb{L}$. The dihedral group $D_n$ acts on $X$ via $\sigma(u, v) = (\zeta u, \zeta^{-1} v)$, $\tau(u, v) = (v, u)$, where $\zeta \in \mathbb{C}^*$ is a primitive root of 1.

**Theorem 6.1.** *Let $Y$ be a smooth variety. Let $\mathbb{L} \to Y$ be a geometric line bundle, $a \in H^0(Y, \mathbb{L}^{\otimes n})$ and $F \in H^0(Y, \mathbb{L}^{\otimes 2})$, such that:*

(i) *the zero locus of $a^2 - F^n \in H^0(Y, \mathbb{L}^{\otimes 2n})$ is smooth in the open set $F \neq 0$;*
(ii) *the divisors $\{a = 0\}$ and $\{F = 0\}$ intersect each other transversely.*

*Then the variety $X$ defined by* Eqs. (26) *is smooth and the restriction to $X$ of the fibre bundle projection $\mathbb{L} \oplus \mathbb{L} \to Y$ is a $D_n$-cover, $\pi : X \to Y$, with branch divisor $\mathcal{B}_\pi = \{F^n - a^2 = 0\}$. Furthermore, if $\{a = 0\} \cap \{F = 0\} \neq \emptyset$, then $X$ is irreducible.*

**Definition 6.2.** A **simple $D_n$-cover** of $Y$ is the $D_n$-cover $\pi : X \to Y$ given as in Theorem 6.1 by the restriction to $X$ of the fibre bundle projection $\mathbb{L} \oplus \mathbb{L} \to Y$.

**Proof.** (Of Theorem 6.1.) Let us define $\Phi_1 := uv - F$ and $\Phi_2 := u^n - 2a + v^n$, so that Eqs. (26) become

$$\begin{cases} \Phi_1 & = 0 \\ \Phi_2 & = 0. \end{cases}$$

Taking partial derivatives along the fibre coordinates we have:

$$\begin{pmatrix} \dfrac{\partial \Phi_1}{\partial u} & \dfrac{\partial \Phi_1}{\partial v} \\ \dfrac{\partial \Phi_2}{\partial u} & \dfrac{\partial \Phi_2}{\partial v} \end{pmatrix} = \begin{pmatrix} v & u \\ nu^{n-1} & nv^{n-1} \end{pmatrix},$$

hence the ramification divisor of $\pi$ is $R = \{v^n - u^n = 0\}$. Notice that $v^n - u^n$ is $\langle \sigma^* \rangle$-invariant, while $\tau^*(v^n - u^n) = -(v^n - u^n)$. Therefore the branch locus $\mathcal{B}_\pi$ is defined by the following equation:

$$\begin{aligned} (v^n - u^n)(u^n - v^n) &= -(v^n - u^n)^2 \\ &= -(v^n + u^n)^2 + 4(uv)^n \\ &= -4a^2 + 4F^n. \end{aligned}$$

Let us consider first the restriction of $\pi$ over the points where $F \neq 0$. In this locus, we have $v = F/u$ and so the equation $\Phi_2 = 0$ is equivalent to

$$\Phi = 0, \quad \text{where} \quad \Phi := u^{2n} - 2au^n + F^n.$$

It follows that $\pi$ is finite over the open set $F \neq 0$. To show the flatness of $\pi$ over $F \neq 0$, by Proposition 2.2 it suffices to prove that $X$ is smooth there. To this aim consider

$$\frac{\partial \Phi}{\partial u} = 2nu^{n-1}(u^n - a),$$

which can vanish only if $u^n - a = 0$, since $F \neq 0 \Rightarrow u \neq 0$. On the other hand, over the locus $u^n - a = 0$,

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= -2u^n \frac{\partial a}{\partial y} + \frac{\partial F^n}{\partial y} \\ &= \frac{\partial (F^n - a^2)}{\partial y}, \end{aligned}$$

where $y$ is any coordinate function on $Y$. Since $F^n - a^2 = 0$ is the branch divisor and, by hypothesis (i), it is smooth in $F \neq 0$, it follows that $X$ is smooth there.

We consider now the restriction of $\pi$ over the locus $F = 0$ and notice that here $uv = 0$. If $u \neq 0$, then as before we see that $\pi$ is finite. The smoothness of $X$ at $(u \neq 0, v = 0)$ follows from the fact that $(u \neq 0, v = 0) \notin R$. The same argument applies if $v \neq 0$. It remains the case where $u = v = 0$, which implies that $a = F = 0$. By hypothesis (ii), $a$ and $F$ are part of a local coordinate system near the points where $a = F = 0$, in the analytic topology. Writing the equations for $X$ in these coordinates, we see that $X$ is smooth at these points. There remains to show that $\pi$ is finite also over the points $a = F = 0$. Here $uv$ and $u^n + v^n$ are precisely the $D_n$-invariants, and $\mathcal{O}_X$ is a free $\mathcal{O}_Y$-module generated by $1, u^n - v^n, u, u^2, \ldots, u^{n-1}, v, v^2, \ldots, v^{n-1}$.

