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Research Article

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Quasilinear elliptic equations with critical potentials

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Abstract: We study Liouville theorems for problems of the form

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + V(x)|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N$$

in the framework of Carnot groups. Here $\mathscr{A}$ is a vector-valued function satisfying Carathéodory condition and ∇_L denotes an horizontal gradient, V is a given singular potential, a is a measurable scalar function and $q > p - 1$. Particular emphasis is given to the case when V is a Hardy or Gagliardo–Nirenberg potential. The results are new even in the canonical Euclidean setting.

Keywords: Quasilinear elliptic inequalities, Liouville theorems, Carnot groups

MSC 2010: 35B45, 35B51, 35B53, 35J10, 35J62, 35J70, 35R03

Dedicated to our dearest Professor Patrizia Pucci on the occasion of her birthday, with great esteem and friendship

1 Introduction

In this paper we study Liouville theorems for a class of possibly singular quasilinear elliptic equations and inequalities of the form

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + V(x)|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N. \tag{1.1}$$

Here $\mathscr{A}$ is a given vector-valued function satisfying Carathéodory conditions (see below for the precise assumptions), $p > 1$, $V \geq 0$ is a singular potential function, $a : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function and $q > p - 1$.

In the last years Liouville theorems for a wide class of weakly elliptic quasilinear problems were studied among others by Farina and Serrin [9] and Pucci and Serrin [18], where sharp interesting results were proved.

Similar problems have been studied among others in the semilinear case in [1], where nonexistence of solutions of the Schrödinger equation

$$\Delta u + \lambda V(x)u = f(x,u) \quad \text{on } \mathbb{R}^N \setminus \{0\} \tag{1.2}$$

was proved by reducing the problem to an ODE inequality by applying the spherical mean operator to (1.2) and using some convexity argument. For our problem (1.1), a radial reduction is in general not possible even if the differential operator is linear. So we need to proceed differently.

In order to achieve our goal, the main technique that we use throughout this paper will be a combination of three ingredients: the quasilinear version of Kato inequalities [8] and a slight modification of the test functions method together with an idea introduced in [15].

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Roughly speaking the proof or our main results will be organised in two steps. The first is to apply Kato's inequality to (1.1) reducing the problem to the study of the nonnegative solutions of

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + V(x)|u|^{p-2}u \geq a(x)u^q, \quad u \geq 0, \quad \text{on } \mathbb{R}^N.$$

The second step will be the application of some a priori estimates proved during the course. These estimates depend on two parameters α and R. By using an idea first introduced in [15, see proof of Theorem 4.1], we can choose α large enough and then by letting $R \to +\infty$ we conclude.

We point out that when dealing with equations or inequalities other fine techniques based on Keller–Osserman ideas ([13] and [17], respectively) are available. However, the application of these later ideas need special strong assumptions on the differential operator and on the nonlinearity, see [6, 7].

Our results allow us to consider, as special case in the Euclidean setting the following:

$$\Delta_p u + \lambda \frac{1}{|x|^p}|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N. \tag{1.3}$$

We have:

Theorem 1.1. *Let $N > p > 1$, $q > p-1$ and let $a \in L^1_{\text{loc}}(\mathbb{R}^N)$ be nonnegative functions such that*

$$a(x) \geq c\frac{1}{|x|^\theta} \quad \textit{for } |x| \textit{ large},$$

with $p > \theta$.

(1) *Let $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ be a weak solution of*

$$\Delta_p u + \lambda \frac{1}{|x|^p}|u|^{p-2}u \geq a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N. \tag{1.4}$$

Let $0 < \lambda \leq (\frac{N-p}{p})^p$ and let x_0 be the unique solution of the equation

$$(x-1+p)^p\lambda = x(Q-p)^p, \quad x \geq 1.$$

If

$$q \leq \frac{(N-\theta)(p-1) + x_0(p-\theta)}{N-p},$$

then $u \leq 0$ a.e. on $\mathbb{R}^N$. In particular, if u is a solution of

$$\Delta_p u + \lambda \frac{1}{|x|^p}|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N,$$

then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

(2) *Let $a(x) = |x|^{-\theta}$. If $\lambda > (\frac{N-p}{p})^p$, then inequality (1.4) has a positive bounded ground state solution.*

(3) *Let $a(x) = |x|^{-\theta}$. If $0 < \lambda \leq (\frac{N-p}{p})^p$ and $q > q_{cr}$, then inequality (1.4) has a positive bounded ground state solution.*

Throughout this paper we endow $\mathbb{R}^N$ with a group law such that it becomes a Carnot group. As usual, ∇_L stands for the horizontal gradient as described in Appendix A.

Considering the operator appearing in (1.1), we shall require that $\mathscr{A}$ is **W**-p-**C** (see Definition 2.1) and

$$a(x) \geq c\frac{\psi^k}{|x|_L^\theta} \qquad \text{for } |x|_L \text{ large}, \tag{1.5}$$

$$C_1\frac{\psi^h}{|x|_L^\nu} \geq V(x) \geq C_2\frac{\psi^p}{|x|_L^p} \quad \text{for } |x|_L \text{ large}. \tag{1.6}$$

In addition we will assume that a Hardy inequality holds for the potential V, that is there exists $\lambda_H > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla_L\phi|^p \geq \lambda_H \int_{\mathbb{R}^N} V|\phi|^p \quad \text{for any } \phi \in \mathscr{C}_0^1(\mathbb{R}^N). \tag{1.7}$$

In what follows, for simplicity, we deal with locally bounded solutions in the setting of Carnot groups. We note that if the function $\mathscr{A}$ is **S**-p-**C** (see Definition 2.1) and V belongs to $L^{Q/p}_{\mathrm{loc}}(\mathbb{R}^N)$ or to the Morrey space $M^{Q/(p-\epsilon)}(\mathbb{R}^N)$, then the positive solutions of (1.1) belong to $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^N)$. This is due to the fact that for **S**-p-**C** operator a weak Harnack inequality holds. See [14] for the Euclidean case and [3] for the Carnot group setting. The validity of (1.7) with $V = \psi^p/|\cdot|_L^p$ is established among other Hardy inequalities in [5], see Theorem A.3.

This paper is organised as follows. In Section 2 we fix some notations and point out some examples of differential operators for which our results apply when considering problems of type (1.1). In Section 3 we prove the main results of this paper for (1.1) when we assume that the potential function V is of Hardy type (see (1.7) below). This section contains also the main a-priori estimates that play a crucial role in the paper and a short discussion on the sharpness of the results proved in this work.

Finally, in Section 4 we consider some quasilinear problems related to a class of weighted Gagliardo–Nirenberg-type inequality. Namely, we consider problems whose prototype in Euclidean setting reads as

$$\Delta_p u + \lambda V(x)u^{p-1} \geq a(x)u^q + \mu W(x)u^{p-1}, \quad u \geq 0 \quad \text{on } \mathbb{R}^N,$$

where the functions V and W are related by a weighted Gagliardo–Nirenberg-type inequality.

In addition, for easy reference, we recall some basic facts that we use throughout the paper in the Appendix.

2 Notations and definitions

As pointed out in the preceding section, a setting in which our results apply is the framework of Carnot groups. For details see Appendix A. In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm. In the Carnot groups framework we denote by $|\cdot|_L$ a homogeneous norm and we set $\psi := |\nabla_L|\cdot|_L|$. For $R > 0$ we define the ball $B_R := \{x \in \mathbb{R}^N : |x|_L < R\}$ and by A_R we denote the annulus $A_R := B_{2R} \setminus B_R$.

Let $\Omega \subset \mathbb{R}^N$ be an open set and $p > 1$. We define the space

$$W^{1,p}_{L,\mathrm{loc}}(\Omega) := \{u \in L^p_{\mathrm{loc}}(\Omega) : |\nabla_L u| \in L^p_{\mathrm{loc}}(\Omega)\}.$$

As a canonical particular setting, our framework contains the Euclidean space $(\mathbb{R}^N, |\cdot|)$ with $|\cdot|$ the Euclidean norm. In this case $\nabla_L = \nabla$ is the isotropic gradient and div_L is the divergence operator. Here, $Q = N$ is the dimension of the space. In this case, $\psi \equiv 1$ and B_R is the Euclidean open ball of radius R centered at the origin. The space $W^{1,p}_{L,\mathrm{loc}}(\Omega)$ is the usual Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$.

