Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications (ISIPTA '17)

Volume 62 of the Proceedings of Machine Learning Research:

ISIPTA '17, 10–14 July 2017
Lugano (Switzerland)
http://isipta.idsia.ch

Conference Organization

Program Chairs

Alessandro Antonucci
Giorgio Corani
Inés Couso
Sébastien Destercke

Steering Committee

Alessandro Antonucci
Giorgio Corani
Inés Couso
Sébastien Destercke
Marco Zaffalon

Program Committee

Thomas Augustin
Alessio Benavoli
Mik Bickis
Seamus Bradley
Andrey Bronevich
Marco Cattaneo
Giulanella Coletti
Fabio G. Cozman
Fabio Cuzzolin
Milan Daniel
Jasper De Bock
Cassio de Campos
Gert de Cooman
Thierry Denœux
Serena Doria
Didier Dubois
Love Ekenberg
Scott Ferson
Thomas Fetz
Brian Hill
Radim Jiroušek
Jim Joyce
Alexander Karlsson
Vladik Kreinovich
Tomáš Kroupa
Francesca Mangili
Denis Mauá
Andrés Masegosa
Enrique Miranda
Ignacio Montes
Serafín Moral
Michael Oberguggenberger
Arthur Paul Pedersen
Renato Pelessoni
Erik Quaeghebeur
Fabrizio Ruggeri
Teddy Seidenfeld
Damjan Škulj
Michael Smithson
Joerg Stoye
David Sundgren
Matthias Troffaes
Lev Utkin
Barbara Vantaggi
Jiřina Vejnarová
Paolo Vicig
Gregory Wheeler
Andrea Wiencierz
Marco Zaffalon

Local Organization

Alessandro Antonucci
Giorgio Corani
Jasper De Bock
David Huber

Organizational Support

We gratefully acknowledge organizational support from Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA), Università della Svizzera Italiana (USI), and Scuola Universitaria Professionale della Svizzera Italiana (SUPSI).

Preface

The ISIPTA (*International Symposium on Imprecise Probability: Theories and Applications*) meetings are the primary forum for research on imprecise probability. They are organized once every two years by SIPTA, the *Society for Imprecise Probability: Theories and Applications*. The first meeting was held in Ghent in 1999. It was followed by meetings in Ithaca, Lugano, Pittsburgh, Prague, Durham, Innsbruck, Compiègne and Pescara. The 2017 edition was held in Lugano (Switzerland) on July 10–14, 2017.

The proceedings of this edition are published for the first time within the *Proceedings of Machine Learning Research* (PMLR) series. This is by itself an acknowledgment of the scientific quality of the symposium.

Each submitted paper has been assigned to three program committee members. Eventually we accepted 32 papers. We would like to thank the 49 members of the program committee for their outstanding job during the reviewing process.

This edition of ISIPTA has been, for the first time, co-located with ECSQARU 2017, *the Fourteenth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. ECSQARU and ISIPTA are both biennial events with a significant overlap in their scopes. The co-location should be regarded as a first attempt to promote a cross-fertilization of work from researchers of these two communities.

Besides the technical program, keynote lectures were given by five distinguished invited speakers, namely: Leila Amgoud (IRIT, France), Alessio Benavoli (IDSIA, Switzerland), Jim Berger (Duke University, USA), Didier Dubois (IRIT, France), and Eyke Hüllermeier (Paderborn University, Germany).

Lugano, July 2017

Alessandro Antonucci
Giorgio Corani
Inés Couso
Sébastien Destercke

The Editorial Team:

Alessandro Antonucci
Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA)
Manno-Lugano (Switzerland)
alessandro@idsia.ch

Giorgio Corani
Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA)
Manno-Lugano (Switzerland)
giorgio@idsia.ch

Inés Couso
University of Oviedo
Oviedo (Spain)
couso@uniovi.es

Sébastien Destercke
University of Technology of Compiègne
Compiègne (France)
sebastien.destercke@hds.utc.fr

PMLR: Proceedings of Machine Learning Research, vol. 62, 85-96, 2017 ISIPTA '17

Weak Dutch Books versus Strict Consistency with Lower Previsions

Chiara Corsato CCORSATO@UNITS.IT
Renato Pelessoni RENATO.PELESSONI@ECON.UNITS.IT
Paolo Vicig PAOLO.VICIG@ECON.UNITS.IT
University of Trieste
Trieste (Italy)

Abstract

Several consistency notions for lower previsions (coherence, convexity, others) require that the suprema of certain gambles, having the meaning of gains, are non-negative. The limit situation that a gain supremum is zero is termed Weak Dutch Book (WDB). In the literature, the special case of WDBs with precise probabilities has mostly been analysed, and strict coherence has been proposed as a radical alternative. In this paper the focus is on WDBs and generalised strict coherence, termed strict consistency, with imprecise previsions. We discuss properties of lower previsions incurring WDBs and conditions for strict consistency, showing in both cases how they are differentiated by the degree of consistency of the given uncertainty assessment.

Keywords: Weak Dutch Books; (Williams') coherence; convex previsions; strict consistency.

1. Introduction

In the coherence approach to the theory of Imprecise Probabilities, consistency of an uncertainty measure is formalised requiring that the supremum of a certain gamble (a bounded random number, called *gain*) is non-negative. This is a common feature to several consistency notions, like *coherence* for lower previsions (Walley, 1991), Williams' coherence (W*-coherence*, Williams, 1975), convexity (Pelessoni and Vicig, 2005b), coherence for precise previsions or dF*-coherence* (de Finetti, 1974), and others. These concepts allow for a behavioural interpretation: the gain has the meaning of an agent's overall gain from a finite number of bets (rules for selecting the admissible gains distinguish the various consistency concepts).

Within this context, the limiting situation that the supremum of some gain G is precisely zero is termed *Weak Dutch Book* (WDB). In fact, under the behavioural interpretation, an agent whose gain is G would at best gain nothing, but otherwise lose, from the corresponding overall bet.

The literature on WDBs is not extended, and mostly focused on WDBs for dF-coherent probabilities. Contributions go back to the fifties of the last century (Kemeny, 1955; Shimony, 1955), when de Finetti's theory was getting widespread. In an attempt to avoid WDBs, the notion of *strict coherence* was introduced, although it became soon clear that it is subject to important constraints.

Properties of an uncertainty assessment incurring a WDB received a lesser attention, and the whole issue was rarely considered outside dF-coherence. The agent's beliefs of incurring a real loss were investigated in Crisma (2006) for dF-coherent probabilities, and in Vicig (2010) for (unconditional) coherent lower/upper previsions.

