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Pareto optimality on compact spaces in a preference-based setting under incompleteness

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We characterize the existence of Pareto optimal elements for a family of not necessarily total preorders on a compact topological space. We identify a rather general semicontinuity assumption, called weak upper semicontinuity, under which there exist Pareto optimal elements. We also show that weak upper semicontinuity of each individual preorder is a necessary and sufficient condition for determining the Pareto optimal elements by solving the classical multi-objective optimization problem in case that each function is upper semicontinuous and order-preserving for the respective preorder, and each preorder satisfies a condition of weak separability.

Keywords: Pareto optimal element; weakly upper semicontinuous preorder; weakly separable preorder.

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1. Introduction

Multi-objective optimization (in particular *Pareto optimality*) is a popular and important tool allowing to choose among various available options in the presence of more that one agent (or criterion). Techniques of this sort appear in many different disciplines ranging from design engineering (see e.g. Das [1]), to economics and risksharing (see e.g. Chateauneuf et al. [2] and Barrieu an Scandolo [3]), and to portfolio selection (see e.g. Xidonas et al. [4]). Pareto optimality has been approached in a purely abstract and general way by using cone orderings (see e.g. Zhu et al. [5]).

It is well known that Pareto optimality is usually formulated as the solution of a *multi-objective optimization problem* (MOP) with the standard notation

$$\max_{x\in X}[u_1(x),\dots,u_m(x)] = \max_{x\in X}\mathbf{u}(x), \quad (1)$$

where $m \geq 2$ is an integer, X is the *choice set* (or the *design space*), u_i is the *decision function* (in this case a *utility function*) associated with the i-th agent (see e.g. Das [1], Florenzano [6], and Lindroth et al. [7]), and $\mathbf{u} : X \to \mathbb{R}^m$ is the vectorvalued function defined by $\mathbf{u}(x) = (u_1(x),\dots,u_m(x))$ for all $x \in X$.

We recall that an element $x_0 \in X$ is a *Pareto optimal* solution to problem (1) as soon as for no $x \in X$ it occurs that $u_i(x_0) \leq u_i(x)$ for all $i \in \{1,\dots,m\}$ and at the same time $u_i(x_0) < u_i(x)$ for at least one index i. In this case, the point $x_0 \in X$ is said to be *Pareto optimal* or an *efficient point* for (MOP).

In the framework of decision theory, the real-valued functions u_i ($i = 1,\dots,m$) are viewed, at least implicitly, as utility functions of the necessarily total preorders $\precsim_i$ expressing the preferences of each agent.

A more correct and realistic approach, which was inaugurated and deeply studied by d'Aspremont and Gevers [8], requires that m not necessarily total preorders $\precsim_i$ are considered on the choice set, and we look for a *Pareto optimal solution* x_0 *with respect to the family* $\{\precsim_i\}_{i\in\{1,\dots,m\}}$ *of preorders* (that is, an element $x_0 \in X$ such that for no point $x \in X$ it occurs that $x_0 \precsim_i x$ for $i \in \{1,\dots,m\}$ with at least one index i such that $x_0 \prec_i x$).

It is clear that when we start from problem (1), and we define, for each $i \in \{1,\dots,m\}$, the total preorder

$$x \precsim_i y \Leftrightarrow u_i(x) \leq u_i(y) \ (x,y \in X),$$

then the Pareto optimal solutions to problem (1) coincide with the Pareto optimal solutions considered with respect to the family $\{\precsim_i\}_{i\in\{1,\dots,m\}}$.

In this paper we precisely consider the much more realistic and general situation when we do not restrict ourselves to the consideration of total preorders. This is the distinctive feature of the present work concerning Pareto optimality in a preference based setting. This is in line with recent works concerning the real representation of nontotal preorders (see e.g. Evren and Ok [9] and Bosi et al. [10]).