Let us now assume that $\{a = 0\} \cap \{F = 0\} \neq \emptyset$. The irreducibility of $X$ is equivalent to the surjectivity of the monodromy of the cover, $\mu : \pi_1(Y \setminus \mathcal{B}_\pi) \to D_n$. Since the branch divisor of the intermediate cover $q : Z \to Y$ coincides with $\mathcal{B}_\pi$, it is reduced, so $Z$ is irreducible and hence $\mathrm{Im}(\mu)$ contains a reflection $\sigma^i\tau$. To see that also $\sigma \in \mathrm{Im}(\mu)$, let us choose local coordinates $(a, F, y_3, \ldots, y_{\dim(Y)})$ for $Y$ at a point in $\{a = 0\} \cap \{F = 0\}$ and consider the path in $X$

$$\gamma(t) = \left( \exp\left(\frac{2\pi\sqrt{-1}}{n} t\right) u_0, \exp\left(-\frac{2\pi\sqrt{-1}}{n} t\right) v_0, F, a, y_3, \ldots, y_{\dim(Y)} \right),$$

where, $(u_0, v_0, F, a, y_3, \ldots, y_{\dim(Y)}) \in X$. Then, $\gamma(1) = \sigma\gamma(0)$, and $\pi \circ \gamma$ is a loop contained in $Y \setminus \mathcal{B}_\pi$, if and only if

$$\exp(2\pi\sqrt{-1}t)u_0^n - \exp(-2\pi\sqrt{-1}t)v_0^n \neq 0, \quad \forall t.$$

Since this condition can be easily achieved, e.g. by choosing $u_0, v_0$ with $|u_0| \neq |v_0|$, the claim follows. □

Notice that the hypothesis (i) in the previous theorem is general, as it follows from the next proposition.

**Proposition 6.3.** *Under the same notation as before, for a general choice of $a \in H^0(Y, \mathbb{L}^{\otimes n})$ and $F \in H^0(Y, \mathbb{L}^{\otimes 2})$, $F^n - a^2 = 0$ is smooth over $F \neq 0$.*

**Proof.** This follows directly from Bertini's theorem (see e.g. [25]). □

### 6.1. *Invariants of simple dihedral covers*

We first determine the eigensheaves decomposition of $\pi_*\mathcal{O}_X$, where $\pi : X \to Y$ is a simple $D_n$-cover. Let $\mathbb{L}$, $F$ and $a$ be as in the statement of Theorem 6.1, and let us denote with $\mathcal{O}_Y(L)$ the sheaf of sections of $\mathbb{L}$. Then, over an open subset $S \subset Y$ where $\mathbb{L}$ is trivial, from (26) we deduce that

$$\begin{aligned}\pi_*\mathcal{O}_{X|S} &\cong \frac{\mathcal{O}_S[\lambda, \mu]}{(\lambda\mu - F, \lambda^n - 2a + \mu^n)} \\ &\cong \mathcal{O}_S \oplus \mathcal{O}_S(\lambda^n - \mu^n) \bigoplus_{i=1}^{n-1} \left(\mathcal{O}_S\lambda^i \oplus \mathcal{O}_S\mu^{n-i}\right),\end{aligned}$$

where $\lambda$ and $\mu$ are fibre coordinates on the dual $\mathbb{L}^\vee$ of $\mathbb{L}$. This local description globalizes and we have:

$$\pi_*\mathcal{O}_X \cong \bigoplus_{i=0}^{n-1} \left[\mathcal{O}_Y(-iL) \oplus \mathcal{O}_Y(-(n-i)L)\right]. \tag{27}$$

Notice that, for the sheaves $\mathcal{L}$ and $U_i$ introduced in Section 5.2, we have:

$$\mathcal{L} = \mathcal{O}_Y(-nL), \quad U_i = \mathcal{O}_Y(-iL) \oplus \mathcal{O}_Y(-(n-i)L).$$

To determine the canonical bundle of $X$, we use the canonical bundle formula for branched covers:

$$\omega_X = \pi^*\omega_Y \otimes \mathcal{O}_X(R),$$

where $R$ is the ramification divisor of $\pi$. If $\pi : X \to Y$ is a simple $D_n$-cover, then $R = \{u^n - v^n = 0\}$ (see the proof of Theorem 6.1). Since $u^n - v^n$ is a generator of the eigensheaf corresponding to the irreducible representation of $D_n$ with character $\chi_2$, $\mathcal{O}_X(R) = \pi^*\mathcal{L}^\vee = \pi^*\mathcal{O}_Y(nL)$. Hence

$$\omega_X = \pi^*\left(\omega_Y(nL)\right). \tag{28}$$

In particular, if $\dim(X) = 2$, then the self-intersection of a canonical divisor of $X$ is:

$$K_X^2 = 2n(K_Y + nL)^2. \tag{29}$$

To compute the Euler characteristic $\chi(\mathcal{O}_X)$, by the finiteness of $\pi$ we have that $\chi(\mathcal{O}_X) = \chi(\pi_*\mathcal{O}_X)$. In particular, if $\dim(X) = 2$, the Riemann–Roch theorem for surfaces yields the following formula:

$$\chi(\mathcal{O}_X) = 2n\chi(\mathcal{O}_Y) + \frac{1}{6}n(2n^2+1)L\cdot L + \frac{1}{2}n^2 L\cdot K_Y. \tag{30}$$

### 6.2. *Almost simple dihedral covers*

In this section we define the almost simple dihedral covers, which can be seen as projectivizations of the simple dihedral covers. The construction follows closely the one of almost simple cyclic covers, introduced and studied in [10].