After introducing some preliminary material in Section 2, in this paper we focus precisely on the properties of WDB assessments, and on how they are differentiated under different consistency assumptions. We especially discuss W-coherence, convexity and dF-coherence. Section 3 is concerned with a 'local precision' property. This means that if the lower prevision $\underline{P}$ satisfies in general

a certain consistency requirement, then $\underline{P}$ complies with stronger properties, that make it closer to a precise prevision, on the set $\mathcal{D}_{\underline{G}}$ of (possibly conditional) gambles appearing in the expression of a WDB gain for $\underline{P}$. We prove one such relationship for the case of conditional convex lower previsions (Proposition 10). This implies that, if $\underline{P}$ is an unconditional convex prevision, $\underline{P}$ is the translation of a dF-coherent prevision on $\mathcal{D}_{\underline{G}}$, while if $\underline{P}$ is a coherent lower prevision on its domain, it is precisely a dF-coherent prevision on $\mathcal{D}_{\underline{G}}$. A result for W-coherence is also supplied. Section 4 discusses the agent's beliefs about incurring a real *Dutch Book* with a WDB gain. Again, these are differentiated by consistency of the assessment, ranging from near-certainty of avoiding any losses bounded away from zero with dF-coherence to no such reassuring beliefs for convexity, with W-coherence somewhat intermediate. We also discuss interdependencies between events of positive probability and maxima for WDB gains. In Section 5 strict consistency, a generalisation of strict coherence, is explored. After recalling a result in Corsato et al. (2017) for W-coherence, the perspective is that of analysing various strict consistency conditions, which are equivalent with dF-coherence, but not necessarily so in an imprecise framework (Proposition 21). Section 6 concludes the paper. Results not proven here may be found in the extended paper Corsato et al. (2017).

2. Preliminaries

Denote with $\mathcal{D}$ an *arbitrary* non-empty set of possibly conditional gambles. In the sequel, $\mathcal{D}$ will be the domain of a (precise or imprecise) conditional or unconditional prevision.

In the *conditional* case, the generic element (conditional gamble) of $\mathcal{D}$ is $X|B$, where X is a gamble and B is a non-impossible event. In the *unconditional* case, we shall simply term X the generic element (gamble) of $\mathcal{D}$.

The simplest non-trivial gamble is the *indicator* I_E of an event E. We shall not distinguish explicitly I_E and E, using the same symbol E for both. Thus we may speak of a set of events (of conditional events) $\mathcal{D}$, when for any $X \in \mathcal{D}$ (for any $X|B \in \mathcal{D}$), X is an (indicator of) event.

We recall the definition of dF-coherence for a precise prevision. In the sequel $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

Definition 1 *Given $P : \mathcal{D} \to \mathbb{R}$, P is a* (conditional) dF-coherent prevision *on $\mathcal{D}$ if, $\forall n \in \mathbb{N}^+$, $\forall X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, defining*

$$G = \sum_{i=1}^{n} s_i B_i \big(X_i - P(X_i|B_i)\big), \quad B = \bigvee_{i=1}^{n} B_i, \tag{1}$$

it holds that $\sup(G|B) \geq 0$.

If $\mathcal{D}$ is made of unconditional *gambles only, then (1) simplifies to*

$$G = \sum_{i=1}^{n} s_i \big(X_i - P(X_i)\big) \quad (B = \Omega), \tag{2}$$

and consequently the coherence condition reduces to $\sup G \geq 0$.

The condition of dF-coherence allows a betting (or behavioural) interpretation, where $g_i = s_i(X_i - P(X_i))$ in (2) is an *elementary gain* with *stake* s_i. It represents the agent's gain from buying (if $s_i > 0$) or selling (if $s_i < 0$) $s_i X_i$ for $s_i P(X_i)$. Thus the condition $\sup G \geq 0$ requires that no finite combination of elementary gains produces an overall uniformly negative gain to the agent.

The other consistency concepts we recall here have a similar betting interpretation. Actually, they can be derived from dF-coherence simply by introducing constraints on the stakes s_i. Their definitions and a few basic properties are laid down below (for more on this topic see e.g. Pelessoni and Vicig (2005b, 2009); Troffaes and de Cooman (2014); Walley (1991); Williams (1975)). Prior to this, let us recall some properties of dF-coherent previsions to be employed later on.

Proposition 2 *If P is a dF-coherent prevision on $\mathcal{D}$, then there exists a dF-coherent extension of P on any $\mathcal{D}' \supseteq \mathcal{D}$. Moreover, the following properties hold whenever their terms are defined:*

(a) $P(aX + bY|B) = aP(X|B) + bP(Y|B)$, $\forall a, b \in \mathbb{R}$ *(linearity).*

(b) $P(AX|B) = P(A|B)P(X|A \wedge B)$, $A \wedge B \neq \emptyset$ *(product rule).*

Definition 3 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be given. $\underline{P}$ is a W-*coherent *lower prevision on $\mathcal{D}$ if, $\forall n \in \mathbb{N}$, $\forall X_0|B_0, X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_i \geq 0$, with $i = 0, 1, \ldots, n$, defining*

$$\underline{G} = \sum_{i=1}^{n} s_i B_i \big(X_i - \underline{P}(X_i|B_i)\big) - s_0 B_0 \big(X_0 - \underline{P}(X_0|B_0)\big), \; B = \bigvee_{i=0}^{n} B_i,$$

it holds that $\sup(\underline{G}|B) \geq 0$.

W-coherence was introduced in Williams (1975); the structure-free form in Definition 3 was employed in Pelessoni and Vicig (2009). In the unconditional case, it is equivalent to Walley's coherence (Walley, 1991, Section 2.5.4 (a)), while it includes (strictly) Walley's definition of coherence in (Walley, 1991, Section 7.1.4 (b)) in the conditional environment.

Proposition 4 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision on $\mathcal{D}$. Then $\underline{P}$ has a least-committal W-coherent extension $\underline{E}$ on any $\mathcal{D}' \supseteq \mathcal{D}$, termed* natural extension*: $\underline{E} = \underline{P}$ on $\mathcal{D}$, and whatever is $\underline{P}^*$, W-coherent extension of $\underline{P}$ on $\mathcal{D}'$, $\underline{E} \leq \underline{P}^*$ on $\mathcal{D}'$. Moreover, for $X|B, Y|B \in \mathcal{D}$,*

(a) *If $X|B \leq Y|B$, then $\underline{P}(X|B) \leq \underline{P}(Y|B)$ (monotonicity).*

(b) $\underline{P}(X|B) \in [\inf(X|B), \sup(X|B)]$ *(internality).*[1]

Proposition 5 (Envelope theorem) *Given $\underline{P} : \mathcal{D} \to \mathbb{R}$, $\underline{P}$ is a W-coherent lower prevision on $\mathcal{D}$ if and only if there exists a non-empty set $\mathcal{P}$ of dF-coherent previsions on $\mathcal{D}$ such that, $\forall X|B \in \mathcal{D}$, it holds that $\underline{P}(X|B) = \min\{P(X|B) : P \in \mathcal{P}\}$.*