We invoke the assumption of *transfer transitive lower continuity* introduced by Rodr´ıguez-Palmero and Garc´ıa-Lapresta [11] in order to characterize the existence of Pareto optimal elements for a family of not necessarily total preorders on a compact topological space. In particular, we refer to the hypothesis of *weak upper semicontinuity* of the individual preorders $\precsim_i$. Such an assumption appears in connection with the existence of an upper semicontinuous order-preserving function for a not necessarily total preorder on a topological space (see e.g., Bosi and Herden[12,13] and Bosi and Zuanon [14]). Recall that a preorder $\precsim$ on a topological space (X,τ) is said to be *weakly upper semicontinuous* if for every pair $(x,y) \in \prec$ there exists some open decreasing subset $O_{x,y}$ of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$. We prove that, if every individual preorder is weakly upper semicontinuous on a compact choice set, then there exists a Pareto optimal element. To this aim, we use a generalization proved in Bosi and Zuanon [15] of a well known theorem concerning the existence of maximal elements for not necessarily total preorders on compact spaces (see e.g. Ward [16], Theorem 1, and the application of paragraph 2 in Evren and Ok [9]). Incidentally, we present different conditions that are equivalent to weak upper semicontinuity. We finally show that weak upper semicontinuity of the individual preorders is a necessary and sufficient condition for using the solutions to the classical multi-objective optimization problem (1) in case that each function in such a formulation is order-preserving and each preorder satisfies a weak separability assumption.

Our results show that weak upper semicontinuity is the most suitable and simple version of upper semicontinuity that can be considered when dealing with Pareto optimality with the purpose of determining the optimal elements by solving the classical multi-objective optimization problem.

2. Notation and preliminaries

Let $\precsim$ be a *preorder* (i.e., a *reflexive* and *transitive* binary relation) on a set X. As usual, the strict part of $\precsim$ will be denoted by $\prec$ (i.e., for all $x,y \in X$, $x \prec y$ if and only if $x \precsim y$ and *not*$(y \precsim x)$). The notations $x \prec y$ and $(x,y) \in \prec$ are equivalent.

A preorder $\precsim$ on X is said to be *total* if for all $x,y \in X$ either $x \precsim y$ or $y \precsim x$.

We set, for every point $x \in X$, the following subsets of X:

$$l(x) = \{y \in X \mid y \prec x\}, \qquad r(x) = \{y \in X \mid x \prec y\}$$

$$d(x) = \{z \in X \mid z \precsim x\},\ i(x) = \{z \in X \mid x \precsim z\}.$$

A point $x_0 \in X$ is said to be *maximal* with respect to $\precsim$ if for no $x \in X$ it occurs that $x_0 \prec x$ (i.e., $r(x_0) = \emptyset$).

A subset D of X is said to be *decreasing* if $d(x) \subset D$ for all $x \in D$. By duality the concept of an *increasing* subset I of X is defined.

If $(X,\precsim)$ is a *preordered set*, then a function $u : X \to \mathbb{R}$ is said to be

(i) *increasing* (*isotone*) if, for every $x,y \in X$, $[x \precsim y \Rightarrow u(x) \leq u(y)]$,
(ii) *order-preserving* if it is increasing and, for every $x,y \in X$, $[x \prec y \Rightarrow u(x) < u(y)]$.

In the economic literature, an order-preserving function is often referred to as a *Richter-Peleg utility function* (see e.g. Richter [17]). Denote by τ_{nat} the *natural topology* on the real line $\mathbb{R}$.

A real-valued function u on a topological space (X,τ) is said to be *upper semicontinuous* if $u^{-1}(]-\infty,\alpha[) = \{x \in X : u(x) < \alpha\}$ is an open set for all $\alpha \in \mathbb{R}$. A popular theorem guarantees that an upper semicontinuous real-valued function attains its maximum on a *compact* topological space.

If $(X,\precsim,\tau)$ is a *preordered topological space*, then denote by $\mathcal{U}(X,\precsim,\tau)$ the set of all the upper semicontinuous isotone functions $u : X \to \mathbb{R}$.

In order to introduce the concept of a *weakly upper semicontinuous preorder*, it is useful to consider the following proposition.