The results we state in this paper in this setting can be proved with slight changes for nonlinear problems associated to more degenerate elliptic operators. For instance for operators generated by the vector field ∇_L such that ∇_L is homogeneous of degree one with respect to an anisotropic dilation δ_R as specified in [8]. However, to simplify the exposition, we prefer to limit ourselves to study our problems in the Carnot groups settings.

In what follows we shall assume that $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Carathéodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \mathrm{div}_L(\mathscr{A}(x, u(x), \nabla_L u(x))).$$

Our model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 2.3 below.

Definition 2.1. Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Carathéodory function. The function $\mathscr{A}$ is called *weakly elliptic* if it generates a weakly elliptic operator L, i.e.

$$\begin{aligned} \mathscr{A}(x, t, \xi) \cdot \xi \geq 0 \quad &\text{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l, \\ \mathscr{A}(x, 0, \xi) = 0 \quad &\text{or} \quad \mathscr{A}(x, t, 0) = 0. \end{aligned} \tag{WE}$$

Let $p \geq 1$. The function $\mathscr{A}$ is called **W**-p-**C** (weakly-p-coercive) if $\mathscr{A}$ is (WE) and it generates a weakly-p-coercive operator L, i.e. if there exists a constant $k_2 > 0$ such that

$$(\mathscr{A}(x,t,\xi)\cdot\xi)^{p-1} \geq k_2|\mathscr{A}(x,t,\xi)|^p \quad \text{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{W-p-C}$$

Let $p > 1$. The function $\mathscr{A}$ is called **S**-p-**C** (strongly-p-coercive) (see [16]) if there exist constants $k_1, k_2 > 0$ such that

$$(\mathscr{A}(x,t,\xi)\cdot\xi) \geq k_1|\xi|^p \geq k_2|\mathscr{A}(x,t,\xi)|^{p'} \quad \text{for each } x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{S-p-C}$$

Definition 2.2. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Carathéodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{L,\mathrm{loc}}(\Omega)$ is a *weak solution* of

$$\mathrm{div}_L(\mathscr{A}(x,u,\nabla_L u)) \geq f(x,u,\nabla_L u) \quad \text{on } \Omega$$

if $\mathscr{A}(\cdot,u,\nabla u) \in L^{p'}_{\mathrm{loc}}(\Omega)$, $f(\cdot,u,\nabla_L u) \in L^1_{\mathrm{loc}}(\Omega)$, and for any nonnegative $\varphi \in \mathscr{C}^1_0(\Omega)$ we have

$$-\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L\varphi \geq \int_\Omega f(x,u,\nabla_L u)\varphi.$$

Examples 2.3. Consider the following examples.

(1) Let $p > 1$. The p-Laplacian operator defined on suitable functions u by

$$\Delta_{p,L}u = \mathrm{div}_L(|\nabla_L u|^{p-2}\nabla_L u)$$

is an operator generated by

$$\mathscr{A}(x,t,\xi) := |\xi|^{p-2}\xi$$

which is **S**-p-**C**.

(2) The mean curvature operator in nonparametric form

$$Tu := \mathrm{div}_L\bigg(\frac{\nabla_L u}{\sqrt{1+|\nabla_L u|^2}}\bigg)$$

is generated by

$$\mathscr{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

In this case $\mathscr{A}$ is **W**-p-**C** with $1 \leq p \leq 2$ and of mean curvature type but it is not **S**-2-**C**.

(3) Let $p > 1$ and define

$$Lu := \sum_{i=1}^N \partial_i(|\partial_i u|^{p-2}\partial_i u).$$

The operator L is **S**-p-**C**.

(4) The operator defined by

$$\mathrm{div}\bigg(\frac{|u|\nabla u}{|u|+|\nabla u|}\bigg)$$

is **W**-2-**C**.

See [8] for further examples.

3 Quasilinear equations related to Hardy inequality

In this section we prove the main results of this paper for (1.1). Here we assume that the potential function V is of Hardy type (see (1.7) below). Our first main result is the following.

Theorem 3.1. *Let $Q > p > 1$. Let $\mathscr{A}$ be* **S**-p-**C** *and let $a, V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions satisfying* (1.5) *and* (1.6) *with $p \geq \nu > \theta$ and $p \geq h \geq k \geq 0$. Assume that* (1.7) *holds and let λ be such that $0 < \lambda \leq \lambda_H k_1$ where λ_H is the best constant in* (1.7) *and k_1 is the constant structure appearing in the definition of* **S**-p-**C** *(see Definition 2.1). Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of*

$$\mathrm{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \lambda V(x)|u|^{p-2}u \geq a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N. \tag{3.1}$$

Then $au \le 0$ a.e. on $\mathbb{R}^N$. Moreover, if $\lambda < \lambda_H k_1$, then $u \le 0$ a.e. on $\mathbb{R}^N$. In particular, if $\mathscr{A}$ is odd[1] *and u solves the equation*

$$\operatorname{div}_L(\mathscr{A}(x, u, \nabla_L u)) + \lambda V(x)|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N, \tag{3.2}$$

then $au = 0$ a.e. on $\mathbb{R}^N$. In addition if $\lambda < \lambda_H k_1$, then $u = 0$ a.e. on $\mathbb{R}^N$.

The proof is one of the consequences of the reduction principles stated in [8]. Indeed, in view of the reduction principles it follows that it suffices to study nonnegative solutions of the inequality related to (3.2). Notice that the case $\lambda \le 0$ has been considered in [8]. Hence, in what follows we shall focus our attention to the case $\lambda > 0$.

Theorem 3.2. *Let $Q > p > 1$. Let $\mathscr{A}$ be* **S**-*p*-**C** *and let $a, V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions satisfying (1.5) and (1.6) with $p \ge \nu > \theta$ and $p \ge h \ge k \ge 0$. Assume that (1.7) holds and let λ be such that $0 < \lambda \le \lambda_H k_1$ where λ_H is the best constant in (1.7) and k_1 is the constant structure appearing in the definition of* **S**-*p*-**C**. *Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a nonnegative weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x, u, \nabla_L u)) + \lambda V(x)u^{p-1} \ge a(x)u^q, \quad u \ge 0 \quad \text{on } \mathbb{R}^N. \tag{3.3}$$

If

$$p - 1 < q \le \frac{(Q-\theta)(p-1) + x_0(\nu-\theta)}{Q-\nu}, \tag{3.4}$$

where $x_0 \ge 1$ is the unique solution of the equation

$$(x - 1 + p)^p \lambda = x\lambda_H k_1 p^p, \quad x \ge 1,$$

then $au \equiv 0$ a.e. on $\mathbb{R}^N$. Moreover, if $a > 0$ or if $\lambda < \lambda_H k_1$, then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

If $L = \Delta_{p,G} = \operatorname{div}_L(|\nabla_L \cdot|^{p-2}\nabla_L \cdot)$ and V is the related Hardy potential, we obtain the following.

Corollary 3.3. *Let $Q > p > 1$ and let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions satisfying (1.5) with $p > \theta$, $p \ge k \ge 0$ and with a homogeneous norm $|\cdot|_L$ such that*

$$-\Delta_{p,G}|x|_L^{\frac{p-Q}{p-1}} = c\delta_0$$

and $\psi := |\nabla_L|\cdot|_L|$.[2] *Suppose that $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of*

$$\Delta_{p,G}u + \lambda\frac{\psi^p}{|x|_L^p}|u|^{p-2}u \ge a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N.$$

If $0 < \lambda \le (\frac{Q-p}{p})^p$ and x_0 is the unique solution of the equation

$$(x - 1 + p)^p \lambda = x(Q-p)^p, \quad x \ge 1,$$

then $u \le 0$ a.e. on $\mathbb{R}^N$, provided

$$p - 1 < q \le \frac{(Q-\theta)(p-1) + x_0(p-\theta)}{Q-p}. \tag{3.5}$$

In particular, if u is a solution of

$$\Delta_{p,G}u + \lambda\frac{\psi^p}{|x|_L^p}|u|^{p-2}u = a(x)|u|^{q-1}u \quad \text{on } \mathbb{R}^N, \tag{3.6}$$

then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

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1 That is $\mathscr{A}(x, -t, -\xi) = -\mathscr{A}(x, t, \xi)$ for any x, t and ξ.