Definition 6 *Given $\underline{P} : \mathcal{D} \to \mathbb{R}$,*

(a) *$\underline{P}$ is a* convex *lower prevision on $\mathcal{D}$ if, $\forall n \in \mathbb{N}^+$, $\forall X_0|B_0, X_1|B_1, \ldots, X_n|B_n \in \mathcal{D}$, $\forall s_i \geq 0$, with $i = 1, \ldots, n$, and $\sum_{i=1}^{n} s_i = 1$ (convexity condition), defining*

$$\underline{G}_c = \sum_{i=1}^{n} s_i B_i \big(X_i - \underline{P}(X_i|B_i)\big) - B_0 \big(X_0 - \underline{P}(X_0|B_0)\big), \quad B = \bigvee_{i=0}^{n} B_i,$$

it holds that $\sup(\underline{G}_c|B) \geq 0$.

1. Being also W-coherent, a dF-coherent prevision satisfies properties (a), (b) too. Property (a) (monotonicity) also holds for a convex lower prevision (Definition 6).

(*b*) $\underline{P}$ *is* centered convex *on* $\mathcal{D}$ *if it is convex on* $\mathcal{D}$ *and* $\forall X|A \in \mathcal{D}, \emptyset|A \in \mathcal{D}$ *and* $\underline{P}(\emptyset|A) = 0$.

Proposition 7 (Envelope theorems with convex previsions) *Let* $\underline{P} : \mathcal{D} \to \mathbb{R}$ *be given. Then*

(*a*) *(Implicit Envelope Theorem, Pelessoni and Vicig (2005a))* $\underline{P}$ *is a convex lower prevision on* $\mathcal{D}$ *if and only if there exists a non-empty set* $\mathcal{P}$ *of* dF*-coherent previsions such that* $\forall X_0|B_0 \in \mathcal{D}, \exists\, P_{X_0|B_0} \in \mathcal{P}$*:* $\forall X|B \in \mathcal{D}$

$$\begin{aligned} &P_{X_0|B_0}(B|B \vee B_0)(P_{X_0|B_0}(X|B) - \underline{P}(X|B)) \geq \\ &P_{X_0|B_0}(B_0|B \vee B_0)(P_{X_0|B_0}(X_0|B_0) - \underline{P}(X_0|B_0)). \end{aligned} \tag{3}$$

(*b*) *(Envelope Theorem) With* unconditional *lower previsions,* $\underline{P}$ *is convex on* $\mathcal{D}$ *if and only if there exist a non-empty set* $\mathcal{P}$ *of* dF*-coherent previsions on* $\mathcal{D}$ *and* $\alpha : \mathcal{P} \to \mathbb{R}$ *such that,* $\forall X \in \mathcal{D}$*, it holds that* $\underline{P}(X) = \min\{P(X) + \alpha(P) : P \in \mathcal{P}\}$.

Moreover, $\underline{P}$ *is centered if and only if* $\min\{\alpha(P) : P \in \mathcal{P}\} = 0$.

Next to lower previsions, upper previsions could be assessed. Customarily, one refers to just one type of previsions by the *conjugacy* relation: $\overline{P}(-X|B) = -\underline{P}(X|B)$. Using conjugacy, the consistency notions for lower previsions and their properties can be expressed for upper previsions.

The various gains we recalled $(G, \underline{G}, \underline{G}_c)$ are gambles themselves, being functions of a finite number of gambles in $\mathcal{D}$ (and, in the conditional case, of indicators of their conditioning events). We mention next some other concepts regarding gains for later use.

Definition 8 *Let* $\underline{G}$ *be the gain in Definition 3.*

Then $\mathcal{D}_{\underline{G}} = \{X_0|B_0, X_1|B_1, \ldots, X_n|B_n\} \subseteq \mathcal{D}$ *is the set of conditional gambles in* $\underline{G}$.

The coarsest partition $\underline{G}|B$ *is defined on is termed* $\mathbb{P}_{\underline{G}}|B$*. In other words, the atoms* $\omega|B$ *of* $\mathbb{P}_{\underline{G}}|B$ *correspond to the distinct jointly possible values of* $X_0, X_1, \ldots, X_n$ *that imply* $B = \bigvee_{i=0}^{n} B_i$*. We say that* $\underline{G}$ *is a* WDB gain *if* $\sup(\underline{G}|B) = 0$.

Analogous definitions apply to the other gains we considered (for instance, $\mathcal{D}_{\underline{G}_c}$ *with* $\underline{G}_c$*).*

Given a partition $\mathbb{P}$, the powerset of $\mathbb{P}$ is called $\mathcal{A}(\mathbb{P})$. With a conditional gamble $X|B$, if X is defined on $\mathbb{P}$ and $B \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\}$, then $X|B$ is defined on $\mathbb{P}|B = \{\omega|B : \omega \in \mathbb{P}, \omega \Rightarrow B\}$.

3. Local Precision Properties of Weak Dutch Books

It is not difficult to obtain WDB gains, see the following simple example.

Example 1 *Let* $E \in \mathcal{D}$*, with* $\emptyset \neq E \neq \Omega$*. Let* $\underline{P}_1, \underline{P}_2 : \mathcal{D} \to \mathbb{R}$ *be such that* $\underline{P}_1(E) = 0$, $\underline{P}_2(E) = 1$*. Then* $\underline{P}_1, \underline{P}_2$ *are coherent lower probabilities on* $\{E\}$*. Consider the gains* $\underline{G}_1 = -s(E - \underline{P}_1(E)) = -sE$ *and* $\underline{G}_2 = s(E - \underline{P}_2(E)) = s(E - 1)$*, with* $s > 0$*. Then* $\max \underline{G}_1 = \underline{G}_1(\neg E) = 0 = \underline{G}_2(E) = \max \underline{G}_2$*, that is* $\underline{G}_1, \underline{G}_2$ *are WDB gains associated with* $\underline{P}_1, \underline{P}_2$*, respectively.*

In this section, we show that the existence of a WDB imposes some constraints both on convex and on coherent imprecise previsions, as for the gambles involved in the corresponding WDB gain.

Let us start with a convex lower prevision $\underline{P}(\cdot|\cdot)$. Its properties on those $\mathcal{D}_{\underline{G}_c}$ derived from WDB gains are investigated in Proposition 10.