Proposition 1. *Let $\precsim$ be a preorder on a topological space (X,τ). Then the following conditions are equivalent:*

(i) *For every pair $(x,y) \in \prec$ there exists some open decreasing subset $O_{x,y}$ of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$.*
(ii) *For every pair $(x,y) \in \prec$ there exists an upper semicontinuous isotone (increasing) function $u_{x,y} : X \to \mathbb{R}$ such that $u_{x,y}(x) < u_{x,y}(y)$.*
(iii) *For every $x \in X$ that is not a minimal element with respect to there exists a uniquely determined open decreasing subset $l^0(x)$ of X such that $x \notin l^0(x)$ and $l(x) \subset l^0(x)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that the preorder $\precsim$ on (X,τ) satisfies condition (i), and consider any pair $(x,y) \in \prec$. Further, let $O_{x,y}$ be an open decreasing subset of X with the indicated property. Then define a function $u_{x,y} : X \to \mathbb{R}$ as follows:

$$u_{x,y}(z) := \begin{cases} 0, \ if \ z \in O_{x,y}, \\ 1, \ if \ z \in X \setminus O_{x,y}, \end{cases}$$

in order to immediately verify that $u_{x,y}$ is an upper semicontinuous isotone function such that $u_{x,y}(x) < u_{x,y}(y)$.

(ii) $\Rightarrow$ (iii). Assume that property (ii) is verified and consider any element $x \in X$ that is not minimal with respect to $\precsim$. Then define the set □

$$l^0(x) := \bigcup_{z \in X, z \prec x} \ \bigcup_{u_{z,x} \in \mathcal{U}(X,\precsim,\tau), u_{z,x}(z) < u_{z,x}(x)} u_{z,x}^{-1}(]-\infty, u_{z,x}(x)[)$$

in order to immediately verify that $l^0(x)$ satisfies the properties of condition (iii).

(iii) ⇒ (i). If property (iii) is verified, and $(x,y) \in \prec$, then define $O_{x,y} := l^0(y)$.

We are now ready to introduce the main definition of this paper.

Definition 1. A preorder - on a topological space (X,τ) is said to be *weakly upper semicontinuous* if it satisfies one of the equivalent conditions in Proposition 1.

From Herden and Levin [18], a preorder - on a topological space (X,τ) is said to be

(i) *upper semicontinuous of type 1* if $l(x) = \{z \in X \mid z \prec x\}$ is an open subset of X for every $x \in X$.

(ii) *upper semicontinuous of type 2* if $i(x) = \{z \in X \mid x - z\}$ is a closed subset of X for every $x \in X$.

Remark 1. It is easily seen that a preorder is weakly upper semicontinuous provided that it is either upper semicontinuous of type 1 or upper semicontinuous of type 2 or else it admits an upper semicontinuous order-preserving function. All the concepts of upper semicontinuity coincide in the case of a total preorder -.

Remark 2. It should be noted that weak upper semicontinuity is a necessary condition for the existence of an upper semicontinuous order-preserving function u for a preorder - on a topological space (X,τ). On the other hand, upper semicontinuity of type 1 or else of type 2 is not necessary for the existence of such an upper semicontinuous order-preserving function. This observation motivates the introduction of the concept of weak upper semicontinuity in our framework.

Let us now consider a simple example of a preorder - on a topological space (X,τ) which is neither upper semicontionus of type 1 nor upper semicontinuous of type 2, but which nevertheless is weakly upper semicontinuous.

Example 1. Let X be the real interval $[0,1]$ and consider the nontotal preorder on X defined as follows:

$$x - y \Leftrightarrow \begin{cases} x \leq y \text{ and } x,y \in \mathbb{Q} \cap [0,1] \\ \text{or} \\ x \leq y \text{ and } x,y \in [0,1] \setminus \mathbb{Q} \end{cases},$$

where, as usual, $\mathbb{Q}$ stands for the set of all the rational numbers.

Then denote by τ the *upper order topology* on X associated to the *natural total preorder* $\leq$ on X (i.e., $\tau = \tau_u^{\leq}$ is the topology generated by the order intervals $l_<(x) = \{z \in X : z < x\}$). It is immediate to check that the identity function $u = i_X$ is an (upper semi)continuous order-preserving function for - on (X,τ), and therefore - is a weakly upper semicontinuous preorder on (X,τ) (see Remark 1). On the other hand, we have that the preorder - is neither upper semicontinuous of type 1 (actually, $l(x) = \{z \in X : z \prec x\}$ is not open for all $x \in X$), nor upper semicontinuous of type 2 (actually, $i(x) = \{z \in X : x - z\}$ is not closed for all $x \in X$). Nevertheless, it should be noted that $l^o(x) = l_<(x) = \{z \in X : z < x\}$ is a τ-open --decreasing subset of X excluding x and containing $l(x)$ for all $x \in X$ (see condition (iii) in Proposition 1).