2 If $\nabla_L = \nabla$, the Euclidean gradient, then $|\cdot|_L$ is the Euclidean norm and $\psi \equiv 1$.

Remark 3.4. (i) If $p = 2$, we know an explicit formula of x_0, namely

$$x_0 = \frac{Q-2}{2\lambda}\Big[Q - 2 + \sqrt{(Q-2)^2 - 4\lambda}\Big] - 1.$$

(ii) If $a \equiv$ constant, then (3.4) becomes

$$p - 1 < q < q_{cr} := \frac{Q(p-1) + x_0 p}{Q - p}$$

and for $\lambda < \lambda_H$, $q_{cr} > q_s := \frac{Q(p-1)+p}{Q-p}$. Notice that q_s is the Sobolev exponent associated to the operator $\Delta_{p,G}$. For the case of the standard Laplacian operator in the Euclidean space and the corresponding problem (3.6) see [1].

The above results can be generalised as follows.

Theorem 3.5. *Let $Q > p > 1$, let $\mathscr{A}$ be* **S**-*p*-**C**, *let $q > p - 1$ and let $a, V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions. Assume that there exist $R_0, M > 0$, $\alpha \ge 1$, $s \ge 1$ such that*

$$R^{-p\frac{q+\alpha}{q-p+1}} \int_{A_R} a^{-\frac{p+\alpha-1}{q-p+1}} \psi^{p\frac{q+\alpha}{q-p+1}} < M \quad \text{for } R > R_0 \tag{3.7}$$

and

$$R^{Q(s-1)} \int_{R_0<|x|_L<R} V^{s\frac{q+\alpha}{q-p+1}} a^{-s\frac{p+\alpha-1}{q-p+1}} < M \quad \text{for } R > R_0. \tag{3.8}$$

Suppose that (1.7) *holds with V satisfying*

$$V(x) \ge C_2 \frac{\psi^p}{|x|_L^p} \quad \text{for } |x|_L \text{ large}.$$

Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of (3.3). *If $\lambda \le k_1 \lambda_H \alpha(\frac{p}{p+\alpha-1})^p$, then*

$$u(x) = 0 \quad \text{for a.e. } |x| > R_0 \qquad \text{and} \qquad a(x)u(x) = 0 \quad \text{for a.e. } |x| < R_0.$$

In particular, $u \equiv 0$ a.e. on $\mathbb{R}^N$ provided a is a.e. positive or $\alpha > 1$.

For the proof, we need the following lemma. Notice that we assume that $\mathscr{A}$ satisfies a **W**-*p*-**C** condition only.

Lemma 3.6. *Let $\mathscr{A}$ be* **W**-*p*-**C** *and let $a, V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions. Assume that for $q > p - 1$ there exist $R_0, M > 0$, $\alpha \ge 1$, $s \ge 1$ such that* (3.7) *and* (3.8) *hold. If $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of* (3.3), *then*

$$\int_{\mathbb{R}^N} a u^{q+\alpha} < +\infty, \tag{3.9}$$

$$\int_{\mathbb{R}^N} \mathscr{A}(x, u\nabla_L u) \cdot \nabla_L u \, u^{\alpha-1} < +\infty, \tag{3.10}$$

$$\int_{\mathbb{R}^N} V u^{p+\alpha-1} < +\infty \tag{3.11}$$

and

$$\int_{\mathbb{R}^N} a u^{q+\alpha} + \alpha \int_{\mathbb{R}^N} \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u \, u^{\alpha-1} \le \lambda \int_{\mathbb{R}^N} V u^{\alpha+p-1}. \tag{3.12}$$

Proof. Since $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, we can apply [8, Lemma 5.1 and Remark 5.6]. Using $\varphi = u^\alpha \phi$ as test function[3] in Definition 2.2, we obtain

$$\int_{\mathbb{R}^N} a u^{q+\alpha}\phi + \alpha \int_{\mathbb{R}^N} \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u \, u^{\alpha-1}\phi \le -\int_{\mathbb{R}^N} \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L \phi \, u^\alpha + \lambda \int_{\mathbb{R}^N} V u^{\alpha+p-1}\phi.$$

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3 Since $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$, we can use $\varphi = u_\epsilon^\alpha \phi$, where u_ϵ is a standard mollified of u (see [8] for the mollification argument on a Carnot group). Then letting $\epsilon \to 0$ we obtain the claim.

Since $\mathscr{A}$ is **W**-p-**C**, by the Hölder and Young inequality, it follows that

$$\int_{\mathbb{R}^N} au^{q+\alpha}\phi + c_1\alpha \int_{\mathbb{R}^N} \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\, u^{\alpha-1}\phi \le c_2\alpha^{1-p}\int_{\mathbb{R}^N} u^{\alpha+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} + \lambda\int_{\mathbb{R}^N} Vu^{\alpha+p-1}\phi, \tag{3.13}$$

where

$$c_1 := 1 - \frac{\epsilon^{p'}}{p'k_2} > 0, \quad c_2 := \frac{p^p}{p\epsilon^p}$$

and $\epsilon > 0$ is sufficiently small. By Hölder's inequality with exponent $\chi := \frac{q+\alpha}{\alpha+p-1}$, from the right-hand side of (3.13) we obtain

$$\begin{aligned}\int_\Omega au^{q+\alpha}\phi &+ c_1\alpha\int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\, u^{\alpha-1}\phi \\ &\le c_2\alpha^{1-p}\Big(\int_S au^{q+\alpha}\phi\Big)^{1/\chi}\Big(\int_S \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}}a^{-\frac{\alpha+p-1}{q-p+1}}\Big)^{1/\chi'} + \lambda\int_\Omega Vu^{\alpha+p-1}\phi. \end{aligned}\tag{3.14}$$

Thus by Young's inequalities, we get

$$c_3\int_{\mathbb{R}^N} au^{q+\alpha}\phi + c_1\alpha\int_{\mathbb{R}^N}\mathscr{A}\cdot\nabla_L u\, u^{\alpha-1}\phi \le c_4\int_{\mathbb{R}^N}\frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}}a^{-\chi'/\chi} + \lambda\int_{\mathbb{R}^N} Vu^{p+\alpha-1}\phi, \tag{3.15}$$

where

$$c_3 := 1 - \frac{\epsilon^\chi c_2\alpha^{1-p}}{\chi}, \quad c_4 := \frac{c_2\alpha^{1-p}}{\epsilon^{\chi'}\chi'}$$

and $\epsilon > 0$.

Now we shall estimate the right-hand side of (3.15). We have

$$\begin{aligned}\int_{\mathbb{R}^N} Vu^{p+\alpha-1}\phi &= \int_{B_{R_0}} Vu^{p+\alpha-1}\phi + \int_{|x|_L>R_0} Vu^{p+\alpha-1}\phi \\ &\le C(\alpha,u,V,R_0) + \Big(\int_{|x|_L>R_0} au^{q+\alpha}\phi\Big)^{1/\chi}\Big(\int_{|x|_L>R_0} V^{\chi'}a^{-\chi'/\chi}\phi\Big)^{1/\chi'},\end{aligned}\tag{3.16}$$

which by Young's inequality yields

$$c_5\int_{\mathbb{R}^N} au^{q+\alpha}\phi + c_1\alpha\int_{\mathbb{R}^N}\mathscr{A}\cdot\nabla_L uu^{\alpha-1}\phi \le c_4\int_{A_R}\frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}}a^{-\chi'/\chi} + \lambda C(\alpha,u,V,R_0) + c_6\int_{|x|_L>R_0} V^{\chi'}a^{-\chi'/\chi}\phi,$$

where

$$c_5 := c_3 - \frac{\lambda\epsilon^\chi}{\chi}, \quad c_6 := \frac{\lambda}{\epsilon^{\chi'}\chi'}$$

and $\epsilon > 0$.

Let $\phi_0 \in \mathscr{C}_0^1(\mathbb{R})$ be a standard cut off function. Setting $\phi(x) := \phi_0(|\delta_{1/R}x|_L)$, for $R > R_0$, by (3.7), we have

$$\int_{A_R}\frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}}a^{-\chi'/\chi} \le c(\phi_0)R^{-p\chi'}\int_{A_R} a^{-\chi'/\chi}\psi^{p\chi'} < M. \tag{3.17}$$

If $s = 1$, then the hypothesis (3.8) assures that $\int_{|x|_L>R_0} V^{\chi'}a^{-\chi'/\chi}\phi$ is uniformly bounded by the constant M. If $s > 1$, then an application of Hölder's inequality with exponent s yields

$$\int_{|x|_L>R_0} V^{\chi'}a^{-\chi'/\chi}\phi \le \int_{R_0<|x|_L<2R} V^{\chi'}a^{-\chi'/\chi} \le |B_{2R}|^{1/s'}\Big(\int_{R_0<|x|_L<2R} V^{s\chi'}a^{-s\chi'/\chi}\Big)^{1/s} < M^s.$$

Therefore, the right-hand side of (3.15) is uniformly bounded with respect to R, that is

$$c_3\int_{B_R} au^{q+\alpha} + c_1\alpha\int_{B_R}\mathscr{A}\cdot\nabla_L u\, u^{\alpha-1} \le C.$$

Letting $R \to +\infty$, we obtain $au^{q+\alpha}$, $\mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\, u^{\alpha-1} \in L^1(\mathbb{R}^N)$. In other words (3.9) and (3.10) hold. Thus (3.16) implies (3.11).