Lemma 9 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional convex lower prevision, $\underline{G}_c, B$ be as in Definition 6. Then, any dF-coherent prevision $P_{X_0|B_0}$ satisfying (3) is such that*

$$\begin{aligned} &P_{X_0|B_0}(\underline{G}_c|B) = \\ &\sum_{i=1}^{n} s_i P_{X_0|B_0}(B_i \vee B_0|B)[P_{X_0|B_0}(B_i|B_i \vee B_0)(P_{X_0|B_0}(X_i|B_i) - \underline{P}(X_i|B_i)) \\ &- P_{X_0|B_0}(B_0|B_i \vee B_0)(P_{X_0|B_0}(X_0|B_0) - \underline{P}(X_0|B_0))] \geq 0. \end{aligned} \tag{4}$$

Proof Let $P_{X_0|B_0}$ satisfy (3). By Proposition 2 (a), recalling also that $\sum_{i=1}^{n} s_i = 1$, any dF-coherent extension of $P_{X_0|B_0}$ (still termed $P_{X_0|B_0}$) on a large enough set is such that

$$P_{X_0|B_0}(\underline{G}_c|B) = \sum_{i=1}^{n} s_i [P_{X_0|B_0}\big(B_i(X_i - \underline{P}(X_i|B_i))|B\big) - P_{X_0|B_0}\big(B_0(X_0 - \underline{P}(X_0|B_0))|B\big)]. \tag{5}$$

In general, we have that, for $i = 0, 1, \ldots, n$,

$$\begin{aligned} P_{X_0|B_0}\big(B_i(X_i - \underline{P}(X_i|B_i))|B\big) &= P_{X_0|B_0}\big(B_i\, X_i|B - \underline{P}(X_i|B_i)\, B_i|B\big) \\ &= P_{X_0|B_0}(B_i\, X_i|B) - \underline{P}(X_i|B_i)\, P_{X_0|B_0}(B_i|B) \\ &= P_{X_0|B_0}(X_i|B_i \wedge B)\, P_{X_0|B_0}(B_i|B) - \underline{P}(X_i|B_i)\, P_{X_0|B_0}(B_i|B) \\ &= \big(P_{X_0|B_0}(X_i|B_i) - \underline{P}(X_i|B_i)\big)\, P_{X_0|B_0}(B_i|B), \end{aligned}$$

using Proposition 2 (a) at the second equality, Proposition 2 (b) at the third, and $B_i \wedge B = B_i$ at the fourth. Using Proposition 2 (b) again, we get also, for any $i = 1, \ldots, n$,

$$\begin{aligned} P_{X_0|B_0}(B_i|B) &= P_{X_0|B_0}(B_i \vee B_0|B) P_{X_0|B_0}(B_i|B_i \vee B_0), \\ P_{X_0|B_0}(B_0|B) &= P_{X_0|B_0}(B_i \vee B_0|B) P_{X_0|B_0}(B_0|B_i \vee B_0). \end{aligned} \tag{6}$$

From (5), these derivations and (3), we obtain (4). ■

Proposition 10 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional convex lower prevision, $\underline{G}_c, B$ be as in Definition 6 and such that $\underline{G}_c$ is a WDB gain. Then, there exist a dF-coherent prevision $P_{X_0|B_0}$ satisfying (3) and $\alpha_{X_0|B_0} \in \mathbb{R}$ such that, for $i = 0$ and for any i such that $s_i > 0$ $(i = 1, \ldots, n)$, exactly one of the following holds*

(a) $P_{X_0|B_0}(B_i|B) = 0$;

(b) $\underline{P}(X_i|B_i) = P_{X_0|B_0}(X_i|B_i) + \dfrac{\alpha_{X_0|B_0}}{P_{X_0|B_0}(B_i|B)}$.

Proof Take $X_0|B_0$ in $\underline{G}_c$. Let $P_{X_0|B_0}$ be the dF-coherent prevision in Proposition 7 (a) and define

$$\alpha_{X_0|B_0} = -P_{X_0|B_0}(B_0|B)\big(P_{X_0|B_0}(X_0|B_0) - \underline{P}(X_0|B_0)\big).$$

Since now $\sup(\underline{G}_c|B) = 0$, using Proposition 4 (b) and Footnote 1 at the first inequality, Lemma 9 at the second, we get $0 = \sup(\underline{G}_c|B) \geq P_{X_0|B_0}(\underline{G}_c|B) \geq 0$, i.e. $P_{X_0|B_0}(\underline{G}_c|B) = 0$.

Let now $s_i > 0$ $(i = 1, \ldots, n)$ such that $P_{X_0|B_0}(B_i|B) > 0$. Since $P_{X_0|B_0}(\underline{G}_c|B) = 0$, by Lemma 9 (the expression in square brackets in (4) is non-negative by (3)), recalling (6) at the second equality, we get

$$\begin{aligned} 0 &= P_{X_0|B_0}(B_i \vee B_0|B)[P_{X_0|B_0}(B_i|B_i \vee B_0)(P_{X_0|B_0}(X_i|B_i) - \underline{P}(X_i|B_i)) \\ &\quad - P_{X_0|B_0}(B_0|B_i \vee B_0)(P_{X_0|B_0}(X_0|B_0) - \underline{P}(X_0|B_0))] \\ &= P_{X_0|B_0}(B_i|B)(P_{X_0|B_0}(X_i|B_i) - \underline{P}(X_i|B_i)) + \alpha_{X_0|B_0}, \end{aligned}$$

hence (b). The case $i = 0$ follows immediately from the definition of $\alpha_{X_0|B_0}$. ■

Thus, for a convex $\underline{P}$, a WDB implies the existence of a dF-coherent prevision $P_{X_0|B_0}$ such that $\underline{P}(X_i|B_i)$ differs from $P_{X_0|B_0}(X_i|B_i)$ by a term $\dfrac{\alpha_{X_0|B_0}}{P_{X_0|B_0}(B_i|B)}$, *for any* $X_i|B_i \in D_{\underline{G}} \setminus \{X_0|B_0\}$ *such that* $P_{X_0|B_0}(B_i|B) \neq 0$. This latter constraint becomes irrelevant when $\underline{P}$ is unconditional, since then $B_i = \Omega$, for $i = 0, 1, \ldots, n$. Therefore $B = \Omega$ as well as $B_i|B = \Omega$ $(i = 0, 1, \ldots, n)$, hence $P_{X_0|B_0}(B_i|B) = P_{X_0|B_0}(\Omega) = 1$. Proposition 10 specialises then to:

Proposition 11 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional convex lower prevision, and $\underline{G}_c$ as in Definition 6 with $B_i = \Omega$, for $i = 0, 1, \ldots, n$, be a WDB gain. Then there exist a dF-coherent prevision P_{X_0} on $\mathcal{D} \cup \{\underline{G}_c\}$ and $\alpha_{X_0} \in \mathbb{R}$ such that $\underline{P} = P_{X_0} + \alpha_{X_0}$ on $\mathcal{D}^+_{\underline{G}_c} = \{X_0\} \cup \{X_i : s_i > 0$, for $i = 1, \ldots, n\}$.*

Hence, convexity of an unconditional lower prevision $\underline{P}$ on $\mathcal{D}$ implies that $\underline{P}$ has a special structure on $\mathcal{D}^+_{\underline{G}_c}$, with WDBs: for each $X_i \in \mathcal{D}^+_{\underline{G}_c}$, $\underline{P}$ differs from a dF-coherent prevision P by the *same* constant α_P. Perhaps surprisingly, if $\underline{P}$ is *centered* convex, the preceding result does not imply that $\alpha_P = 0$ in all cases, but only if $\emptyset \in \mathcal{D}^+_{\underline{G}_c}$.

