Quaternionic Kleinian modular groups and arithmetic hyperbolic orbifolds over the quaternions

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Abstract Using the rings of Lipschitz and Hurwitz integers $\mathbb{H}(\mathbb{Z})$ and $\mathbb{H}ur(\mathbb{Z})$ in the quaternion division algebra $\mathbb{H}$, we define several Kleinian discrete subgroups of $PSL(2, \mathbb{H})$. We define first a Kleinian subgroup $PSL(2, \mathfrak{L})$ of $PSL(2, \mathbb{H}(\mathbb{Z}))$. This group is a generalization of the modular group $PSL(2, \mathbb{Z})$. Next we define a discrete subgroup $PSL(2, \mathfrak{H})$ of $PSL(2, \mathbb{H})$ which is obtained by using Hurwitz integers. It contains as a subgroup $PSL(2, \mathfrak{L})$. In analogy with the classical modular case, these groups act properly and discontinuously on the hyperbolic quaternionic half space. We exhibit fundamental domains of the actions of these groups and determine the isotropy groups of the fixed points and describe the orb-ifold quotients $\mathbf{H}^1_{\mathbb{H}}/PSL(2, \mathfrak{L})$ and $\mathbf{H}^1_{\mathbb{H}}/PSL(2, \mathfrak{H})$ which are quaternionic versions of the classical modular orbifold and they are of finite volume. Finally we give a thorough study of their descriptions by Lorentz transformations in the Lorentz–Minkowski model of hyperbolic 4-space.

Keywords Modular groups · Arithmetic hyperbolic 4-manifolds · 4-Orbifolds · Quaternionic hyperbolic geometry

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Dedicated to Bill Goldman on occasion of his 60th birthday.

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1 Introduction

Since the time of Carl Friedrich Gauss and the foundational papers by Richard Dedekind and Felix Klein the classical modular group $PSL(2, \mathbb{Z})$ and its action on the hyperbolic (complex) upper half plane $\mathbf{H}^1_{\mathbb{C}} = \{z \in \mathbb{C} \; : \; \Im(z) > 0\}$ have played a central role in different branches of mathematics and physics like number theory, Riemann surfaces, elliptic curves, hyperbolic geometry, theory of modular forms and automorphic forms, crystallography, string theory and others. Similarly, discrete subgroups of $PSL(2, \mathbb{C})$ are very important in the construction of lattices to study of arithmetic hyperbolic 3-orbifolds (see, for instance, [20]) and many other fields of mathematics.

We consider the action defined by isometries of the modular group $PSL(2, \mathbb{Z})$ on the hyperbolic plane $\mathbf{H}^1_{\mathbb{C}}$. A fundamental domain is a triangle with one ideal point an two other vertices were the sides have an angle of $\pi/3$. This is the triangle with Coxeter notation $\triangle(3, 3, \infty)$. See the Fig. 1. The modular group $PSL(2, \mathbb{Z})$ is a subgroup of the group of symmetries of the regular tessellation of $\mathbf{H}^1_{\mathbb{C}}$ whose tiles are congruent copies of the triangle $\triangle(3, 3, \infty)$. We can describe the Cayley graph and a presentation of the group $PSL(2, \mathbb{Z})$ in terms of 2 generators and 2 relations to obtain:

$$PSL(2, \mathbb{Z}) \; = \; \langle a, b | a^2 = (ab)^3 = 1 \rangle \; = \; \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

The quotient $\mathcal{O}^2 := \mathbf{H}^1_{\mathbb{C}}/PSL(2, \mathbb{Z})$ has underlying topological space the plane $\mathbb{R}^2$ (or $\mathbb{C}$) and $\Sigma_{\mathcal{O}^2}$ consists of two distinguished conical points. The local groups of the singular points are $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ modeled on groups of two and three elements which consist of hyperbolic rotations of angles π and $2\pi/3$, respectively. The Euler characteristic of the orbifold $\mathcal{O}^2$ is $-1/6$. Thus a minimal surface Selberg cover is of order 6.

Now we consider the action defined by isometries of the modular group $PSL(2, \mathbb{Z}[\mathbf{i}])$ which is the Picard group related to the Gauss integers on the hyperbolic real 3-space $\mathbf{H}^3_{\mathbb{R}}$. The quotient $\mathcal{O}^3 := \mathbf{H}^3_{\mathbb{R}}/PSL(2, \mathbb{Z}[\mathbf{i}])$ has underlying space the 3-space $\mathbb{R}^3$. Its singular locus $\Sigma_{\mathcal{O}^3}$ is the 1-skeleton of a squared pyramid without the apex. The Euler characteristic of the orbifold $\mathcal{O}^3$ is 0.