The covering space $X$ of an almost simple dihedral cover $\pi : X \to Y$, over the smooth variety $Y$, is defined as a complete intersection in the fibre product $\mathbb{P}_1 \times_Y \mathbb{P}_2 \to Y$ of two $\mathbb{P}^1$-bundles over $Y$ in the following way. Let $\mathbb{L} \to Y$ be a geometric line bundle and $\underline{\mathbb{C}} = Y \times \mathbb{C} \to Y$ be the trivial geometric line bundle. For each $i = 1, 2$, let $\mathbb{P}_i := \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L})$ be the $\mathbb{P}^1$-bundle associated to $\underline{\mathbb{C}} \oplus \mathbb{L} \to Y$, and let $p_i : \mathbb{P}_i \to Y$ be the corresponding projection. Here, each fibre of $p_i$ is the projective space of 1-dimensional subspaces in the corresponding fibre of $\underline{\mathbb{C}} \oplus \mathbb{L}$, hence, using the notation in [21], $\mathbb{P}_i = \mathrm{Proj}\,(\mathcal{O}_Y \oplus \mathcal{O}_Y(-L))$ and in particular $(p_i)_*\mathcal{O}_{\mathbb{P}_i}(1) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$, where $L$ is a divisor in $Y$ such that $\mathcal{O}_Y(L)$ is the sheaf of sections of $\mathbb{L}$ (hence $\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}(\mathcal{O}_Y(-L)))$). On each $\mathbb{P}_i$ there are projective fibre coordinates: $[u_0 : u_1]$ for $\mathbb{P}_1$, $[v_0 : v_1]$ for $\mathbb{P}_2$. $u_0, u_1$ are defined as follows ($v_0, v_1$ are defined in the same way): let $U \subset p_1^*(\underline{\mathbb{C}} \oplus \mathbb{L})$ be the universal sub-bundle, then $u_0 : U \to p_1^*(\underline{\mathbb{C}})$ is the composition of the inclusion with the projection; similarly for $u_1 : U \to p_1^*\mathbb{L}$. Notice that, since $U = \mathrm{Spec}(\mathrm{Sym}\,\mathcal{O}_{\mathbb{P}_1}(1))$, then $u_0 \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1))$ and $u_1 \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1) \otimes p_1^*\mathcal{O}_Y(L))$.

Let $F \in H^0(Y, \mathbb{L}^{\otimes 2})$ and let $A_0, A_\infty$ be effective divisors in $Y$ such that

$$A_0 \equiv nL + A_\infty.$$

Let $a_\infty \in H^0(Y, \mathcal{O}_Y(A_\infty))$ and $a_0 \in H^0(Y, \mathcal{O}_Y(A_0))$ be such that $A_\infty = \{a_\infty = 0\}$ and $A_0 = \{a_0 = 0\}$.

Consider the subvariety $X \subset \mathbb{P}_1 \times_Y \mathbb{P}_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L})$ defined by the following equations:

$$\begin{cases}\Phi_1 &= 0 \\ \Phi_2 &= 0,\end{cases} \tag{31}$$

where

$$\Phi_1 := u_1v_1 - u_0v_0F, \quad \Phi_2 := a_\infty v_1^n u_0^n - 2a_0 v_0^n u_0^n + a_\infty v_0^n u_1^n,$$

$u_0, u_1, v_0, v_1$ are defined above and $([u_0 : u_1], [v_0 : v_1])$ are projective fibre coordinates on $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L})$.

Notice that the dihedral group $D_n$ acts on $X$ in the following way:

$$\sigma([u_0 : u_1], [v_0 : v_1]) = ([u_0 : \zeta u_1], [v_0 : \zeta^{-1} v_1]),$$
$$\tau([u_0 : u_1], [v_0 : v_1]) = ([v_0 : v_1], [u_0 : u_1]),$$

where $\zeta \in \mathbb{C}^*$ is a primitive $n$th root of 1. Then we have the following result.

**Theorem 6.4.** *Let $Y$ be a smooth variety, $\mathbb{L} \to Y$ be a geometric line bundle, $\underline{\mathbb{C}} = Y \times \mathbb{C} \to Y$ be the trivial geometric line bundle, and $F \in H^0(Y, \mathbb{L}^{\otimes 2})$. Let $A_0, A_\infty$ be effective divisors in $Y$ such that*

$$A_0 \equiv nL + A_\infty,$$

*where $L$ is a divisor in $Y$ with $\mathcal{O}_Y(L)$ being the sheaf of sections of $\mathbb{L}$. Let $a_\infty \in H^0(Y, \mathcal{O}_Y(A_\infty))$ and $a_0 \in H^0(Y, \mathcal{O}_Y(A_0))$ be such that $A_\infty = \{a_\infty = 0\}$ and $A_0 = \{a_0 = 0\}$.*

*Assume that the following conditions are satisfied:*

(i) *$A_0$ intersects $\{F = 0\}$ transversely, $A_0 \cap A_\infty = \emptyset$ and $A_\infty$ is smooth;*
(ii) *the locus $\{a_0^2 - F^n a_\infty^2 = 0\}$ is smooth on the open set $F \neq 0$.*

*Then $X$, defined by (31), is smooth and the restriction of the projection $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \to Y$ to $X$ is a $D_n$-cover $\pi : X \to Y$ with branch divisor $\mathcal{B}_\pi = \{a_\infty(a_0^2 - a_\infty^2 F^n) = 0\}$. Furthermore, if $A_0 \cap \{F = 0\} \neq \emptyset$, then $X$ is irreducible.*

**Definition 6.5.** An **almost simple $D_n$-cover** of $Y$ is the $D_n$-cover $\pi : X \to Y$ given as in Theorem 6.4 by the restriction to $X$ of the fibre bundle projection $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \times_Y \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}) \to Y$.