Next, from (3.9) we have

$$\int_{A_R} au^{q+\alpha} \to 0 \quad \text{as } R \to +\infty,$$

which, in turn by (3.17) and (3.14), implies

$$\int_{\mathbb{R}^N} au^{q+\alpha} + \alpha\Big(1 - \frac{\epsilon^{p'}}{p'k_2}\Big) \int_{\mathbb{R}^N} \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u\, u^{\alpha-1} \le \lambda \int_{\mathbb{R}^N} Vu^{\alpha+p-1}.$$

Letting $\epsilon \to 0$ in the above inequality, (3.12) follows. □

Lemma 3.7. *Let the hypotheses of Lemma 3.6 be fulfilled. Set*

$$W^{1,p}_{A,\alpha} := \Big\{ v \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \mathscr{A}(x, v, \nabla_L v) \cdot \nabla_L v\, v^{\alpha-1} < +\infty,\ \int_{\mathbb{R}^N} Vv^{p+\alpha-1} < +\infty \Big\}.$$

Suppose that the following Hardy-type inequality holds:

$$\int_{\mathbb{R}^N} \mathscr{A}(x, v, \nabla_L v) \cdot \nabla_L v\, v^{\alpha-1} \ge \lambda_{H,\alpha} \int_{\mathbb{R}^N} Vv^{p+\alpha-1} \quad \text{for any } v \in W^{1,p}_{A,\alpha} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N),\ v \ge 0. \tag{3.18}$$

Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of (3.3). *If $\lambda \le \alpha\lambda_{H,\alpha}$, then*

$$au^{q+\alpha} \equiv 0 \quad \text{a.e. on } \mathbb{R}^N.$$

Proof. From (3.12) and (3.18) we have

$$\int_{\mathbb{R}^N} au^{q+\alpha} + \Big(\alpha - \frac{\lambda}{\lambda_{H,\alpha}}\Big) \int_{\mathbb{R}^N} \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u\, u^{\alpha-1} \le 0.$$

Since each addendum in the above inequality is nonnegative the claim follows. □

In order to prove Theorem 3.5 we need the following generalization of the weak maximum principle. This results is essentially based on the validity of (1.7). Indeed, we are in a position to apply some of the results of [8] obtaining a weak maximum principle for (3.19). For sake of completeness we state it here.

Theorem 3.8 (Generalized weak maximum principle). *The following statements hold.*

(i) *Assume that* (1.7) *holds and $\mathscr{A}$ is* **S**-*p*-**C**. *Let $u \in W^{1,p}_L(B_R)$ be a weak solution of*

$$\mathrm{div}_L(\mathscr{A}(x, u, \nabla_L u)) + \lambda V(x)|u|^{p-2}u \ge 0, \quad u \le 0 \quad \text{on } \partial B_{R_0}$$

with $\lambda < \lambda_H k_1$. Then $u \le 0$ a.e. on B_R.

(ii) *Let $Q > p > 1$ and let S be a homogeneous norm such that $\Delta_{p,L} S^{\frac{p-Q}{p-1}} = c\delta_0$ on $\mathbb{G}$. Let $u \in W^{1,p}_L(B_R)$ be a weak solution of*

$$\Delta_{p,L} u + \lambda \frac{|\nabla_L S|^p}{S^p} |u|^{p-2}u \ge 0 \quad \text{on } B_R, \qquad u \le 0 \quad \text{on } \partial B_{R_0},$$

with $\lambda \le (\frac{Q-p}{p})^p$. Then $u \le 0$ a.e. on B_R.

Proof of Theorem 3.5. First of all we observe that by density argument the Hardy inequality (1.7) holds for functions in $D^{1,p}_L(\mathbb{R}^N)$ (see Appendix A.2 for definition).

Set $\beta := 1 + \frac{\alpha-1}{p}$. We claim that $u^\beta \in D^{1,p}_L(\mathbb{R}^N)$. Indeed, from (3.10) we have

$$\int_{\mathbb{R}^N} |\nabla_L u^\beta|^p = \beta^p \int_{\mathbb{R}^N} |\nabla_L u|^p |u|^{\alpha-1} < +\infty.$$

From (3.11) it follows that

$$\int_{\mathbb{R}^N} V|u^\beta|^p = \int_{\mathbb{R}^N} V|u|^{p+\alpha-1} < +\infty.$$

Therefore by Theorem A.1 we deduce that $u^\beta \in D^{1,p}_L(\mathbb{R}^N)$.

Since $\mathscr{A}$ is **S**-p-**C**, from (1.7) we obtain

$$\int_{\mathbb{R}^N} \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\, u^{\alpha-1} \geq k_1 \int_{\mathbb{R}^N} |\nabla_L u|^p u^{\alpha-1} = \frac{k_1}{\beta^p}\int_{\mathbb{R}^N} |\nabla_L u^\beta|^p \geq \frac{k_1}{\beta^p}\lambda_H \int_{\mathbb{R}^N} V|u|^{p+\alpha-1}.$$

By the above Lemma 3.7, we deduce that if

$$\lambda \leq \alpha\frac{k_1}{\beta^p}\lambda_H = k_1\lambda_H\alpha\Big(\frac{p}{p+\alpha-1}\Big)^p,$$

then $au^{q+\alpha}\equiv 0$.

Now if a is positive, it is clear that $u\equiv 0$. Otherwise (3.7) implies $a(x)\neq 0$ for a.e. $|x|_L > R_0$. Therefore $u\equiv 0$ on $\mathbb{R}^N\setminus B_{R_0}$. Hence,

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \lambda V(x)u^{p-1}\geq 0,\quad u\geq 0 \quad\text{on } B_{R_0},\qquad u=0\quad \text{on } \partial B_{R_0}. \tag{3.19}$$

Therefore, since $\alpha > 1$, we have

$$\lambda \leq k_1\lambda_H\alpha\Big(\frac{p}{p+\alpha-1}\Big)^p < k_1\lambda_H.$$

Indeed, it is simple to verify that the function

$$f(\alpha) := \alpha\Big(\frac{p}{p+\alpha-1}\Big)^p$$

has a negative first derivative

$$f'(\alpha) = \Big(\frac{p}{p+\alpha-1}\Big)^p\frac{(p-1)(1-\alpha)}{p+\alpha-1}$$

for $\alpha > 1$ and $f(1) = 1$. Hence $\lambda \leq k_1\lambda_H f(\alpha) < k_1\lambda_H f(1)$. Therefore, from i) of Theorem 3.8 it follows that $u\equiv 0$ provided $\lambda < k_1\lambda_H$. □

Proof of Theorem 3.2. From (1.5) we have

$$R^{-p\frac{q+\alpha}{q-p+1}}\int_{A_R} a^{-\frac{p+\alpha-1}{q-p+1}}\psi^{p\frac{q+\alpha}{q-p+1}} \leq R^{-p\frac{q+\alpha}{q-p+1}+\theta\frac{p+\alpha-1}{q-p+1}}c\int_{A_R}\psi^{p\frac{q+\alpha}{q-p+1}-k\frac{p+\alpha-1}{q-p+1}} \leq CR^{\Theta_1}$$

with

$$\Theta_1 := Q + \frac{\theta(\alpha+p-1) - p(q+\alpha)}{q-p+1}.$$

Condition (3.7) is fulfilled provided $\Theta_1 \leq 0$, that is if

$$q \leq \frac{(Q-\theta)(p-1)+\alpha(p-\theta)}{Q-p}.$$

On the other hand, if $s > 1$ we have

$$\begin{aligned} R^{Q(s-1)}\int_{R_0<|x|_L<R} V^{s\frac{q+\alpha}{q-p+1}}a^{-s\frac{p+\alpha-1}{q-p+1}} &\leq CR^{Q(s-1)}\int_{R_0<|x|_L<R}|x|_L^{-s\nu\frac{q+\alpha}{q-p+1}+s\theta\frac{p+\alpha-1}{q-p+1}}\psi^{sh\frac{q+\alpha}{q-p+1}-sk\frac{p+\alpha-1}{q-p+1}} \\ &\leq CR^{Q(s-1)}\int_{R_0}^{R}\rho^{Q-1-s\nu\frac{q+\alpha}{q-p+1}+s\theta\frac{p+\alpha-1}{q-p+1}}. \end{aligned}$$

Condition (3.8) holds provided $Qs - s\nu\frac{q+\alpha}{q-p+1} + s\theta\frac{p+\alpha-1}{q-p+1} \leq 0$, that is if

$$q \leq \frac{(Q-\theta)(p-1)+\alpha(\nu-\theta)}{Q-\nu}. \tag{3.20}$$

Therefore, taking into account that $p\geq\nu>\theta$, inequalities (3.7) and (3.8) follow provided (3.20) holds.