If - is a preorder on a set X, $x \in X$ and A is a subset of X, then the scripture "$A \prec x$" means "$z \prec x$ for all $z \in A$".

Definition 2. (Rodr´ıguez-Palmero and Garc´ıa-Lapresta[11]) A preorder - on a topological space (X,τ) is said to be *transfer transitive lower continuous* if for every element $x \in X$ such that $r(x)$ 6= $\emptyset$ there exist an element $y \in X$ and a neighbourhood $N(x)$ of x such that $y \prec z$ implies that $N(x) \prec z$ for all $z \in X$.

Proposition 2. (Rodr´ıguez-Palmero and Garc´ıa-Lapresta[11], Theorem 3) *A preorder - on a compact topological space* (X,τ) *has a maximal element if and only if it is transfer transitive lower continuous.*

3. Existence of Pareto optimal elements

Let X be a (nonempty) choice set and consider m preorders $-_i$ ($i = 1,...,m$) on X.

Define the *social preorder* - on X as the intersection of the *individual preorders* $-_i$, i.e. $- = \bigcap_{i=1}^{m} \precsim_i$. Observe that - is not necessarily total, even in case that all the preorders $-_i$ are total. We are not restricted to the consideration of total preorders throughout the present paper.

Definition 3. A point $x_0 \in X$ is said to be *Pareto optimal with respect to the family* $\{-_i\}_{i\in\{1,...,m\}}$ *of preorders* if for no point $x \in X$ it occurs that $x_0 -_i x$ for $i \in \{1,...,m\}$ with at least one index i such that $x_0 \prec_i x$.

In order to reduce ourselves to a, so to say, more familiar situation, let us consider the following proposition which reduces the problem of finding the Pareto optimal elements to the problem of solving the multi-objective optimization problem relative to upper semicontinuous functions.

Proposition 3. *If X is endowed with a compact topology τ, then there exists a Pareto optimal element $x_0 \in X$ with respect to the family $\{-_i\}_{i\in\{1,...,m\}}$ of preorders provided that every preorder $-_i$ ($i \in \{1,...,m\}$) admits an upper semicontinuous order-preserving function u_i. In particular, a Pareto optimal element x_0 is any solution to the multi-objective optimization problem*

□

$$\max_{x\in X}[u_1(x),...,u_m(x)].$$

Proof. Since u_i is an upper semicontinuous function for $-_i$ for every $i \in \{1,...,m\}$, then $u=\sum_{i=1}^{m} u_i$ is also upper semicontinuous, and therefore it attains its maximum on the compact topological space (X,τ) at some point x_0. Then x_0 is a Pareto optimal element with respect to the family $\{-_i\}_{i\in\{1,...,m\}}$ of preorders since otherwise the existence of some point $x \in X$ such that $x_0 -_i x$ for $i \in \{1,...,m\}$ with at least one index i such that $x_0 \prec_i x$ would imply that $u(x_0) < u(x)$. Indeed, u_i is an order-preserving function for $-_i$ ($i \in \{1,...,m\}$). This consideration completes the proof.

The next corollary concerns the case when all the individual preorders are total.

Corollary 1. *Consider a family $-_i$ ($i \in \{1,...,m\}$) of individual total preorders on a set X endowed with a compact metrizable topology τ. If all the preorders are upper semicontinuous of type 1, then there exist Pareto optimal elements with respect to the family $\{-_i\}_{i\in\{1,...,m\}}$ which can be determined as the solutions to problem (1), where u_i is an order-preserving function for $-_i$ for all $i \in \{1,...,m\}$.*

Proof. Consider that the topology τ is separable, and therefore second countable. Hence, from the classical Rader theorem (see Rader [19]), there exists an orderpreserving function u_i for $-_i$ ($i \in \{1,...,m\}$. Now apply Proposition 3. □

The following proposition also holds, which reduces the problem of finding the Pareto optimal elements to the problem of determining the maximal elements of the social preorder.