**Proof.** We first prove that the intersection of $X$ with each one of the standard open subsets $v_0 u_0 \neq 0$, $v_1 u_0 \neq 0$ and $v_1 u_1 \neq 0$ is a smooth variety. This suffices because $\tau(\{v_0 u_1 \neq 0\}) = \{v_1 u_0 \neq 0\}$.

To this aim, observe that the restriction of $\pi$ to the locus where $u_0 v_0 \neq 0$ is a simple $D_n$-cover. Indeed, setting $u = u_1/u_0$ and $v = v_1/v_0$, Eqs. (31) reduce to

$$\begin{cases} uv - F & = 0 \\ a_\infty v^n - 2a_0 + a_\infty u^n & = 0. \end{cases}$$

Since $A_\infty \cap A_0 = \emptyset$, on this locus $a_\infty$ never vanishes, and setting $a = a_0/a_\infty$ we obtain Eqs. (26). Under our hypotheses Theorem 6.1 applies, hence $X$ is smooth if $u_0 v_0 \neq 0$.

Consider now the locus $v_1 u_1 \neq 0$ and observe that there $F$ never vanishes. Let us define $u = u_0/u_1$, $v = v_0/v_1$ and $g = 1/F$. Then Eqs. (31) become

$$\begin{cases} uv & = g \\ a_\infty u^n - 2a_0 g^n + a_\infty v^n & = 0. \end{cases}$$

Since $A_\infty \cap A_0 = \emptyset$, $a_\infty$ never vanishes in this locus. So, defining $a = a_0/a_\infty$, we get the following equations for $X$:

$$u^{2n} - 2ag^n u^n + g^n = 0, \quad v = g/u.$$

Notice that this is the equation of a $D_n$-cover, with action $\sigma(u) = \zeta u$, $\tau(u) = g/u$. Since $g \neq 0$ everywhere, $\langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$ acts freely and so $X$ is smooth if and only if the intermediate double cover is smooth. The intermediate double cover has equation $z^2 - 2ag^n z + g^n = 0$ and its branch divisor is $\{g^n(g^n a^2 - 1) = 0\} = \{a^2 - F^n = 0\}$. By hypothesis (ii) this locus is smooth, so $X$ is smooth where $v_1 u_1 \neq 0$.

It remains to consider the case where $v_1 u_0 \neq 0$. Here, setting $v_1 = 1 = u_0$, Eqs. (31) reduce to

$$\begin{cases} u_1 - v_0 F & = 0 \\ a_\infty - 2a_0 v_0^n + a_\infty v_0^n u_1^n & = 0. \end{cases}$$

Substituting $u_1 = v_0 F$ in the second equation we obtain:

$$a_\infty - 2a_0 v_0^n + a_\infty F^n v_0^{2n} = 0.$$

Notice that, if $v_0 \neq 0$, then we have already seen that $X$ is smooth. On the other hand, when $v_0 = 0$, we must have $a_\infty = 0$, and then the smoothness of $A_\infty$ implies that of $X$.

The finiteness of $\pi$ follows from the fact that $\pi$ has finite fibres and its restriction to each open subset of $Y$ where $\mathbb{L}$ is trivial is projective ([21], III, Ex. 11.1).

To describe the branch divisor, recall that the restriction of $\pi$ to the open set $u_0 v_0 \neq 0$ is a simple dihedral cover and so its branch divisor is $F^n - a^2 = 0$, where $a := a_0/a_\infty$. Notice that, if $u_0 v_0 \neq 0$, then $a_\infty \neq 0$ everywhere, on the other hand, if $u_0 v_0 = 0$, then $u_1 v_1 = 0$ by (31), so $X \cap \{u_0 v_0 = 0\} \subset \{u_1 v_0 \neq 0\} \cup \{u_0 v_1 \neq 0\}$. The claim now follows from the previous explicit description of $\pi$ on $u_0 v_1 \neq 0$.

Finally, if $A_0 \cap \{F = 0\} \neq \emptyset$, then $X$ is irreducible since the open subset $X \cap \{u_0 v_0 \neq 0\}$ is irreducible by Theorem 6.1. □

The invariants of almost simple $D_n$-covers $\pi : X \to Y$ can be computed in the same way as in the simple case, once a description of $\pi_*\mathcal{O}_X$ in terms of $L$ and $A_\infty$ is provided. In the remaining part of this section we show that there is an isomorphism as follows:

$$\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-nL - A_\infty) \oplus \left(\bigoplus_{i=1}^{n-1}[\mathcal{O}_Y(-iL) \oplus \mathcal{O}_Y(-(n-i)L)]\right)(-A_\infty). \tag{32}$$

Notice that, on the open subset $Y \setminus A_\infty$ where $\pi$ is a simple cover, the previous formula reduces to (27).