The conclusion of Theorem 3.5 holds if for some $\alpha \geq 1$ we have $\alpha k_1 \geq \frac{\lambda(\alpha-1+p)^p}{p^p \lambda_H}$, or equivalently,

$$f(\alpha) := \frac{(\alpha-1+p)^p}{\alpha} \leq p^p \frac{\lambda_H k_1}{\lambda}.$$

Notice that the function f for $\alpha \geq 1$ is increasing, $f(1) = p^p$ and $f(+\infty) = +\infty$. Therefore $f(\alpha) \leq p^p \frac{\lambda_H k_1}{\lambda}$ provided $\alpha \in [1, x_0]$, where $x_0 \geq 1$ is the only solution of

$$f(\alpha) = p^p \frac{\lambda_H k_1}{\lambda}.$$

If $\lambda < \lambda_H k_1$, then $x_0 > 1$. Hence $\alpha > 1$. From Theorem 3.5 we complete the proof. □

Proof of Corollary 3.3. By Kato inequality (or reduction principle [8]) it is enough to consider nonnegative solutions. In order to apply Theorem 3.2, we need to check that inequality (1.7) holds for $V = \frac{\psi^p}{|x|_L^p}$ and that the best constant therein is given by

$$\lambda_H = \left(\frac{Q-p}{p}\right)^p.$$

This fact is proved in [5] (see Theorem A.3 below). Therefore, from Theorem 3.2, it follows that $au \equiv 0$ a.e. on $\mathbb{R}^N$. Further, if $\lambda < \lambda_H$, we are done. Otherwise assume $\lambda = \lambda_H$. Since (1.5) holds, we have that $u(x) = 0$ for a.e. $|x|_L > R_0$ and u is a solution of

$$\Delta_{p,G} u + \lambda_H \frac{\psi^p}{|x|_L^p} u^{p-1} \geq 0, \qquad u \geq 0 \quad \text{on } B_{R_0}, \quad u = 0 \quad \text{on } \partial B_{R_0}.$$

By (ii) of Theorem 3.8 we complete the proof. □

3.1 Sharpness of the results

In this subsection we point out that many of our assumptions (the bounds on q and λ, $N > p$, $\nu \geq p$ and $\theta < p$) are necessary in order to have a Liouville-type result. Namely, we prove that if an assumption is not satisfied, then there are nontrivial solution, that is we show the sharpness of our results.

We claim that the exponents provided in Theorems 3.2, 3.1 Corollary 3.3 are sharp. To this end, we deal with the inequality

$$\Delta_p u + \lambda \frac{u^{p-1}}{|x|^p} \geq \frac{u^q}{|x|^\theta} \quad \text{on } \mathbb{R}^N. \tag{3.21}$$

We begin proving that:

Claim 1. *If $\lambda > \lambda_H$, then* (3.21) *has a bounded positive solution for any $q > p-1$.*

Claim 2. *If $\lambda \leq \lambda_H$ and $q > q_{cr} := \frac{(N-\theta)(p-1)+x_0(p-\theta)}{N-p}$, then* (3.21) *has a bounded positive solution.*

Obviously, a positive function u solves inequality (3.21) if we show that

$$\frac{\Delta_p u + \frac{\lambda u^{p-1}}{|x|^p}}{\frac{u^q}{|x|^\theta}} \geq 1.$$

Indeed, let $q > p-1$, $N > p > \theta$. We define

$$u(x) := c(1+|x|^\beta)^{-\frac{p-\theta}{\beta(q-p+1)}}$$

for $\beta > 0$ and $c > 0$ that we choose later. Clearly, $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap \mathscr{C}(\mathbb{R}^N)$, it is bounded and vanishes at infinity. Now, by computation, for $r = |x|$, we have

$$\begin{aligned}\frac{\Delta_p u + \lambda u^{p-1} r^{-p}}{u^q r^{-\theta}} &= \left(\frac{p-\theta}{q-p+1}\right)^p c^{p-1-q} \frac{(1+r^\beta)^{\frac{p-\theta}{\beta}}}{r^{p-\theta}} R\left(\frac{r^\beta}{1+r^\beta}\right)\\ &\geq \left(\frac{p-\theta}{q-p+1}\right)^p c^{p-1-q} R\left(\frac{r^\beta}{1+r^\beta}\right),\end{aligned}$$

where $R(t) := a_2 t^p + a_1 t^{p-1} + a_0$, and

$$a_0 := \left(\frac{q-p+1}{p-\theta}\right)^p \lambda > 0,$$
$$a_1 := -(q-p+1)\frac{\beta(p-1)+N-p}{p-\theta} < 0,$$
$$a_2 := (p-1)\frac{\beta(q-p+1)+p-\theta}{p-\theta} > 0.$$

Therefore setting $m := \min_{[0,1]} R(t)$, we have

$$\frac{\Delta_p u + \lambda u^{p-1} r^{-p}}{u^q r^{-\theta}} \geq \left(\frac{p-\theta}{q-p+1}\right)^p c^{p-1-q}\, m.$$

If $m > 0$, then u is a solution of (3.21) for $c > 0$ small enough. Therefore, our goal is to choose $\beta > 0$ so that $m = m(\beta) > 0$.

Now, observing that $R(0) = a_0 > 0$ and $R(+\infty) = +\infty$, it follows that the function R has a minimum. Since R has only one critical point

$$t_0 = \frac{(q-p+1)(\beta(p-1)+N-p)}{p(\beta(q-p+1)+p-\theta)},$$

we have

$$m(\beta) = R(t_0) = \left(\frac{q-p+1}{p}\right)^p \left[\left(\frac{p}{p-\theta}\right)^p \lambda - \frac{1}{p-\theta}\frac{(\beta(p-1)+N-p)^p}{(\beta(q-p+1)+p-\theta)^{p-1}}\right],$$

Proof of Claim 1. Let $\lambda > \lambda_H = (\frac{N-p}{p})^p$. It is easy to check that

$$m(0) = \left(\frac{q-p+1}{p}\right)^p \left[\left(\frac{p}{p-\theta}\right)^p \lambda - \frac{(N-p)^p}{(p-\theta)^p}\right] = \left(\frac{q-p+1}{p-\theta}\right)^p [\lambda - \lambda_H] > 0.$$

Therefore, by continuity, for $\beta > 0$ small enough we have that $m(\beta) > 0$, and hence we obtain the claim.

Proof of Claim 2. For the sake of simplicity we set $\beta := \gamma(N-p)$. Let $q > q_{cr}$. We write q as

$$q := \frac{(N-\theta)(p-1) + x_0(p-\theta) + \epsilon(p-\theta)(p-1)}{N-p} > q_{cr}.$$

By computation we have

$$m(\beta) = m(\gamma(N-p)) = \left(\frac{(q-p+1)(N-p)}{p(p-\theta)}\right)^p \left[\left(\frac{p}{N-p}\right)^p \lambda - \frac{(\gamma(p-1)+1)^p}{(\gamma(x_0+(\epsilon+1)(p-1)+1^{p-1})}\right]$$
$$=: \left(\frac{(q-p+1)(N-p)}{p(p-\theta)}\right)^p g(\gamma).$$

We reach our goal by choosing $\gamma > 0$ such that $g(\gamma) > 0$. It is easy to see that the function g has a maximum at

$$\gamma_0 := \frac{\epsilon(p-1)+x_0-1}{p+\epsilon(p-1)+x_0-1}.$$

From the definition of x_0, that is $\lambda := \frac{x_0(N-p)^p}{(x_0-1+p)^p}$, we have

$$g(\gamma_0) = \left(\frac{p}{x_0-1+p}\right)^p x_0 - \left(\frac{p}{p+\epsilon(p-1)+x_0-1}\right)^p (x_0+\epsilon(p-1)).$$

Therefore, $g(\gamma_0) > 0$ if and only if

$$\left(1+\frac{\epsilon(p-1)}{p+x_0-1}\right)^p > 1+\frac{\epsilon(p-1)}{x_0}.$$

Next by using Bernoulli's inequality and the fact that $x_0 \geq 1$, we obtain

$$\left(1+\frac{\epsilon(p-1)}{p+x_0-1}\right)^p > 1+p\frac{\epsilon(p-1)}{p+x_0-1} \geq 1+\frac{\epsilon(p-1)}{x_0}.$$

This completes the proof.