When $\underline{P}$ is a conditional W-coherent prevision, Proposition 5 can be applied in the place of Proposition 7 (a). We get the following proposition (Corsato et al., 2017).

Proposition 12 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision, $\underline{G}, B$ be as in Definition 3 and such that $\underline{G}$ is a WDB gain. Suppose $B_1|B, \ldots, B_n|B \in \mathcal{D}$. Define*

$$\mathcal{D}^+_{\underline{G}} = \{X_0|B_0\} \cup \{X_i|B_i \in \mathcal{D}_{\underline{G}} : s_i\underline{P}(B_i|B) > 0, \text{ for } i = 1, \ldots, n\}.$$

Then $\underline{P}$ is dF-coherent on $\mathcal{D}^+_{\underline{G}}$.

The condition $B_1|B, \ldots, B_n|B \in \mathcal{D}$ in Proposition 12 is not overly restrictive. If it is not met, we may consider a W-coherent extension $\underline{P}'$ of $\underline{P}$ on $\mathcal{D}' = \mathcal{D} \cup \{B_i|B : i = 1, \ldots, n\}$ and apply Proposition 12 to $\underline{P}'$ on $\mathcal{D}'$. However, the set on which $\underline{P}'$ is dF-coherent depends then on the specific extension. It is minimal when the natural extension of $\underline{P}$ is selected.

The result is subject to a second, more significant restriction. In fact, assuming $\underline{P}(B_i|B)$ positive is a sufficient but not necessary condition for dF-coherence of $\underline{P}$, i.e. $\underline{P}$ may be dF-coherent on a set larger than $\mathcal{D}^+_{\underline{G}}$.

The important special case that $\underline{P}$ is *unconditional*, i.e $B_i = \Omega$, for $i = 0, 1, \ldots, n$, hence $B = \Omega$, reinforces the result in Proposition 12. If all the stakes s_i, for $i = 1, \ldots, n$, are non-zero, since necessarily $\underline{P}(\Omega) = 1$, we get $\mathcal{D}_{\underline{G}} = \mathcal{D}^+_{\underline{G}}$ and Proposition 12 specialises to the following statement.

Proposition 13 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional coherent lower prevision. Let $\underline{G}$ be given by $\underline{G} = \sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - s_0\big(X_0 - \underline{P}(X_0)\big)$, with $s_0 \geq 0$, $s_i > 0$, for $i = 1, \ldots, n$, $\{X_0, X_1, \ldots, X_n\} = \mathcal{D}_{\underline{G}} \subseteq \mathcal{D}$, and assume that $\underline{G}$ is a WDB gain. Then $\underline{P}$ is dF-coherent on $\mathcal{D}_{\underline{G}}$.*

Thus a WDB implies that a coherent lower prevision is a dF-coherent prevision on $\mathcal{D}_{\underline{G}}$.

4. Further Features of Weak Dutch Books

Which are the agent's beliefs, with a WDB assessment, about suffering from a real *Dutch Book* (meaning a loss, if the gain has a maximum of zero, or a uniformly negative loss, if it does not achieve its supremum of zero)? In the simplest case, i.e. of a dF-coherent *probability* P, it was shown in (Crisma, 2006, Section 9.5.4) that $P(G < 0) = 0$. The generalisation to (unconditional) dF-coherent *previsions* has been investigated in Vicig (2010):

Proposition 14 *Given a dF-coherent prevision P on $\mathcal{D}$, let the WDB gain G be as in (2). Then (any dF-coherent extension of) P is uniquely determined on certain events concerning G, and precisely:*

(a) $P(G \leq -\varepsilon) = 0, \forall \varepsilon > 0$*;*

(b) if in addition $X_1, \ldots, X_n$ are all simple, we also have that $P(G < 0) = 0$.[2]

Thus, the results with precise previsions are rather reassuring. Take for instance case (b): although the agent cannot get any positive reward, whatever happens, she/he does not even believe that the bet will end up with a loss. However, the judgement on the potential vulnerability to Dutch books of a WDB assessment depends crucially on the kind of imprecise prevision being assessed.

In fact, the following result holds with W-coherent lower/upper previsions:

Proposition 15 *Given a W-coherent lower prevision $\underline{P}$ on $\mathcal{D}$, let $\underline{G}, B$ be as in Definition 3 such that $\underline{G}$ is a WDB gain. Then, for any W-coherent extension of $\underline{P}$ (still termed $\underline{P}$)*

(a) $\underline{P}(\underline{G}|B \leq -\varepsilon) = 0, \forall \varepsilon > 0$*;*

(b) *if $\mathcal{D}_{\underline{G}}$ is made of simple conditional gambles, $\underline{P}(\underline{G}|B < 0) = 0$.*

Proposition 15, which includes also W-coherence for unconditional lower/upper previsions as a special instance, is formally analogous to Proposition 14. Yet, it replaces precise with lower previsions, as for the Dutch book evaluations. The upper probability of, say, $(\underline{G}|B < 0)$ in case (b) may well be even 1, as shown in Corsato et al. (2017). One may wonder whether it is at least *always* possible to put $\overline{P}(\underline{G}|B < 0) = 0$ or more generally (for an arbitrary $\underline{G}|B$) $\overline{P}(\underline{G}|B \leq -\varepsilon) = 0$, if wished. The answer is negative even in an unconditional environment:

Proposition 16 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional W-coherent lower prevision, and $\underline{G}$ defined in Proposition 13, with $s_i > 0$ $(i = 1, \ldots, n)$ be a WDB gain. Then*

(a) *if $\mathcal{D}_{\underline{G}} = \mathcal{D}$, it is coherent to put $\overline{P}(\underline{G} \leq -\varepsilon) = 0, \forall \varepsilon > 0$;*

(b) *otherwise this choice may be incoherent.*

2. The dF-coherent extension of P is mentioned explicitly because (the indicators of) the events $(G \leq -\varepsilon)$ and $(G < 0)$ need not belong to $\mathcal{D}$. Similar specifications will be omitted hereafter.