In this paper, we introduce two generalizations of the modular group in the four-dimensional setting of the quaternions and the rings of Lipschitz and Hurwitz integers (see [14,15],[19]) and then focus our attention to their actions on hyperbolic (quaternionic) half space $\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \; : \; \Re(\mathbf{q}) > 0\}$ with metric $\frac{d|q|^2}{(\Re \mathbf{q})^2}$. We then explore new results which give a very detailed description of the orbifolds related to the corresponding quaternionic modular groups. Secondly we define the group generated by translations by the imaginary
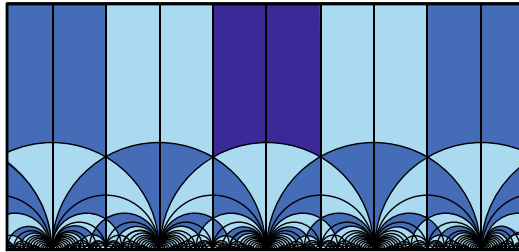

Fig. 1 A fundamental domain for the action of the modular group $PSL(2, \mathbb{Z})$ on the hyperbolic plane $\mathbf{H}^1_{\mathbb{C}}$

parts of Lipschitz integers, the inversion T and a finite group related to the Hurwitz units. We give a thorough description of its properties and the corresponding orbifolds.

Furthermore, we also give a geometric description of the fundamental domains for the actions of the quaternionic modular groups and a detailed analysis of the topology and of the isotropy groups of the singularities of the orbifolds introduced. We study the corresponding modular Lipschitz and Hurwitz groups in the Lorentz–Minkowski model.

2 The quaternionic hyperbolic 4-space $\mathbf{H}^1_{\mathbb{H}}$ and its isometries

2.1 Isometries in the half-space model of the hyperbolic 4-space $\mathbf{H}^1_{\mathbb{H}}$

Consider the quaternions

$$\mathbb{H} := \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \: : \: x_n \in \mathbb{R}, \, n = 0, 1, 2, 3, \, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\}.$$

If $\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$ then $\Re(\mathbf{q}) := x_0 \in \mathbb{R}$, $\overline{\mathbf{q}} := x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k} \in \mathbb{H}$ and $|\mathbf{q}|^2 := \mathbf{q}\overline{\mathbf{q}} \in \mathbb{R}^+$.

Let $\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \: : \: \Re(\mathbf{q}) > 0\}$ be the half-space model of the *one-dimensional quaternionic hyperbolic space*. This set is isometric to the hyperbolic real space in four dimensions, namely $\mathbf{H}^1_{\mathbb{H}} \cong \mathbf{H}^4_{\mathbb{R}} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \: : \: x_0 > 0\}$ with the element of hyperbolic metric given by $(ds)^2 = \frac{(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2}{x_0^2}$ where s measures length along a parametrized curve. Even though the (natural) algebraic structures carried by the two sets are deeply different.

Let $GL(2, \mathbb{H})$ denote the *general linear group* of 2×2 invertible[1] matrices with entries in the quaternions $\mathbb{H}$. The next definitions can be found in [3,10,16] and [26].

Definition 2.1 For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H})$, the associated real analytic function $F_A : \mathbb{H} \cup \{\infty\} \to \mathbb{H} \cup \{\infty\}$ defined by $F_A(\mathbf{q}) = (a\mathbf{q} + b) \cdot (c\mathbf{q} + d)^{-1}$ is called the *Möbius transformation* associated with A.

We set $F_A(\infty) = \infty$ if $c = 0$, $F_A(\infty) = ac^{-1}$ if $c \neq 0$ and $F_A(-c^{-1}d) = \infty$.

Let $\mathbb{F} := \{F_A \: : \: A \in GL(2, \mathbb{H})\}$ the group of *Möbius transformations*.

Definition 2.2 Let $SL(2, \mathbb{H})$ be the *special linear group* which consists of matrices in $GL(2, \mathbb{H})$ with Dieudonnè determinant 1.