Claim 3. *If $p > N$, we can construct a positive solution of* (3.1).

We first observe that in this case a Hardy inequality as (1.7) cannot hold. Moreover, the potentials V and a cannot be very singular, that is for x close to the origin, $V(x)$ cannot behave as $|x|^p$. Having this in mind, let us consider

$$\Delta_p u + \lambda V(x)u^{p-1} \geq a(x)u^q, \quad u \geq 0 \quad \text{on } \mathbb{R}^N.$$

We claim that the above inequality admits a positive solution for potentials V satisfying (1.6) with $\nu = p$ and any $\theta < p$, $\lambda > 0$, $q > p-1$. Let

$$V(x) := \frac{1}{(1+|x|^\beta)^{p/\beta}}, \quad a(x) := \frac{1}{(1+|x|^\beta)^{\theta/\beta}}$$

with $\beta := \frac{p-N}{p-1}$. Indeed, by a direct computation it is easy to see that the function

$$u(x) := \frac{c}{(1+|x|^\beta)^\alpha} \quad \text{with } \alpha := \frac{p-\theta}{(q-p+1)\beta}$$

satisfies

$$\begin{aligned}\frac{\Delta_p u + \lambda u^{p-1}V}{au^q} &= \alpha^p\beta^p c^{p-1-q}\Big[\lambda\Big(\frac{q-p+1}{p-\theta}\Big)^p + a_2 t^{p-\frac{p}{\beta}}\Big]\\ &\geq \alpha^p\beta^p c^{p-1-q}\lambda\Big(\frac{q-p+1}{p-\theta}\Big)^p\\ &= c^{p-1-q}\lambda\end{aligned}$$

where we have set

$$t := \frac{|x|^\beta}{1+|x|^\beta} \quad \text{and} \quad a_2 := \frac{(p-1)(p+\beta(q-p+1))}{p-\theta} = (p-1)\Big(1+\frac{1}{\alpha}\Big).$$

By choosing a suitable $c > 0$ we complete the proof.

As we can see we have not considered the case $N = p$. This particular situation will be studied elsewhere.

Claim 4. *We observe that the assumption $\theta < p$ in Theorem 3.2 is necessary in order to prove a nonexistence result.*

Indeed, if $V \geq a$, inequality (3.3) has positive nonconstant solutions. To see this, consider the special case $\operatorname{div}_L(\mathscr{A}(x,\cdot,\nabla_L\cdot)) = \Delta_p$. Let v be a nonnegative bounded solution of $\Delta_p v \geq 0$ (see [8, Remark 11.8]). By choosing a suitable $c > 0$ and $\epsilon > 0$, it follows that the function $u := \epsilon + cv$ is a bounded positive solution of (3.3).

Claim 5. *Assumption $\nu \geq p$ is also necessary in order to prove a nonexistence result. Dealing with a potential $V(x) = \frac{1}{|x|^\nu}$ with $\nu < p$, our original inequality* (3.3) *has solutions.*

Indeed, the inequality

$$\Delta u + \lambda u\frac{1}{|x|} \geq u^q, \quad u \geq 0 \quad \text{on } \mathbb{R}^3,$$

possesses a positive $\mathscr{C}^\infty(\mathbb{R}^3)$ ground state solution for any $\lambda > 0$ and $q > 1$. Indeed, defining

$$u(x) := ce^{-\frac{\lambda^2}{64}|x|^2}, \quad \textit{with} \quad c^{q-1} = \frac{3\lambda^2}{32},$$

we have

$$\begin{aligned}\Delta u + \lambda u\frac{1}{|x|} &= u^q\frac{\lambda}{1024c^{q-1}}e^{\frac{(q-1)\lambda^2}{64}|x|^2}\frac{1024-96\lambda|x|+\lambda^3|x|^3}{|x|}\\ &\geq u^q\frac{\lambda}{1024c^{q-1}}\frac{1024-96\lambda|x|+\lambda^3|x|^3}{|x|}\\ &\geq u^q\frac{\lambda}{1024c^{q-1}}96\lambda \geq u^q.\end{aligned}$$

4 Quasilinear equations related to weighted Gagliardo–Nirenberg-type inequality

Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of

$$\mathrm{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \lambda V(x)u^{p-1} \geq a(x)u^q + \mu W(x)u^{p-1}, \quad u \geq 0 \quad \text{on } \mathbb{R}^N. \tag{4.1}$$

In our main result of this section we require that the following weighted Gagliardo–Nirenberg inequality holds:

$$C_{V,W} \int_{\mathbb{R}^N} V(x)|u|^p \leq \Big(\int_{\mathbb{R}^N} |\nabla_L u|^p\Big)^{1/\gamma}\Big(\int_{\mathbb{R}^N} W(x)|u|^p\Big)^{1/\gamma'} \tag{4.2}$$

for any function $u \in D^{1,p}_L(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} W(x)|u|^p < +\infty$. Notice that if $\gamma = 1$, then (4.2) coincides with (1.7).

A concrete example of (4.2) is the classical weighted Gagliardo–Nirenberg inequality, that is choosing $V(x) := |x|^{-1}$ inequality (4.2) holds with $W \equiv 1$ and $\gamma = 2$. In this case the best constant is given by

$$C_{V,W} = \frac{N-1}{2}.$$

For potentials V and W satisfying (4.2) we have the following result, which is the analogue of Theorem 3.5.

Theorem 4.1. *Let $Q > p > 1$, let $\mathscr{A}$ be* **S**-p-**C**, *$q > p-1$ and let $a, V, W \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions. Assume that there exist $R_0, M > 0$, $\alpha \geq 1$, $s \geq 1$ such that* (3.7) *and* (3.8) *holds. Suppose that* (4.2) *holds with V satisfying*

$$V(x) \geq C_2 \frac{\psi^p}{|x|_L^p} \quad \text{for } |x|_L \text{ large}. \tag{4.3}$$

Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of (4.1). *If $\lambda > 0$ and*

$$\mu \geq \Big(\frac{\lambda}{\gamma C_{V,W}}\Big)^{\gamma'}(\gamma-1)\Big(\frac{(p+\alpha-1)^p}{p^p\alpha k_1}\Big)^{\gamma'-1}, \tag{4.4}$$

then

$$a u^{q+\alpha} \equiv 0 \quad \text{a.e. on } \mathbb{R}^N.$$

Proof. If u is a solution of (4.1), then u solves (3.3) as well. Therefore Lemma 3.6 applies and it is easy to check that under the same hypotheses we also have

$$\int_{\mathbb{R}^N} W(x)u^{p+\alpha-1} < +\infty,$$

and

$$\int_{\mathbb{R}^N} a u^{q+\alpha} + \mu \int_{\mathbb{R}^N} W u^{\alpha+p-1} + \alpha \int_{\mathbb{R}^N} \mathscr{A}(x,u,\nabla_L u)\cdot \nabla_L u\, u^{\alpha-1} \leq \lambda \int_{\mathbb{R}^N} V u^{\alpha+p-1}. \tag{4.5}$$

Again we argue as in the proof of Theorem 3.5. Setting $\beta := 1 + \frac{\alpha-1}{p}$, using (4.3) and (3.10) and arguing as in the proof of Theorem 3.5, we obtain $u^\beta \in D^{1,p}_L(\mathbb{R}^N)$. Therefore, (4.2) applies for $v := u^\beta = u^{\frac{1}{p}(\alpha+p-1)}$ and from (4.5) and the fact that $\mathscr{A}$ is **S**-p-**C**, we obtain

$$\int_{\mathbb{R}^N} a u^{q+\alpha} + \mu \int_{\mathbb{R}^N} W v^p + \alpha k_1 \int_{\mathbb{R}^N} |\nabla_L u|^p\, u^{\alpha-1} \leq \frac{\lambda}{C_{V,W}}\Big(\int_{\mathbb{R}^N} |\nabla_L v|^p\Big)^{1/\gamma}\Big(\int_{\mathbb{R}^N} W(x)|v|^p\Big)^{1/\gamma'}.$$