We may conclude that the (conditional) *p-box* of a WDB gain $\underline{G}|B$ for a W-coherent $\underline{P}$ has a special structure, as for its lower distribution function $\underline{F}(x) = \underline{P}(\underline{G}|B \leq x)$, $x \in \mathbb{R}$. $\underline{F}$ is a single-step function, identically equal to 0 for any $x < 0$, to 1 for any $x \geq 0$. On the contrary, the upper distribution function $\overline{F}(x) = \overline{P}(\underline{G}|B \leq x)$, $x \in \mathbb{R}$, is essentially unconstrained and need not coincide with $\underline{F}(x)$ if $(\underline{G}|B \leq x)$ is a non-trivial event.

Weaker notions than (W-)coherence may allow for even weaker implications about the non-occurrence of Dutch Books. In particular, in the case of centered convexity, the agent may have no strong belief that a Dutch Book associated with the gain $\underline{G}$ will be avoided. In fact, examples may be built to show that not even $\underline{P}(\underline{G} \leq -\varepsilon)$ may be forced to be zero.

Summing up, when an uncertainty assessment incurs a WDB the agent's evaluation about avoiding a real Dutch Book depends on the degree of precision of the consistency notion the assessment satisfies. The self-protection offered by dF-coherence is maximal, whilst convexity does not ensure that $\underline{P}(\underline{G} \leq -\varepsilon)$ may be consistently set to zero.

Another facet of WDBs is concerned with conditions for the gain supremum of zero to be attained. Of course it is, if the gain involves only simple gambles, and in particular events. To explore this issue in more general situations the next result proves to be useful.

Proposition 17 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be an unconditional W-coherent lower prevision and $\underline{G}$ as in Definition 3, with $B_i = \Omega$, for $i = 0, 1, \ldots, n$. Let also $\underline{G}$ be a WDB gain. Then, for any event $E \in \mathcal{D}$ with $\underline{P}(E) > 0$, it holds that $\sup(\underline{G}|E) = 0$.*

Now suppose that $\mathcal{D}$ includes some atom $\omega \in \mathbb{P}_{\underline{G}}$, the coarsest partition $\underline{G}$ is defined on. If $\underline{P}(\omega) > 0$, Proposition 17 implies (with $E = \omega$) that

$$\sup(\underline{G}|\omega) = \underline{G}(\omega) = 0,$$

hence $\underline{G}$ achieves its supremum (at least) at ω. More generally, it holds that

Corollary 18 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$, $\underline{G}$ be as in Proposition 17. Let $\mathbb{P} \subseteq \mathcal{D}$ be either $\mathbb{P}_{\underline{G}}$ or a partition finer than $\mathbb{P}_{\underline{G}}$, $e \in \mathbb{P}, \omega \in \mathbb{P}_{\underline{G}}$ be such that $e \Rightarrow \omega$ and $\underline{P}(e) > 0$. Then $\underline{G}(e) = \underline{G}(\omega) = 0$.*

Corollary 18 implies also that if $\sup \underline{G}$ is not achieved, then necessarily $\underline{P}(\omega) = 0$, $\forall \omega \in \mathbb{P}_{\underline{G}}$. Yet, there may be some $E \in \mathcal{D}$ such that $\underline{P}(E) > 0$, hence implying $\sup(\underline{G}|E) = 0$ by Proposition 17, but $E \notin \mathbb{P}_{\underline{G}}$. Letting $\mathcal{P} = \{\omega \in \mathbb{P}_{\underline{G}} : \underline{P}(\omega) > 0\}$ and $\mathcal{N} = \{\omega \in \mathbb{P}_{\underline{G}} : \underline{G}(\omega) = 0\}$, it is $\mathcal{P} \subseteq \mathcal{N}$, by Corollary 18. Thus the cardinality of $\mathcal{P}$ is a lower bound to that of the set of atoms where $\underline{G}$ achieves the value of zero. However, it is interesting to note that other causes may be influencial too. In the next example the number of such atoms depends on the choice of the stakes.

Example 2 *Let $\mathcal{D} = \{E_1, E_0, \neg E_0, \neg E_1 \wedge E_0\}$, with $E_1 \wedge \neg E_0 = \emptyset$, $\underline{P} : \mathcal{D} \to \mathbb{R}$ be the vacuous lower probability and $\mathcal{D}_{\underline{G}} = \{E_1, E_0\}$. It may be checked that $\max \underline{G} = \max(s_1(E_1 - 0) - s_0(E_0 - 0)) = 0$ if $s_0 \geq s_1 > 0$. Here $\mathbb{P}_{\underline{G}} = \{E_1, \neg E_0, \neg E_1 \wedge E_0\}$, $\mathcal{P} = \emptyset$, while there are one or two atoms of $\mathbb{P}_{\underline{G}}$ where $\underline{G}$ attains its maximum of zero, according to whether, respectively, $s_1 < s_0$ or $s_1 = s_0$. In fact $\underline{G}(E_1) = s_1 - s_0 \leq 0$ iff $s_1 \leq s_0$, in particular $\underline{G}(E_1) = 0$ iff $s_1 = s_0$, $\underline{G}(\neg E_0) = 0$, $\underline{G}(\neg E_1 \wedge E_0) = -s_0 < 0$.*

Testing Weak Dutch Books. If it is not known whether, given a coherent $\underline{P}$, $\sup \underline{G} = 0$ or not, we can try to rule out the possibility of a WDB by checking the sign of $\underline{G}$ at some $\omega \in \mathbb{P}_{\underline{G}}$ such that $\underline{P}(\omega) > 0$ (if any). In fact:

- if $\underline{G}(\omega) > 0$, then obviously $\sup \underline{G} > 0$;
- if $\underline{G}(\omega) < 0$, then $\sup \underline{G} > 0$ by Corollary 18.

This method is very simple, but allows no conclusion when $\underline{G}(\omega) = 0$.[3] In fact, it is clearly possible that $\underline{G}(\omega) = 0$ and $\sup \underline{G} = 0$, but even when $\underline{G}(\omega) = 0$ for all $\omega \in \mathbb{P}_{\underline{G}}$ such that $\underline{P}(\omega) > 0$, $\sup \underline{G}$ may be strictly positive.