Then $\Phi : GL(2, \mathbb{H}) \to \mathbb{F}$ defined as $\Phi(A) = F_A$ is a surjective group antihomomorphism with $\ker(\Phi) = \{t\mathcal{I} \: : \: t \in \mathbb{R} \setminus \{0\}\}$. Furthermore, the restriction of Φ to the special linear group $SL(2, \mathbb{H})$ is still surjective and whose kernel is $\{\pm\mathcal{I}\}$.

Let $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ be the set of Möbius transformations that leave invariant $\mathbf{H}^1_{\mathbb{H}}$. Any transformation $F_A \in \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is conformal and preserves orientation, moreover is an isometry of $\mathbf{H}^1_{\mathbb{H}}$. We conclude that $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ is isomorphic to the groups $Conf_+(\mathbf{H}^1_{\mathbb{H}})$ and $Isom_+(\mathbf{H}^1_{\mathbb{H}})$ of conformal diffeomorphisms and isometries orientation–preserving of the half-space model $\mathbf{H}^1_{\mathbb{H}}$ (see [2] and [1]). Moreover, the group $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ acts by orientation-preserving conformal transformations on the sphere at infinity of the hyperbolic 4-space defined as $\partial\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \: : \: \Re(\mathbf{q}) = 0\} \cup \{\infty\}$. In other words $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \cong Conf_+(\mathbb{S}^3) \cong Conf_+(\mathbf{H}^1_{\mathbb{H}}) \cong Isom_+(\mathbf{H}^1_{\mathbb{H}})$.

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[1] By this we mean that a 2×2 quaternionic matrix A has a right and left inverse; in [4] it is shown that this is equivalent for A to have non zero Dieudonné determinant (see [3]).

Finally we recall the conditions found by Ahlfors (see [2]) and then applied by Bisi and Gentili in the next form (see [4]):

Proposition 2.3 [**Ahlfors conditions**] *The subgroup* $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$ *of* $PSL(2, \mathbb{H})$ *can be characterized as the group induced by matrices which satisfy one of the following (equivalent) conditions:*

$$\begin{cases} \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} a, b, c, d \in \mathbb{H} \; : \; \overline{A}^t K A = K \right\} \text{ with } K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\[2mm] \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} a, b, c, d \in \mathbb{H} \; : \; \Re(a\overline{c}) = 0, \; \Re(b\overline{d}) = 0, \; \overline{b}c + \overline{d}a = 1 \right\}, \\[2mm] \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} a, b, c, d \in \mathbb{H} \; : \; \Re(c\overline{d}) = 0, \; \Re(a\overline{b}) = 0, \; a\overline{d} + b\overline{c} = 1 \right\}. \end{cases}$$

2.2 The affine subgroup $\mathcal{A}(\mathbb{H})$ of the isometries of $\mathbf{H}^1_\mathbb{H}$

Consider now the *affine subgroup* $\mathcal{A}(\mathbb{H})$ of $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$ consisting of transformations which are induced by matrices of the form $\begin{pmatrix} \lambda a & b \\ 0 & \lambda^{-1}a \end{pmatrix}$ with $|a| = 1$, $\lambda > 0$ and $\Re(\overline{b}a) = 0$. The group $\mathcal{A}(\mathbb{H})$ is the maximal subgroup of $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$ which fixes the point at infinity and its a Lie group of real dimension 7.

Each matrix in $\mathcal{A}(\mathbb{H})$ acts as a conformal transformation on the hyperplane at infinity $\partial \mathbf{H}^1_\mathbb{H}$. Moreover $\mathcal{A}(\mathbb{H})$ is the group of conformal and orientation preserving transformations acting on the space of pure imaginary quaternions at infinity which can be identified with $\mathbb{R}^3$ so that $\mathcal{A}(\mathbb{H})$ is isomorphic to the conformal group $Conf_+(\mathbb{R}^3)$.

2.3 Isotropy subgroup of the isometries of $\mathbf{H}^1_\mathbb{H}$ which fixed one point

Let $\mathcal{K} := \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \in \mathcal{M}_{\mathbf{H}^1_\mathbb{H}} \right\}$ be the subgroup of symmetric matrices in $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$. For the matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ the conditions $|\alpha|^2 + |\beta|^2 = 1$ and $\Re(\alpha\overline{\beta}) = 0$ are equivalent to Ahlfors conditions in Proposition 2.3. The group $\mathcal{K}$ is the isotropy subgroup at $1 \in \mathbf{H}^1_\mathbb{H}$ of the action of $PSL(2, \mathbb{H})$ by orientation preserving isometries on $\mathbf{H}^1_\mathbb{H}$. Moreover the group $\mathcal{K}$ is a maximal compact Lie subgroup of $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$ which is isomorphic to the special orthogonal group $SO(4)$.