Now by using Young inequality with $\epsilon > 0$,

$$\int_{\mathbb{R}^N} a u^{q+\alpha} + \Big(\mu - \frac{\lambda}{C_{V,W}\gamma'\epsilon^{\gamma'}}\Big)\int_{\mathbb{R}^N} W v^p + \Big(\frac{\alpha k_1}{\beta^p} - \frac{\lambda\epsilon^\gamma}{C_{V,W}\gamma}\Big)\int_{\mathbb{R}^N} |\nabla_L v|^p \leq 0. \tag{4.6}$$

Choosing ϵ so that $\epsilon^{-\gamma} = \frac{\lambda\beta^p}{\alpha k_1 C_{V,W}\gamma}$, from (4.6) and our assumption on μ, we have the thesis. □

The following result is the analogue of Theorem 3.2.

Theorem 4.2. *Let $Q > p > 1$. Let $\mathscr{A}$ be* **S**-*p*-**C** *and let a, V, $W \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be nonnegative functions satisfying* (1.5) *and* (1.6) *with $p \geq \nu > \theta$ and $p \geq h \geq k \geq 0$. Assume that* (4.2) *holds. Let $\lambda > 0$ and*

$$\mu \geq \mu_1 := \Big(\frac{\lambda}{C_{V,W}\gamma k_1}\Big)^{\gamma'}(\gamma - 1)k_1,$$

where $C_{V,W}$ is the best constant in (4.2) *and k_1 is the constant structure appearing in the definition of* **S**-*p*-**C** *(see Definition 2.1). Let $u \in W^{1,p}_{L,\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of* (4.1). *If*

$$p - 1 < q \leq \frac{(Q - \theta)(p - 1) + x_0(\nu - \theta)}{Q - \nu},$$

where $x_0 \geq 1$ is the unique solution of the equation

$$\frac{(x - 1 + p)^p}{x} = \Big(\frac{\mu}{\mu_1}\Big)^{\gamma - 1} p^p, \quad x \geq 1,$$

then $au \equiv 0$ a.e. on $\mathbb{R}^N$.

Proof. In order to apply Theorem 4.1 we argue as in proof of Theorem 3.2. Hence, conditions (3.7) and (3.8) are fulfilled if (3.20) holds for some $\alpha \geq 1$. On the other hand the hypotheses we made on μ, x_0 and q assure that condition (4.4) holds. An application of Theorem 4.1 completes the proof. □

A Carnot groups

A.1 Basic facts

Here, we quote some facts on Carnot groups and refer the interested reader to [2, 4, 10, 11] for a more detailed information on this structures.

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ of dimension N with graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $i = 1, \dots, r - 1$ and $[V_1, V_r] = 0$. Such an integer r is called the *step* of the group. We set $l = n_1 = \dim V_1$, $n_2 = \dim V_2, \dots, n_r = \dim V_r$. A Carnot group $\mathbb{G}$ of dimension N can be identified, up to an isomorphism, with the structure of a *homogeneous Carnot group* $(\mathbb{R}^N, \circ, \delta_R)$ defined as follows: We identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split in r subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)}, \dots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that for any $R > 0$ the dilation $\delta_R(\xi) = (R\xi^{(1)}, R^2\xi^{(2)}, \dots, R^r\xi^{(r)})$ is a Lie group automorphism. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $\mathcal{G}$. For $i = 1, \dots, n_1 = l$ let Z_i be the unique vector field in $\mathcal{G}$ that coincides with $\partial/\partial\xi_i^{(1)}$ at the origin. We require that the Lie algebra generated by $Z_1, \dots, Z_l$ is the whole $\mathcal{G}$.

We denote with $\nabla_{\mathcal{L}}$ the vector field $\nabla_{\mathcal{L}} := (Z_1, \dots, Z_l)^T$ and we call it the *canonical horizontal vector field* in $\mathbb{G}$. The *canonical sub-Laplacian* on $\mathbb{G}$ is the second order differential operator defined by $\sum_{i=1}^l Z_i^2$.

Along the paper we choose $X_1, \dots, X_l$ stands for a basis of $\mathrm{span}\{Z_1, \dots, Z_l\}$. We denote with ∇_L the vector field $\nabla_L := (X_1, \dots, X_l)^T$ and we call it the a *horizontal vector field* in $\mathbb{G}$. Moreover, the vector fields $X_1, \dots, X_l$ are homogeneous of degree 1 with respect to δ_R. In this case

$$Q = \sum_{i=1}^r i n_i = \sum_{i=1}^r i \dim V_i$$

is called the *homogeneous dimension* of $\mathbb{G}$.

For $i = 1, \dots, l$, X_i^* stands for the formal adjoint of X_i. Hence, we shall use the notation

$$\mathrm{div}_L(h) = -\sum_{i=1}^l X_i^* h_i$$

for any vector field $h = (h_1, \dots, h_l)^T \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$.

A *sub-Laplacian* on $\mathbb{G}$ is the second order differential operator defined by $\Delta_G = \sum_{i=1}^{l} X_i^2$ and for $p > 1$ the p-sub-Laplacian operator is given by

$$\Delta_{p,G} u := \sum_{i=1}^{l} -X_i^*(|\nabla_L u|^{p-2} X_i u).$$

Since $X_1, \ldots, X_l$ generate the whole graded Lie algebra $\mathcal{G}$, the sub-Laplacian Δ_G satisfies the Hörmander hypoellipticity condition.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm* on $\mathbb{G}$ if $S(\xi^{-1}) = S(\xi)$, $S(\xi) = 0$ if and only if $\xi = 0$, and it is homogeneous of degree 1 with respect to δ_R (i.e. $S(\delta_R(\xi)) = RS(\xi)$). A homogeneous norm S defines on $\mathbb{G}$ a *pseudo-distance* defined as $d(\xi, \eta) := S(\xi^{-1}\eta)$, which in general is not a distance. If S and $\tilde{S}$ are two homogeneous norms, then they are equivalent, that is there exists a constant $C > 0$ such that $C^{-1}S(\xi) \le \tilde{S}(\xi) \le CS(\xi)$. Let S be a homogeneous norm, then there exists a constant $C > 0$ such that $C^{-1}|\xi| \le S(\xi) \le C|\xi|^{1/r}$, for $S(\xi) \le 1$. Examples of homogeneous norms are $S_\delta(\cdot)$ defined as

$$S_\delta(x) := \left(\sum_{i=1}^{N} (x_i^m)^{\frac{d}{\delta_i}}\right)^{\frac{1}{md}},$$

where $d := \delta_1\delta_2 \cdots \delta_N$ and m is the lowest even integer such that $m \ge \max\{\frac{\delta_1}{d}, \ldots, \frac{\delta_N}{d}\}$, or as

$$S(\xi) := \left(\sum_{i=1}^{r} |\xi_i|^{\frac{2r!}{i}}\right)^{\frac{1}{2r!}}.$$

Notice that if S is a homogeneous norm differentiable a.e., then $|\nabla_L S|$ is homogeneous of degree 0 with respect to δ_R; hence $|\nabla_L S|$ is bounded.

Special examples of Carnot groups are the Euclidean spaces $\mathbb{R}^Q$. Moreover, if $Q \le 3$, then any Carnot group is the ordinary Euclidean space $\mathbb{R}^Q$.