To prove (32), we consider the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\tilde{p}_2} & \mathbb{P}_2 \\ {\scriptstyle \tilde{p}_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{P}_1 & \xrightarrow{p_1} & Y \end{array}$$

where $\mathcal{Q} := \mathbb{P}_1 \times_Y \mathbb{P}_2$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle with projection $p : \mathcal{Q} \to Y$, $p = p_i \circ \tilde{p}_i$, $\forall\, i = 1, 2$. Recall that the Picard group of $\mathcal{Q}$ is isomorphic to $\mathrm{Pic}(Y) \times \mathbb{Z}^{\oplus 2}$ via the usual isomorphism that sends $(\mathcal{L}, m, n) \in \mathrm{Pic}(Y) \times \mathbb{Z}^{\oplus 2}$ to $p^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{Q}}(m, n) := p^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}_1}(m) \boxtimes \mathcal{O}_{\mathbb{P}_2}(n)$.

Let us define $D_i := \{\Phi_i = 0\}$ to be the divisor of $\mathcal{Q}$ given by the equation $\Phi_i = 0$ in (31), for $i = 1, 2$, and notice that

$$\begin{aligned} &\Phi_1 \in H^0\left(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1,1) \otimes p^*\mathcal{O}_Y(2L)\right), \\ &\Phi_2 \in H^0\left(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(n,n) \otimes p^*\mathcal{O}_Y(nL + A_\infty)\right). \end{aligned} \tag{33}$$

Then we consider the usual short exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathcal{Q}} \to \mathcal{O}_X \to 0, \tag{34}$$

where $\mathcal{I}_X$ is the sheaf of ideals of $X$. Since $X$ is the complete intersection of two divisors in $\mathcal{Q}$, the Koszul resolution of $\mathcal{I}_X$ is as follows:

$$0 \to \mathcal{O}_{\mathcal{Q}}(-D_1 - D_2) \to \mathcal{O}_{\mathcal{Q}}(-D_1) \oplus \mathcal{O}_{\mathcal{Q}}(-D_2) \to \mathcal{I}_X \to 0. \tag{35}$$

Applying $p_*$ to (34) we obtain the following split short exact sequence:

$$0 \to \mathcal{O}_Y \cong p_*\mathcal{O}_{\mathcal{Q}} \to p_*\mathcal{O}_X \to R^1p_*\mathcal{I}_X \to 0,$$

where we have used the fact that $p_*\mathcal{I}_X = 0 = R^1p_*\mathcal{O}_{\mathcal{Q}}$.

In order to compute $R^1p_*\mathcal{I}_X$, we apply $p_*$ to (35) and we obtain the following exact sequence:

$$0 \to R^1p_*\mathcal{I}_X \to R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_1 - D_2) \to R^2p_*\left(\mathcal{O}_{\mathcal{Q}}(-D_1) \oplus \mathcal{O}_{\mathcal{Q}}(-D_2)\right),$$

where we have used the equality $R^1p_*\left(\mathcal{O}_{\mathcal{Q}}(-D_1) \oplus \mathcal{O}_{\mathcal{Q}}(-D_2)\right) = 0$ that follows from the Künneth formula. Furthermore, since $R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_1) = 0$, we have that

$$R^1p_*\mathcal{I}_X = \ker\left[R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_1 - D_2) \xrightarrow{\mu} R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_2)\right],$$

where $\mu$ is induced by the morphism $\mathcal{O}_{\mathcal{Q}}(-D_1 - D_2) \to \mathcal{O}_{\mathcal{Q}}(-D_2)$ in the Koszul resolution of $\mathcal{I}_X$, given by $\psi_1 \wedge \psi_2 \mapsto \psi_1(\Phi_1)\psi_2$, for $\psi_i \in \mathcal{O}_{\mathcal{Q}}(-D_i)$.

To describe $\mu$ explicitly, we use the following isomorphisms:

$$\begin{aligned} R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_1 - D_2) &\cong S^{n-1}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L))^{\otimes 2} \otimes \mathcal{O}_Y(-nL - A_\infty), \\ R^2p_*\mathcal{O}_{\mathcal{Q}}(-D_2) &\cong S^{n-2}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L))^{\otimes 2} \otimes \mathcal{O}_Y(-(n-2)L - A_\infty), \end{aligned}$$

that follow applying the projection formula, Künneth formula, and the standard isomorphisms (see e.g. [21, Ex. 8.4, III]). Then, if we choose local sections $x_0, y_0$ of $\mathcal{O}_Y$, and $x_1, y_1$ of $\mathcal{O}_Y(L)$, that generate the corresponding sheaves, we obtain the following local basis for $S^{n-1}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L))^{\otimes 2}$:

$$E_{ij} := x_0^i x_1^{n-1-i} \otimes y_0^j y_1^{n-1-j}, \qquad 0 \le i, j \le n-1.$$

Similarly, $G_{km} := (x_0^k x_1^{n-2-k} \otimes y_0^m y_1^{n-2-m}) \otimes (x_1 \otimes y_1)$, for $0 \le k, m \le n-2$, is a local basis for $S^{n-2}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L))^{\otimes 2} \otimes \mathcal{O}_Y(2L)$. The morphism $\mu$ in these basis is given as follows:

$$\mu(E_{ij}) = G_{ij} - fG_{i-1,j-1},$$

where $F = fx_1 \otimes y_1$, and $G_{km} := 0$, for $k, m \notin \{0, \ldots, n-2\}$. Hence from elementary linear algebra, we have that $\ker(\mu)$ (twisted by $\mathcal{O}_Y(nL + A_\infty)$) is generated by $E_{n-1,j}$ and $E_{i,n-1}$, for $0 \le i, j \le n-1$. So

$$\ker(\mu) = \left( \left[ \bigoplus_{j=0}^{n-1} \mathcal{O}_Y((n-1-j)L) \right] \oplus \left[ \bigoplus_{i=0}^{n-2} \mathcal{O}_Y((n-1-i)L) \right] \right) \otimes \mathcal{O}_Y(-nL - A_\infty),$$

and hence (32) follows.