5. Strict Consistency

As soon as the behavioural interpretation of de Finetti's theory of subjective probabilities became more widespread, the issue of whether WDBs could possibly be avoided was investigated. Thus, as early as the mid-fifties of the last century Kemeny (1955) and Shimony (1955) proposed the most immediate solution: rule out WDBs by redefining coherence. They referred to (precise) probabilities only, replacing the condition $\sup G \geq 0$ with $\sup G > 0$, for any admissible $G \neq 0$, which is what is called *strict coherence* today. As well-known, strict coherence has non-negligible drawbacks, like that of being confined to a denumerable environment. Alternative ways of tackling WDBs have also been developed. We discuss in Corsato et al. (2017) that due to Wagner (2007) and based on the interpretation of buying/selling prices in betting schemes, going back to Walley (1991), and the one resorting to desirability concepts (see e.g. Quaeghebeur (2014)). Alternatively, Pedersen (2014) introduces a qualitative version of strict coherence for comparing (not necessarily bounded, unconditional) gambles.

However, little has been said about the role and properties of (some extended version of) strict coherence with imprecise rather than precise probabilities. It is relatively simple to extend the strict coherence approach (Corsato et al., 2017):

Definition 19 *Let $\mu : \mathcal{D} \to \mathbb{R}$ be a measure, whose consistency requires that $\sup(G|B) \geq 0$ for any conditional gain $G|B$ admissible according to certain rules. Then μ is* strictly consistent *if, for each such $G|B$, either $G|B = 0$ or $\sup(G|B) > 0$.*

As for the issue of characterising strict consistency, the case of conditional coherence was hinted in Williams (1975) and is tackled for W-coherent previsions in the next proposition.

Proposition 20 *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a W-coherent lower prevision. Then,*

(a) If $\underline{P}$ is strictly W-coherent on $\mathcal{D}$,

$$\underline{P}(A|B) > 0, \text{ for all events } A|B \in \mathcal{D}, A|B \neq \emptyset|B. \tag{7}$$

(b) If $\underline{P}$ is not strictly W-coherent on $\mathcal{D}$ and, for any WDB gain $\underline{G}|B \neq 0$ as in Definition 8, $\exists\, \varepsilon > 0 : (\underline{G}|B \leq -\varepsilon) \in \mathcal{D}$ is non-impossible, then $\exists\, A|B \in \mathcal{D}, A|B \neq \emptyset|B : \underline{P}(A|B) = 0$.

Clearly, Proposition 20 concerns the special case of (unconditional) coherent lower previsions too. For these previsions, (7) requires *strict positivity* (sP) of any non-impossible event in $\mathcal{D}$. It can be seen that this again limits the effectiveness of strict coherence to denumerable settings. Turning to another question, in today's language the necessary and sufficient condition for strict dF-coherence

3. Alternatively, linear programming could potentially be employed in some special cases.

of a dF-coherent probability in Kemeny (1955) asks instead for *strict normalisation* (sN), i.e., that $P(E) < 1$, for any $E \neq \Omega$. In the realm of dF-coherent probabilities, this is obviously equivalent to (sP). However, as often happens, the equivalence in a precise framework conceals a more complex situation in the field of imprecision. To see this, consider the following conditions for an uncertainty measure μ on a domain $\mathcal{D}$ of events:

$$
\begin{array}{lrll}
(sM) & \forall E, F \in \mathcal{D}, & \text{if } E \Rightarrow F, E \neq F \quad \text{then } \mu(E) < \mu(F) & \text{(strict Monotonicity);} \\
(sP) & \forall E \in \mathcal{D}, & \text{if } E \neq \emptyset \quad \text{then } \mu(E) > 0 & \text{(strict Positivity);} \\
(sN) & \forall E \in \mathcal{D}, & \text{if } E \neq \Omega \quad \text{then } \mu(E) < 1 & \text{(strict Normalisation).}
\end{array}
$$

Then, it holds that:

Proposition 21 *Let $\mathcal{A}$ be an algebra of events (i.e., $\forall E \in \mathcal{A}$, $\neg E \in \mathcal{A}$, $\forall E, F \in \mathcal{A}$, $E \wedge F \in \mathcal{A}$), and let $\underline{P} : \mathcal{A} \to \mathbb{R}$ be a lower probability.*

(a) *If $\underline{P}$ is* W-coherent, *then* $(sM) \Leftrightarrow (sP) \Rightarrow (sN)$,
while (sN) implies neither of $(sM), (sP)$.

(b) *If $\underline{P}$ is* centered convex, *then* $(sM) \Rightarrow (sP) \Rightarrow (sN)$,
while (sP) does not imply (sM), nor does (sN) imply (sP).

Proof Recall that $\emptyset, \Omega \in \mathcal{A}$ and that for any $E \in \mathcal{A}$, $\emptyset \Rightarrow E \Rightarrow \Omega$.
Proof of (a). $(sM) \Leftrightarrow (sP)$. Let (sM) hold, and let $E \in \mathcal{A}$, $E \neq \emptyset$. We have $\underline{P}(E) > \underline{P}(\emptyset) = 0$. Assume now (sP) is satisfied. Let $E, F \in \mathcal{A}$ such that $E \Rightarrow F$ and $E \neq F$. We have $\neg E \wedge F \in \mathcal{A} \setminus \{\emptyset\}$. Since then $\underline{P}(\neg E \wedge F) > 0$ by (sP), by superlinearity we get $\underline{P}(E) < \underline{P}(E) + \underline{P}(\neg E \wedge F) \leq \underline{P}(F)$.
$(sP) \Rightarrow (sN)$. By the previous step, it is equivalent to $(sM) \Rightarrow (sN)$, which holds: taking $E \neq \Omega$, by (sM) $\underline{P}(E) < \underline{P}(\Omega) = 1$.
$(sN) \not\Rightarrow (sP)$. Let us consider a non-trivial event E, $\mathcal{A} = \{\emptyset, E, \neg E, \Omega\}$ and $\underline{P} : \mathcal{A} \to \mathbb{R}$ given by $\underline{P}(\emptyset) = \underline{P}(E) = 0$, $\underline{P}(\neg E) = \varepsilon$, for some $\varepsilon \in]0, 1[$, $\underline{P}(\Omega) = 1$. Then $\underline{P}$ is a coherent lower probability on $\mathcal{A}$ satisfying (sN) but not (sP) (nor its equivalent condition (sM)).

Proof of (b). $(sM) \Rightarrow (sP)$: same as in the proof of (a).
$(sP) \not\Rightarrow (sM)$. Can be shown by means of a (counter)example. For this, let $\emptyset \neq E \Rightarrow F \neq \Omega$, $E \neq F$. Thus $\mathbb{P} = \{E, \neg E \wedge F, \neg F\}$ is a partition. Let $\mathcal{A} = \mathcal{A}(\mathbb{P})$, and $\underline{P} = \min\{P_1, P_2 + 0.2\}$, where P_1, P_2 are defined in Table 1 (only the relevant events in $\mathcal{A}$ are displayed).

	$\emptyset$	E	$\neg E \wedge F$	$\neg F$	F
P_1	0	0.1	0.2	0.7	0.3
P_2	0	0	0	1	0
$P_2 + 0.2$	0.2	0.2	0.2	1.2	0.2
$\underline{P}$	0	0.1	0.2	0.7	0.2

Table 1: Data for the (counter)example.