Let $\mathcal{D} := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in \mathcal{M}_{\mathbf{H}^1_\mathbb{H}} \right\}$ be the subgroup of $\mathcal{K}$ whose elements are diagonal matrices in $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$. Then the Ahlfors conditions imply that $|\alpha| = 1$. The action at infinity is given by $\mathbf{q} \mapsto \alpha\mathbf{q}\bar{\alpha}$, which is the usual action of $SO(3)$ on the purely imaginary quaternions. Therefore $\mathcal{D}$ is isomorphic to $SO(3)$.

2.4 Iwasawa decomposition of the isometries of $\mathbf{H}^1_\mathbb{H}$

In analogy with the complex and real case, we can state a generalization of Iwasawa decomposition for any element of $\mathcal{M}_{\mathbf{H}^1_\mathbb{H}}$ as follows

Proposition 2.4 *Every element of $\mathcal{M}_{\mathbf{H}_{\mathbb{H}}^1}$ which is represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written in an unique way as follows*

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \tag{1}$$

with $\lambda > 0$, $\Re(\omega) = 0$, $|\alpha|^2 + |\beta|^2 = 1$ and $\Re(\alpha\overline{\beta}) = 0$.

Proof We'll give explicit expressions for α, β, λ and ω in terms of $a, b, c,$ and d. Indeed, from direct computations, one easily obtains that $\lambda d = \alpha$ and $\lambda c = \beta$; therefore, from the equations $a = \lambda^2(d + \omega c)$ $b = \lambda^2(c + \omega d)$ it is a matter of calculations to conclude that

$$\lambda = \frac{1}{\sqrt{|c|^2 + |d|^2}} \quad \text{and} \quad \omega = a\overline{c} + b\overline{d}.$$

Therefore, from Ahlfors conditions of Proposition 2.3, it follows that $\Re(\omega) = 0$ and $\Re(\alpha\overline{\beta}) = 0$. $\qquad\square$

3 The quaternionic modular groups

In this section we investigate a class of hyperbolic isometries of $\mathbf{H}_{\mathbb{H}}^1$ which will play a crucial role in the definition of the quaternionic modular groups.

3.1 Quaternionic Translations

A translation fixing ∞ is the isometry $\tau_b : \mathbf{H}_{\mathbb{H}}^1 \to \mathbf{H}_{\mathbb{H}}^1$ defined as $\mathbf{q} \mapsto \mathbf{q} + b$ associated with the matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{\mathbf{H}_{\mathbb{H}}^1}$, the Ahlfors conditions implies that $\Re(b) = 0$.

Hence if the entries are integers then b is an integer linear combination of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. Therefore the group of such translations is isomorphic to $\mathbb{Z}^3$.

We recall the definitions of quaternionic integers and refer to [7]. The ring of *Lipschitz integers* $\mathbb{H}(\mathbb{Z})$ is the subset of quaternions with integer coefficients, *i.e.* $\mathbb{H}(\mathbb{Z}) := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} : a, b, c, d \in \mathbb{Z}\}$.

This is a subring of the ring of *Hurwitz integers*:

$$\mathbb{H}ur(\mathbb{Z}) := \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}.$$

As a group, $\mathbb{H}ur(\mathbb{Z})$ is free abelian with generators $1/2(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}), \mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore $\mathbb{H}ur(\mathbb{Z})$ forms a lattice in $\mathbb{R}^4$ which is the root lattice of the semisimple Lie algebra $\mathcal{F}_4$. The Lipschitz quaternions $\mathbb{H}(\mathbb{Z})$ form an index 2 sublattice of $\mathbb{H}ur(\mathbb{Z})$.

In what follows we consider translations where b is the imaginary part of a Lipschitz or Hurwitz integer in order to satisfy the Ahlfors conditions. We remark that the imaginary part of a Lipschitz integer is still a Lipschitz integer but the imaginary part of a Hurwitz integer is not necessarily a Hurwitz integer.