## 7. Deformations of simple dihedral covers

Let $X$ be a simple dihedral covering of $Y$: this means that $X$ is the subvariety of the vector bundle $V = \mathbb{L} \oplus \mathbb{L}$ which is (see (26)) the complete intersection of two hypersurfaces, one in $|p^*(2L)|$, the other in $|p^*(nL)|$; here $p : V \to Y$ is the natural projection.

Observe now that the cotangent sheaf of $V$ is an extension of $p^*\Omega^1_Y$ by $p^*(\mathcal{O}_Y(-L)^{\oplus 2})$,

$$0 \to p^*\Omega^1_Y \to \Omega^1_V \to p^*(\mathcal{O}_Y(-L)^{\oplus 2}) \to 0, \tag{36}$$

where $p^*\Omega^1_Y \to \Omega^1_V$ is the cotangent map of $p$.

Then the conormal sheaf exact sequence of $X$ reads out, if we denote by $L' := \pi^*(L)$, as

$$0 \to N^*_{X|V} = \mathcal{O}_X(-2L') \oplus \mathcal{O}_X(-nL') \to \Omega^1_V \otimes \mathcal{O}_X \to \Omega^1_X \to 0,$$

and the dual sequence is

$$0 \to \Theta_X \to \Theta_V \otimes \mathcal{O}_X \to N_{X|V} = \mathcal{O}_X(2L') \oplus \mathcal{O}_X(nL') \to 0,$$

whose direct image under $\pi_*$ yields the tangent exact sequence:

$$0 \to \pi_*\Theta_X \to \pi_*(\Theta_V \otimes \mathcal{O}_X) \to (\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X) \to 0.$$

Passing to the long exact cohomology sequence, we get the Kodaira–Spencer exact sequence

$$H^0((\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X)) \to H^1(\Theta_X) \to H^1(\pi_*(\Theta_V \otimes \mathcal{O}_X)) \to . \tag{37}$$

The meaning of the first linear map is given through the following definition.

**Definition 7.1.** The space of **natural deformations** of a simple dihedral covering is the family of complete intersections of $V$:

$$\begin{cases} uv - F & = 0 \\ u^n - 2a + v^n + \sum_1^{n-1}(b_i u^i + c_i v^i) + d(u^n - v^n) & = 0, \end{cases} \tag{38}$$

where $b_i, c_i \in H^0(\mathcal{O}_Y(n-i)L)$, $d \in H^0(\mathcal{O}_Y)$. (39)

**Remark 7.2.** The reader can see that in the second equation

$$-2a + v^n + \sum_1^{n-1}(b_i u^i + c_i v^i) + d(u^n - v^n)$$

can be any section in $H^0(\mathcal{O}_Y(nL) \otimes \pi_*(\mathcal{O}_X))$, in view of our basic formulae; instead, any section in $H^0(\mathcal{O}_Y(2L) \otimes \pi_*(\mathcal{O}_X))$ is of the form

$$-F + \beta u + \alpha v + \lambda u^2 + \mu u^2, \text{ where } \alpha, \beta \in H^0(\mathcal{O}_Y(L)), \lambda, \mu \in \mathbb{C}.$$

But the new equation $uv - F + \beta u + \alpha v + \lambda v^2 + \mu u^2$ is the old form $u'v' = F'$ (up to a multiplicative constant) if we choose new variables

$$u' := u + \alpha + \lambda v, \ v' := v + \beta + \mu u.$$

Of course, one can deform not only the equations, but also simultaneously the base $Y$, the vector bundle $V$, and the equations; this however leads in general to a deformation with non smooth base.

We have at any rate an easy result which says that all small deformations are obtained by natural deformations.

**Theorem 7.3.** *Assume that $\pi : X \to Y$ is a simple dihedral covering. Then all small deformations of $X$ are natural deformations of $\pi : X \to Y$, provided $H^1(\pi_*(\Theta_V \otimes \mathcal{O}_X)) = 0$ (that happens, for example if $H^1((\Theta_Y \oplus \mathcal{O}_Y(L)^{\oplus 2}) \otimes \pi_*(\mathcal{O}_X)) = 0$). In particular, the Kuranishi family of $X$ is smooth, and the Kuranishi space* Def$(X)$ *is locally analytically isomorphic to*

$$\mathrm{Def}' := \mathrm{coker}\left(H^0(\pi_*(\Theta_V \otimes \mathcal{O}_X)) \to H^0((\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X))\right).$$