Proposition 4. *The following conditions are equivalent:*

(i) *$x_0 \in X$ is Pareto optimal with respect to the family $\{-_i\}_{i\in\{1,...,m\}}$ of preorders;*

(ii) *$x_0 \in X$ is maximal with respect to the social preorder $- = \setminus_{i=1}^{m} -_i$.*

Proof. In order to prove the implication (i) ⇒ (ii), assume by contraposition that $x_0 \in X$ is not maximal with respect to $- = \setminus_{i=1}^{m} -_i$. Then the existence of some point $x \in X$ such that $x_0 \prec x$ precisely means that $x_0 -_i x$ for $i \in \{1,...,m\}$ with at least one index i such that $x_0 \prec_i x$. Hence, x_0 is not a Pareto optimal element. The proof of the implication (ii) ⇒ (i) is perfectly analogous. □

As an application of Proposition 2 and Proposition 4, we can characterize the existence of Pareto optimal elements on compact spaces. Indeed, the following theorem holds true.

Theorem 1. *If X is endowed with a compact topology τ, then there exists a Pareto optimal element $x_0 \in X$ with respect to the family $\{\precsim_i\}_{i\in\{1,...,m\}}$ of preorders if and only if for every element $x \in X$ which is not Pareto optimal there exist an element $y \in X$ and a neighbourhood $\mathrm{N}(x)$ of x such that, for all $z \in X$, if $y \precsim_i z$ for all $i \in \{1,...,m\}$ and there exists at least one index i such that $y \prec_i z$, then, for all $y^0 \in \mathrm{N}(x)$, $y^0 \precsim_i z$ for all $i \in \{1,...,m\}$ and there exists at least one index i such that $y^0 \prec_i z$.*

It is not immediate to identify conditions on the individual preorders $\precsim_i$, ensuring that the associated social preorder $\precsim$ has a maximal element on a compact topological space. Nevertheless, we are now ready to prove that the concept of weak upper semicontinuity is extremely useful for our purposes. □

Theorem 2. *If X is endowed with a compact topology τ, then there exists a Pareto optimal element $x_0 \in X$ with respect to the family $\{\precsim_i\}_{i\in\{1,...,m\}}$ of preorders provided that every preorder $\precsim_i$ is weakly upper semicontinuous.*

Proof. From Bosi and Zuanon [15], since every weakly upper semicontinuous preorder on a compact topological space admits a maximal element, we only have to show that under our assumptions the social preorder $\precsim = \bigcap_{i=1}^{m} \precsim_i$ is weakly upper semicontinuous. Consider any two elements $x,y \in X$ such that $x \prec y$. Then there exists some index $i \in \{1,...,m\}$ such that $x \prec_i y$ and this implies, by condition (ii) of Proposition 1, the existence of a $\precsim_i$-increasing upper semicontinuous function $u_{x,y}$ such that $u_{x,y}(x) < u_{x,y}(y)$. Now, just observe that $u_{x,y}$ is also $\precsim$-increasing in order to immediately realize that $\precsim$ is weakly upper semicontinuous. This consideration completes the proof.

Corollary 2. *If X is endowed with a compact topology τ, then there exists a Pareto optimal element $x_0 \in X$ provided that every preorder $\precsim_i$ is either upper semicontinuous of type 1 or upper semicontinuous of type 2.*

Remark 3. The assumption according to which there are finitely many agents with preferences expressed by the preorders $\{\precsim_i\}_{i\in\{1,...,m\}}$ is only made for the ease of the present exposition. Indeed, it is easily seen that we could have considered an arbitrary (not necessarily finite) family of preorders $\{\precsim_\alpha\}_{\alpha\in I}$, with a corresponding social preorder $\precsim = \bigcap_{\alpha\in I} \precsim_\alpha$. In this much more general situation Proposition 4, Theorem 2 and Corollary 2 are still true.

4. Pareto optimality from maximization of upper semicontinuousorder-preserving functions

In this section we show that the consideration of a weak separability assumption referred to all the individual preorders allows us to immediately recover the Pareto optimal elements by solving problem (1).

Definition 4. We say that a preorder $\precsim$ on a set X is *weakly separable* if there exists a countable subset D of X such that for all $x,y \in X$ such that $x \prec y$ there exists an element $d \in D$ such that $x \prec d$ and $not(y \prec d)$ (equivalently, $x \in l(d), y \notin l(d)$).
In this case, we say that the set D is *weakly dense*.