We denote by $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ the abelian group of translations by imaginary parts of all Lipschitz integers $\Im\mathbb{H}(\mathbb{Z})$. The group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ acts freely on $\mathbf{H}_{\mathbb{H}}^1$ since its representation is the abelian group $\mathbb{Z}^3$.

3.2 Inversion

Let us consider now the isometry $T : \mathbf{H}^1_{\mathbb{H}} \to \mathbf{H}^1_{\mathbb{H}}$ defined as

$$T(\mathbf{q}) = \mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

The representative matrix of T is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The only fixed point of T in $\mathbf{H}^1_{\mathbb{H}}$ is 1. We also notice here that in the topological closure of $\mathbf{H}^1_{\mathbb{H}}$ (denoted by $\overline{\mathbf{H}^1_{\mathbb{H}}}$) the points 0 and ∞ are periodic (of period 2) for T. Furthermore T is an isometric involution[2] of $\mathbf{H}^1_{\mathbb{H}}$. In particular T is an inversion on $\mathbb{S}^3$ which becomes the antipodal map on any copy of $\mathbb{S}^2$ obtained as intersection of $\mathbb{S}^3$ with a perpendicular 3-plane to the line passing through 0 and 1. Finally, this isometry T leaves invariant the hemisphere (which is a hyperbolic 3-dimensional hyperplane) $\Pi := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \; : \; |\mathbf{q}| = 1\}$. Each point of Π different from 1 (which is fixed by T) is a periodic point of T of period 2.

3.3 Composition of translations and inversion

We observe that if $\tau_b(\mathbf{q}) := \mathbf{q} + b$, $b \in \mathbb{H}$, then $L_b := \tau_b T$ has as corresponding matrix $\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$. Similarly $R_b := T \tau_b$ has as corresponding matrix $\begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix}$.

Therefore R_b is represented by interchanging the elements on the diagonal of the matrix which represents L_b.

The order of the matrix L_b depends on b; in particular L_b has order 6 if b is $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$ or $\pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$ and L_b has order 4 if b is $\pm\mathbf{i} \pm \mathbf{j}$, $\pm\mathbf{j} \pm \mathbf{k}$ or $\pm\mathbf{i} \pm \mathbf{k}$.

Each of the six transformations L_b with $b = \pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$, has order 6 but when restricted to the plane $S_b := \{\mathbf{q} = x_1 + x_{\mathbf{i}}\mathbf{i} + x_{\mathbf{j}}\mathbf{j} + x_{\mathbf{k}}\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} \; : \; x_\alpha = 0 \text{ if } \alpha \neq b, 0\}$, with $b = \mathbf{i}, \mathbf{j}, \mathbf{k}$ has order 3. Furthermore $\mathbf{q}_0$ is a fixed point for $L_b = \tau_b T$ with $b = 0, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, if and only if $\mathbf{q}_0$ is a root of $\mathbf{q}^2 - b\mathbf{q} - 1 = 0$. If $b = 0$ there is only one root in $\mathbf{H}^1_{\mathbb{H}}$ (and so only one fixed point for T), namely $\mathbf{q}_0 = 1$. If $b = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, then it is easily verified that if α and β are two roots of $\mathbf{q}^2 - b\mathbf{q} - 1 = 0$, it follows that $\Re(\alpha + \beta) = 0$ or $\Re(\alpha) = -\Re(\beta)$. Since a root of $\mathbf{q}^2 - \omega\mathbf{q} - 1 = 0$ is $\alpha = \frac{\sqrt{3}}{2} + \frac{b}{2}$ ($\alpha = \frac{\sqrt{3}}{2} - \frac{b}{2}$) any other possible root β of the above given equation would not sit in $\mathbf{H}^1_{\mathbb{H}}$. In the same way $\mathbf{q}_0$ is a fixed point for $R_b = T \tau_b$ with $b = 0, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, if and only if $\mathbf{q}_0$ is a root of $\mathbf{q}^2 + \mathbf{q}b - 1 = 0$. If $b = 0$ there is only one root in $\mathbf{H}^1_{\mathbb{H}}$ (and so only one fixed point for T), namely $\mathbf{q}_0 = 1$. If $b = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, then it is easily verified that if α and β are two roots of $\mathbf{q}^2 + \mathbf{q}b - 1 = 0$, it follows that $\Re(\alpha) = -\Re(\beta)$. Since a root of $\mathbf{q}^2 + \mathbf{q}b - 1 = 0$ is $\alpha = \frac{\sqrt{3}}{2} - \frac{b}{2}$ any other possible root β of the above given equation would not sit in $\mathbf{H}^1_{\mathbb{H}}$.