**Proof.** By our assumption, and the Kodaira–Spencer exact sequence, we have an isomorphism of vector spaces Def′ $\cong H^1(\Theta_X)$. Moreover, by the previous remark, the family of natural deformations has Kodaira–Spencer map which is surjective onto Def; therefore, by the implicit functions theorem, Def$(X)$ is the germ of the analytic space $H^1(\Theta_X)$ at the origin, hence our claim. □

## 8. Examples and applications

### *8.1. Simple dihedral covers of projective spaces*

We first consider the case where $Y = \mathbb{P}^2$, the complex projective plane, $n = 3$ and $\mathcal{O}_Y(L) = \mathcal{O}_{\mathbb{P}^2}(1)$. Then $a \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ and $F \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are a cubic and a quadric curve respectively, such that the sextic curve $\{F^3 - a^2 = 0\}$ is smooth in the locus $F \neq 0$, and $\{a = 0\}$ intersects $\{F = 0\}$ transversely. By Theorem 6.1, we have a smooth $D_3$-cover $\pi : X \to \mathbb{P}^2$ branched over $\mathcal{B} = \{F^3 - a^2 = 0\}$. Notice that $\mathcal{B} \in |-2K_{\mathbb{P}^2}|$ and $\omega_{\mathbb{P}^2} = \mathcal{O}_Y(-3L)$. Hence

$$\omega_X = \pi^*(\omega_{\mathbb{P}^2}(3L)) \cong \mathcal{O}_X.$$

Furthermore, $q(X) = 0$, hence $X$ is a K3 surface.

Let $W = X/\langle\tau\rangle$ and let $f : W \to \mathbb{P}^2$ be the induced triple cover (see Section 5.3). Then $W$ can be realized as a cubic surface in $\mathbb{P}^3$ in such a way that $f$ is the projection from a point in $\mathbb{P}^3 \setminus W$. Indeed, consider Eqs. (26) which define $X$. If we define $w := u + v$, then

$$\begin{aligned} w^3 &= u^3 + v^3 + 3uv(u + v) \\ &= 2a + 3Fw. \end{aligned}$$

The branch divisor of $f$ is $\mathcal{B} = \{F^3 - a^2\}$. So, under the hypotheses of Theorem 6.1, $\mathcal{B}$ is a sextic with 6 cusps lying on a conic.

The fundamental group of the complement $\mathbb{P}^2 \setminus \mathcal{B}$ of a sextic curve $\mathcal{B}$ as above has been studied in [11], where in particular it is proven that $\pi_1(\mathbb{P}^2 \setminus \mathcal{B})$ is generated by two elements of order 2 and 3 respectively. From this it follows that there exists a surjective group homomorphism $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \to D_3$, and hence it follows from the generalized Riemann existence theorem of Grauert and Remmert, that there exists a $D_3$-cover $\pi : X \to \mathbb{P}^2$. From Theorem 6.1 we have an explicit construction of such a cover.

Let now $n > 3$ and consider the simple $D_n$-cover $\pi : X \to \mathbb{P}^2$ associated to $\mathcal{O}_Y(L) = \mathcal{O}_{\mathbb{P}^2}(1)$, $a \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ and $F \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Under the hypotheses of Theorem 6.1, $X$ is a smooth surface with

$$\begin{aligned} \omega_X &= \pi^*\mathcal{O}_{\mathbb{P}^2}(n-3), \\ K_X^2 &= 2n(n-3)^2, \\ \chi(\mathcal{O}_X) &= \frac{1}{3}n^3 - \frac{3}{2}n^2 + \frac{13}{6}n, \end{aligned}$$

in particular $X$ is a surface of general type, which is minimal since $K_X \cdot C > 0$ for any curve $C \subset X$.

Finally, let $n = 2$, $Y = \mathbb{P}^2$ and $\mathcal{O}_Y(L) = \mathcal{O}_{\mathbb{P}^2}(1)$. Under the hypotheses of Theorem 6.1, $X$ is a smooth surface with the following invariants:

$$\begin{aligned} \omega_X &= \pi^*\mathcal{O}_{\mathbb{P}^2}(-1) \\ \chi(\mathcal{O}_X) &= 1 \\ q &= 0 \\ K_X^2 &= 4 \\ p_n &= \dim H^0(X, \omega_X^{\otimes n}) = 0. \end{aligned}$$

Hence $X$ is a rational, non-minimal surface. Indeed, $X$ is isomorphic to a del Pezzo surface of degree 4 in $\mathbb{P}^4$, the complete intersection of the quadrics $uv = F(x_0 : x_1 : x_2)$, $u^2 + v^2 = 2a(x_0 : x_1 : x_2)$, where $(x_0 : x_1 : x_2 : u : v)$ are now homogeneous coordinates in $\mathbb{P}^4$.

In a similar way one can construct examples of dihedral covers of $\mathbb{P}^2$ with $\mathcal{O}_Y(L) = \mathcal{O}_{\mathbb{P}^2}(d)$, $d > 1$.

As an application of Theorem 7.3, we have that all the small deformations of a simple $D_n$-cover $\pi : X \to \mathbb{P}^2$ associated to $\mathcal{O}_Y(L) = \mathcal{O}(m)$, $m \geq 1$, are natural deformations, if $(m, n) = (1, 2)$, or $m \geq 2$ and any $n \geq 2$. This follows directly from Theorem 7.3, formula (27) and the computation of the cohomology of $\Omega^q_{\mathbb{P}^d}(k)$ ([26, p. 256]).