Lemma 1. *Let $\precsim$ be a weakly separable preorder on a topological space (X,τ). Then the following conditions are equivalent:*

(i) *There exists an upper semicontinuous order-preserving function u for $\precsim$;* (ii) $\precsim$ *is weakly upper semicontinuous.*

Proof. It has been already observed in Remark 2 that the implication (i) ⇒ (ii) holds (even without any separability asssumption). Conversely, in order to show that also the implication (ii) ⇒ (i) is valid, assume that the preorder $\precsim$ on the topological space (X,τ) is weakly separable and weakly upper semicontinuous. Then, if $D = \{d_n : n \in \mathbb{N}^+\}$ is a weakly dense subset of X, consider for every $n \in \mathbb{N}^+$ the upper semicontinuous function $u_n : X \to \mathbb{R}$ defined as follows:

$$u_n(z) := \begin{cases} 0, \ if \ z \in l^0(d_n), \\ 1, \ if \ z \in X \setminus l^0(d_n) \end{cases},$$

where the sets $l^0(d_n)$ $(z \in X)$ are those defined in condition (iii) of Proposition 1. It is easy to verify that $u := \sum_{n \in \mathbb{N}^+} 2^{-n} u_n$ is an upper semicontinuous order-preserving function for $\precsim$. The proof is now complete. □

Proposition 5. *If X is endowed with a compact topology τ, and every preorder $\precsim_i$ $(i \in \{1,...,m\})$ is weakly separable, then the following conditions are equivalent:*

(i) *There exists a Pareto optimal element $x_0 \in X$ determined as a solution to the multi-objective optimization problem*

$$\max_{x \in X} [u_1(x),...,u_m(x)],$$

where u_i is an order-preserving function for $\precsim_i$ for every $i \in \{1,...,m\}$; (ii) $\precsim_i$ *is weakly upper semicontinuous for every $i \in \{1,...,m\}$.*

Proof. The validity of the implication (i) ⇒ (ii) is an immediate consequence of the corresponding implication in Lemma 1. Conversely, if $\precsim_i$ is weakly separable and weakly upper semicontinuous for every $i \in \{1,...,m\}$, then for every $i \in \{1,...,m\}$ there exists an upper semicontinuous order-preserving function u_i for $\precsim_i$ by Lemma 1, and the existing solution (by compactness) of the multi-objective optimization problem (1) furnishes a Pareto optimal element by Proposition 3. This consideration completes the proof. □

Since it is immediate to check that every preorder on a countable set X is weakly separable, we finally get the following corollary to Proposition 5.

Corollary 3. *Let X be a countable set endowed with a compact topology τ, and consider m preorders $\precsim_i$ ($i \in \{1,...,m\}$). Then the following conditions are equivalent:*

(i) *There exists a Pareto optimal element $x_0 \in X$ determined as a solution to the multi-objective optimization problem*

$$\max_{x \in X}[u_1(x),...,u_m(x)],$$

where u_i is an order-preserving function for $\precsim_i$ for every $i \in \{1,...,m\}$;

(ii) $\precsim_i$ *is weakly upper semicontinuous for every* $i \in \{1,...,m\}$*.*

5. Conclusions

In the present paper, we have followed a preference-based approach to Pareto optimality which refers both to compactness of the choice set and the consideration of individual preferences represented by not necessarily total preorders. At least in principle, such an approach appears more correct that the usual *functional approach* based on the classical multi-objective optimization problem. Nevertheless, it can be reduced to this latter approach in the particular case when each individual preorder admits an upper semicontinuous order-preserving function. The consideration of the *social preorder*, which is defined as the intersection of the *individual preorders*, allows us to use classical results on the existence of maximal elements for a preorder on a compact space. The introduction of a weak separability condition for all the individual preorders allows us to translate the problem of finding Pareto optimal elements into the problem of solving the classical multi-objective optimization problem. The problem is still open, of finding suitable upper semicontinuity conditions on the individual preorders characterizing the existence of Pareto optimal elements on a compact topological space. We hope to be able to solve this problem in the future.

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