The simplest nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$. For an integer $n \ge 1$, the Heisenberg group $\mathbb{H}^n$ is defined as follows: let $\xi = (\xi^{(1)}, \xi^{(2)})$ with $\xi^{(1)} := (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $\xi^{(2)} := t$. We endow $\mathbb{R}^{2n+1}$ with the group law $\hat{\xi} \circ \tilde{\xi} := (\hat{x} + \tilde{x}, \hat{y} + \tilde{y}, \hat{t} + \tilde{t} + 2\sum_{i=1}^{n}(\tilde{x}_i\hat{y}_i - \hat{x}_i\tilde{y}_i))$. We consider the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t} \quad \text{for } i = 1, \ldots, n,$$

and the associated Heisenberg gradient $\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^T$. The Kohn Laplacian Δ_H is then the operator defined by

$$\Delta_H := \sum_{i=1}^{n} X_i^2 + Y_i^2.$$

The family of dilations is given by $\delta_R(\xi) := (Rx, Ry, R^2t)$ with homogeneous dimension $Q = 2n + 2$. In $\mathbb{H}^n$ a canonical homogeneous norm is defined as

$$|\xi|_H := \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2\right)^2 + t^2\right)^{\frac{1}{4}}.$$

A.2 Mollifiers

On a Carnot Group there is a "good" notion of *mollifier*. Let $\mathbb{G}$ be a homogeneous Carnot group on $\mathbb{R}^N$ and let S be a fixed homogeneous norm on $\mathbb{G}$. For every $x \in \mathbb{G}$ and every $r > 0$, the set

$$B_S(x, r) := \{y \in \mathbb{G} : S(x^{-1} \circ y) < r\}$$

is called the *S-ball with center at x and radius r*. For a fixed point $x \in \mathbb{G}$ and a set $A \subset \mathbb{G}$, the number

$$\text{dist}_S(x, A) := \inf\{S(x^{-1} \circ a) : a \in A\}.$$

is called *S-distance of x from A*. Let $\Omega \subset \mathbb{G}$ and $\epsilon > 0$ we define

$$\Omega_{S,\epsilon} := \{x \in \Omega : \mathrm{dist}_S(x, \partial\Omega) > \epsilon\}.$$

In order to avoid cumbersome notations we shall omit the norm S in the above symbols.

Let $m \in \mathscr{C}_0^\infty(\mathbb{G})$, $m \geq 0$, be given such that

$$\mathrm{supp}(m) \subset B_S(0,1) \quad \text{and} \quad \int m = 1.$$

For any $\eta > 0$ we set $m_\eta := \eta^{-Q} m(\delta_{1/\eta}(x))$. The family $(m_\eta)_\eta$ will be called a *family of mollifiers*.

Let $\Omega \subset \mathbb{G}$ be an open set and let $u \in L^1_{\mathrm{loc}}(\Omega)$. For any $x \in \Omega_\eta$ we define

$$u_\eta := (u \star_{\mathbb{G}} m_\eta)(x) := \int\limits_{B(x,\eta)} u(y) m_\eta(x \circ y^{-1})\, dy = \int\limits_{B(0,\eta)} u(y^{-1} \circ x) m_\eta(y)\, dy$$

calling u_η a *mollified of u related to the homogeneous norm S*.

It is easy that check that if $u \in L^1_{\mathrm{loc}}(\Omega)$, then

$$u_\eta \to u \quad \text{as } \eta \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega).$$

See [2].

A.3 A characterization of the space $D_L^{1,p}(\mathbb{R}^N)$

As in the Euclidean case, for $1 < p < Q$ the space $D_L^{1,p}(\mathbb{R}^N)$ is defined as the closure of $\mathscr{C}_0^1(\mathbb{R}^N)$ with respect to the norm $|\nabla_L u|_p = (\int |\nabla_L u|^p)^{1/p}$. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. It is clear that the assumption that the distribution $|\nabla_L u|$ belongs to $L^p(\mathbb{R}^N)$ does not guarantee that $u \in D_L^{1,p}(\mathbb{R}^N)$.

We have the following.

Theorem A.1. *Let $1 < p < Q$ and let $|\cdot|_L$ be a homogeneous norm. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be such that $|\nabla_L u| \in L^p(\mathbb{R}^N)$. If there exists $R_0 > 0$ such that*

$$\int\limits_{|x|_L > R_0} \frac{\psi^p}{|x|_L^p} |u|^p < \infty, \tag{A.1}$$

then $u \in D_L^{1,p}(\mathbb{R}^N)$.

Proof. Let $\phi_0 : \mathbb{R} \to \mathbb{R}$ be a usual cut off function and let $\phi_R := \phi_0(|\delta_{1/R} x|_L)$. We have

$$|\nabla_L(\phi_R u) - \nabla_L u|_p = |u \nabla_L \phi_R + \phi_R \nabla_L u - \nabla_L u|_p \leq |u \nabla_L \phi_R|_p + |\phi_R \nabla_L u - \nabla_L u|_p.$$

Now by Lebesgue dominated convergence theorem it follows that

$$|\phi_R \nabla_L u - \nabla_L u|_p^p = \int\limits_{\mathbb{R}^N} (1 - \phi_R)^p |\nabla_L u|^p \to 0.$$

On the other hand,

$$|u \nabla_L \phi_R|_p^p = \int\limits_{R < |x|_L < 2R} |u|^p |\phi_0'|^p \Big(\frac{|x|_L}{R}\Big) \frac{\psi^p}{R^p} \leq \int\limits_{R < |x|_L < 2R} |u|^p C(\phi_0, p) \frac{\psi^p}{|x|_L^p}$$

which, by (A.1), implies

$$\lim_{R \to +\infty} |u \nabla_L \phi_R|_p^p = 0.$$

Therefore for any $R > 0$, $\phi_R u \in L^1(\mathbb{R}^N)$ has compact support and $\phi_R u \to u$ in $D_L^{1,p}(\mathbb{R}^N)$.

Finally, if u is smooth we are done. Otherwise, by a standard mollification argument we complete the proof. □

A.4 The Kato inequality for quasilinear weakly elliptic operators

In this subsection we shall recall that suitable versions of the Kato inequality [12] proved in [8] hold for some *quasilinear weakly elliptic operators*.

Let Ω be an open set contained in $\mathbb{R}^N$, let $p \geq 1$ and let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. As usual, we denote by sign, sign$^+$ and u^+ the functions defined as follows:

$$\operatorname{sign}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0, \end{cases} \quad \operatorname{sign}^+(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \quad u^+ := \operatorname{sign}^+(u)u.$$

Theorem A.2 (Kato inequality: The quasilinear case). *Let $\mathscr{A}$ be* (WE). *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and let $u \in W^{1,p}_{L,\mathrm{loc}}(\Omega)$ be a weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x, u, \nabla_L u)) \geq f \quad \text{on } \Omega.$$

Then u^+ is a weak solution of

$$\operatorname{div}_L(\mathscr{A}(x, u^+, \nabla_L u^+)) \geq \operatorname{sign}^+ u\, f \quad \text{on } \Omega.$$

Moreover, if

$$\operatorname{div}_L(\mathscr{A}(x, u, \nabla_L u)) = f \quad \text{on } \Omega,$$

and $\mathscr{A}$ is odd, i.e.

$$\mathscr{A}(x, -t, -\xi) = -\mathscr{A}(x, t, \xi),$$

then $|u|$ satisfies

$$\operatorname{div}_L(\mathscr{A}(x, |u|, \nabla_L |u|)) \geq \operatorname{sign} u\, f \quad \text{on } \Omega.$$

A.5 Hardy inequality in the Carnot framework

The following result has been proved in [5].

Theorem A.3. *Let $p > 1$. Let $d : \Omega \to \mathbb{R}$ be a nonnegative nonconstant measurable function and let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that*

$$d^{-p}|\nabla_L d|^p, \; d^{(\alpha-1)(p-1)}|\nabla_L d|^{p-1} \in L^1_{\mathrm{loc}}(\Omega).$$

If $-\Delta_{L,p}(d^\alpha) \geq 0$ in the weak sense, then for every $u \in \mathscr{C}^1_0(\Omega)$ we have

$$\left(\frac{|\alpha|(p-1)}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} |\nabla_L d|^p \, dx \leq \int_\Omega |\nabla_L u|^p \, dx.$$

In particular, if S is a homogeneous norm such that[4] $-\Delta_{L,p} S^{\frac{p-Q}{p-1}} = c\delta_0$ *on $\mathbb{G}$ with $Q > p > 1$, then*[5]

$$\left(\frac{Q-p}{p}\right)^p \int_{\mathbb{G}} \frac{|u|^p}{S^p} |\nabla_L S|^p \, dx \leq \int_{\mathbb{G}} |\nabla_L u|^p \, dx, \quad u \in D^{1,p}_L(\mathbb{G}),$$

where the constant $(\frac{Q-p}{p})^p$ is sharp and it is not achieved.

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4 In the Euclidean setting S is the Euclidean norm.

5 See Appendix A.2 for the definition of $D^{1,p}_L(\mathbb{G})$.

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