**Remark 8.1.** Notice that, in general, a simple $D_n$-cover of $\mathbb{P}^d$, $\pi : X \to \mathbb{P}^d$, associated to $\mathcal{O}_Y(L) = \mathcal{O}(m)$, $F \in H^0(\mathbb{P}^d, \mathcal{O}(2m))$ and $a \in H^0(\mathbb{P}^d, \mathcal{O}(nm))$, is isomorphic to a complete intersection $X'$ in the weighted projective space $\mathbb{P}^{d+2}(1, \ldots, 1, m, m)$, where

$$X' := \{(x_0 : \ldots : x_d : u : v) \in \mathbb{P}^{d+2}(1, \ldots, 1, m, m) | uv = F, u^n + v^n = 2a\}.$$

To see this, observe that $X \subset \mathbb{L} \oplus \mathbb{L}$ is the quotient of $\tilde{X} \subset (\mathbb{C}^{d+1} \setminus \{0\}) \times \mathbb{C}^2$ via the linear diagonal action of $\mathbb{C}^*$ with weights $(1, \dots, 1, m, m)$, where

$$\tilde{X} := \{(x_0, \dots, x_d, u, v) | uv = F, u^n + v^n = 2a\}.$$

The claim now follows since $(\mathbb{C}^{d+1} \setminus \{0\}) \times \mathbb{C}^2 \subset \mathbb{C}^{d+3} \setminus \{0\}$, and the action of $\mathbb{C}^*$ on $(\mathbb{C}^{d+1} \setminus \{0\}) \times \mathbb{C}^2$ is the restriction of the linear diagonal action on $\mathbb{C}^{d+3} \setminus \{0\}$ with weights $(1, \dots, 1, m, m)$.

If $d \geq 3$, then Theorem 7.3 implies that all the small deformations of $X$ are natural deformations of the simple $D_n$-cover $\pi : X \to \mathbb{P}^d$. Indeed, in this case, we have that

$$H^1 \left( (\Theta_{\mathbb{P}^d} \oplus \mathcal{O}_{\mathbb{P}^d}(m)^{\oplus 2}) \otimes \pi_* \mathcal{O}_X \right) = 0,$$

as it follows from the fact that $H^1(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(k)) = 0, \forall k$, and that $H^1(\mathbb{P}^d, \Theta_{\mathbb{P}^d}(k)) \cong H^{d-1}(\mathbb{P}^d, \Omega^1_{\mathbb{P}^d}(-k-d-1))^\vee = 0, \forall k$ [26].

### *8.2. An application to fundamental groups*

According to the generalized Riemann existence theorem of Grauert and Remmert [27], coverings $\pi : X \to Y$ of a normal variety $Y$, of degree $n$, with branch locus contained in a divisor $\mathcal{B} \subset Y$, and with $X$ normal, correspond to conjugacy classes of group homomorphisms $\mu : \pi_1(Y \setminus \mathcal{B}) \to \mathfrak{S}_n$. In this situation, $X$ is irreducible, if and only if $\mathrm{Im}(\mu)$ is a transitive subgroup of $\mathfrak{S}_n$; $\pi : X \to Y$ is Galois with group $G := \mathrm{Im}(\mu)$, if and only if $G$ coincides with the group of automorphisms of $\pi$. In particular, if $\pi : X \to Y$ is a $G$-cover with $G$ non-abelian and $X$ irreducible, then $\pi_1(Y \setminus \mathcal{B})$ is necessarily non-abelian.

Fundamental groups of complements of divisors in projective varieties have been extensively studied by many authors. As a direct consequence of Theorem 6.1, we have the following result.

**Proposition 8.2.** *Let $Y$ be a smooth variety and $L \subset Y$ be a divisor. Assume that there exist $a \in H^0(Y, \mathcal{O}_Y(nL))$ and $F \in H^0(Y, \mathcal{O}_Y(2L))$, such that the conditions* (i), (ii) *of* Theorem 6.1 *are satisfied and $\{a = 0\} \cap \{F = 0\} \neq \emptyset$. Then $\pi_1(Y \setminus \mathcal{B})$ admits an epimorphism onto $D_n$, in particular it is non-abelian, where $\mathcal{B} = \{a^2 - F^n = 0\}$.*

Notice that similar results have been obtained using different methods ([28, Lemma 3], [29]). Briefly, one considers the pencil $\{\lambda 2(a = 0) + \mu n(F = 0)\}_{(\lambda:\mu) \in \mathbb{P}^1} \subset |2nL|$ and the induced morphism $Y \setminus \mathcal{B} \to \mathbb{P}^1 \setminus \{(1:1)\}$. This gives a group homomorphism $\pi_1(Y \setminus \mathcal{B}) \to \pi_1^{orb}(\mathbb{P}^1 \setminus \{(1:1)\})$, where $\pi_1^{orb}$ is the orbifold fundamental group of $\mathbb{P}^1 \setminus \{(1:1)\}$ with two orbifold points, $(1:0)$ of order 2 and $(0:1)$ of order $n$. Now, using the long exact homotopy sequence, one concludes that $\pi_1(Y \setminus \mathcal{B}) \to \pi_1^{orb}(\mathbb{P}^1 \setminus \{(1:1)\})$ is surjective and so $\pi_1(Y \setminus \mathcal{B})$ is not abelian.


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