- $\underline{P}$ is centered convex on $\mathcal{A}$ (by Proposition 7), but not coherent: $\underline{P}(E) + \underline{P}(\neg E \wedge F) > \underline{P}(F)$, thus $\underline{P}$ does not comply with superadditivity.

- $\underline{P}$ satisfies (sP) (on the events in $\mathbb{P}$ and hence, by monotonicity of convex lower previsions, on all events in $\mathcal{A}$).
- $\underline{P}$ does not satisfy (sM): $\neg E \wedge F \Rightarrow F$ (and $\neg E \wedge F \neq F$), while $\underline{P}(\neg E \wedge F) = \underline{P}(F) = 0.2$.

$(sP) \Rightarrow (sN)$. Let $\underline{P}(E) > 0$, $\forall E \neq \emptyset$. Recall that a centered convex lower prevision avoids sure loss (Pelessoni and Vicig, 2005b), and as such satisfies the inequality $\underline{P}(X) + \underline{P}(\mu - X) \leq \mu, \forall \mu \in \mathbb{R}$ (Walley, 1991, Section 2.4.7 (c)). Putting $X = E$, $\mu = 1$, the inequality boils down to $\underline{P}(E) + \underline{P}(\neg E) \leq 1$. This implies $\underline{P}(E) < 1$ if $E \neq \Omega$ since then $\underline{P}(\neg E) > 0$ by assumption.

$(sN) \not\Rightarrow (sP)$. Indeed the implication is not valid under the stronger assumption that $\underline{P}$ is coherent, as proven in (a). ■

Comments. When $\mu = P$, a dF-coherent probability, in $(sM), (sP), (sN)$ and P is defined on an algebra $\mathcal{A}$, then $(sM) \Leftrightarrow (sP) \Leftrightarrow (sN)$, by normalisation and linearity of P.

We may summarise the situation in the next figure (only valid implications are displayed).

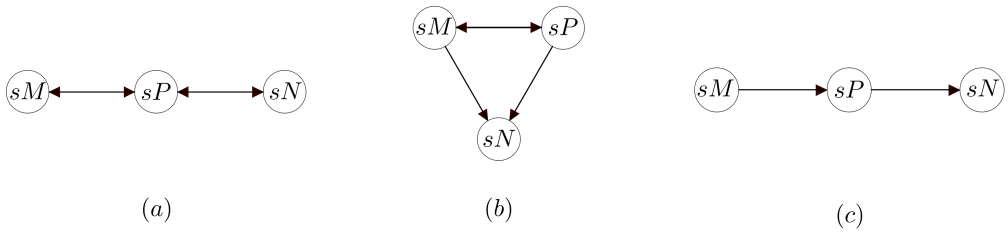

Figure 1: Comparison among the conditions $(sM), (sP), (sN)$ for either a dF-coherent probability in case (a), or a lower probability which is coherent (b) or centered convex (c).

It is clear from Proposition 21 why Proposition 20 refers to strict positivity instead of strict normalisation as in Kemeny (1955): even in an unconditional frame, strict positivity is tighter. To put it differently, requiring (sN) does not prevent $\underline{P}$ from incurring a WDB: it suffices that there is a possible E with $\underline{P}(E) = 0$ to which the WDB gain $\underline{G} = -s_0 E \leq 0$ is associated. Instead, (sM) could replace (sP) on algebras, while on more general domains (sP) is easier to work with. However, (sM) remains the only relevant condition for strict convexity.

Interestingly, these relationships may change with upper probabilities. Thus, when $\overline{P}$ is a coherent upper probability on an algebra $\mathcal{A}$, (sN) and (sP) exchange their roles. Using the conjugacy relation $\underline{P}(E) = 1 - \overline{P}(\neg E)$, we deduce that

$$(sM) \Leftrightarrow (sN) \Rightarrow (sP), \quad (sP) \not\Rightarrow (sM), \quad (sP) \not\Rightarrow (sN).$$

6. Conclusions

In this paper the properties of assessments incurring WDBs have been explored by their degree of consistency. The results point out a certain differentiation and a number of perhaps surprising features of such assessments. By contrast, the more known special case of dF-coherent precise

probabilities often flattens these differences. The situation is similar for strict consistency, the generalisation of strict coherence that avoids WDBs, even though its domain of application remains restricted even with W-coherence. Thus, in general WDBs are something to coexist with.

Acknowledgements

R. Pelessoni and P. Vicig acknowledge partial support by the FRA2015 grant 'Mathematical Models for Handling Risk and Uncertainty'.

References

C. Corsato, R. Pelessoni, and P. Vicig. Weak Dutch Books with imprecise previsions. *International Journal of Approximate Reasoning*, 88:72–90, 2017.

L. Crisma. *Introduzione alla teoria delle probabilità coerenti*. EUT, 2006.

B. de Finetti. *Theory of Probability*. Wiley, 1974.

J. G. Kemeny. Fair bets and inductive probabilities. *J. Symb. Logic*, 20:263–273, 1955.

A. P. Pedersen. Comparative Expectations. *Stud. Logica*, 102:811–848, 2014.

R. Pelessoni and P. Vicig. Envelope theorems and dilation with convex conditional previsions. In F. Cozman, R. Nau, and T. Seidenfeld, editors, *Proc. ISIPTA '05*, pages 266–275. SIPTA, 2005a.

R. Pelessoni and P. Vicig. Uncertainty modelling and conditioning with convex imprecise previsions. *Int. J. Approx. Reason.*, 39:297–319, 2005b.

R. Pelessoni and P. Vicig. Williams coherence and beyond. *Int. J. Approx. Reason.*, 50(4):612–626, 2009.

E. Quaeghebeur. Desirability. In T. Augustin, F. P. A. Coolen, G. d. Cooman, and M. C. M. Troffaes, editors, *Introduction to Imprecise Probabilities*, pages 1–27. J. Wiley & Sons, 2014.

A. Shimony. Coherence and the axioms of confirmation. *J. Symb. Logic*, 20:1–28, 1955.

M. C. M. Troffaes and G. de Cooman. *Lower Previsions*. Wiley, 2014.

P. Vicig. A gambler's gain prospects with coherent imprecise previsions. In E. Hüllermeier, R. Kruse, and F. Hoffmann, editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Methods*, pages 50–59. Springer, 2010.

C. G. Wagner. The Smith-Walley Interpretation of Subjective Probability: An Appreciation. *Stud. Logica*, 86:343–350, 2007.

P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.

P. M. Williams. *Notes on conditional previsions. Research Report*. School of Math. and Phys. Science, University of Sussex, 1975.