In short, the only fixed point of L_b in $\mathbf{H}^1_{\mathbb{H}}$ is $\frac{\sqrt{3}}{2} + \frac{b}{2}$ and the only fixed point of R_b is $\frac{\sqrt{3}}{2} - \frac{b}{2}$.

3.4 The Lipschitz and Hurwitz quaternionic modular group $PSL(2, \mathfrak{L})$

We are now in the position of introducing the following:

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[2] In the following sense; T sends every point of a hyperbolic geodesic parametrized by arc length $\gamma(s)$, passing through 1 at time 0 (i.e. such that $\gamma(0) = 1$), to its opposite $\gamma(-s)$.

Definition 3.1 The *Lipschitz* and *Hurwitz quaternionic modular groups* are the groups of quaternionic Möbius transformations whose entries are Lipschitz and Hurwitz integers, respectively, and which also satisfy Ahlfors conditions. They will be denoted by $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$, respectively.

A diagonal matrix satisfying the Ahlfors conditions has $ad = 1$. If $|a| = 1$ then $a = d$ is a unit. There are 8 units in the Lipschitz integers and 24 in the Hurwitz integers.

Definition 3.2 Let $\mathfrak{L}_u$ be the group (of order 8) of *Lipschitz units* $\mathfrak{L}_u := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ and let $\mathfrak{H}_u$ be the group (of order 24) of *Hurwitz units* $\mathfrak{H}_u := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}, \frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}) : \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{k}\}$, where in $\frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ all 16 possible combinations of signs are allowed.

The Lipschitz units are the elements of the non-abelian *quaternion group*. Moreover, its elements are the 8 vertices of a 16-cell in the 3-sphere $\mathbb{S}^3$ and the 8 barycentres of the faces of its dual polytope which is a hypercube also called 8-cell.

The Hurwitz units are the elements of a group known as the *binary tetrahedral* group. Its elements can be seen as the vertices of the 24-cell. We recall that the 24-cell is a convex regular 4-polytope, whose boundary is composed of 24 octahedral cells with six meeting at each vertex, and three at each edge. Together they have 96 triangular faces, 96 edges, and 24 vertices. It is possible to give an (ideal) model of the 24-cell by considering the convex hull (of the images) of the 24 unitary Hurwitz numbers via the Cayley transformation $\Psi(\mathbf{q}) = (1 + \mathbf{q})(1 - \mathbf{q})^{-1}$.

Definition 3.3 The subgroups $\mathcal{U}(\mathfrak{L})$ and $\mathcal{U}(\mathfrak{H})$ of $PSL(2, \mathbb{H})$ whose elements are the diagonal matrices $D_{\mathbf{u}} := \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$ with $\mathbf{u}$ a Lipschitz unit or Hurwitz unit is called *Lipschitz unitary group* or *Hurwitz unitary group*, respectively.

The Lipschitz unitary group is isomorphic to the so called *Klein-4 group* which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, since $\mathbf{ij} = \mathbf{k}$. The epimorphism from units to unitary groups $\mathfrak{H}_u \to \mathcal{U}(\mathfrak{H})$ given by $\mathbf{u} \mapsto D_{\mathbf{u}}$ has kernel $\{1, -1\}$ so it is of order two. Any matrix in $\mathcal{U}(\mathfrak{H})$ satisfies the Ahlfors conditions and is an isometry which represents a rotation in $\mathbf{H}^1_{\mathbb{H}}$. Moreover, we observe that the action on $\mathbf{H}^1_{\mathbb{H}}$ of the transformation $D_{\mathbf{u}}$ defines the conjugation and sends a quaternion $\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}}$ to $\mathbf{uqu}^{-1}$. If $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ it acts as a rotation of angle π fixing opposite faces of a cube as in Fig. 3, and if $\mathbf{u} = \frac{1}{2}(\pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ it acts as a rotation of angle $\frac{2\pi}{3}$ fixing each of its main diagonals as in Fig. 4. The axis of rotation of the transformation $D_{\mathbf{u}}$ is the vertical hyperbolic 2-plane $S_{\mathbf{u}} = \{x + y\mathbf{u} : x, y \in \mathbb{R}, x > 0\}$.

The group $\mathcal{U}(\mathfrak{H})$ is of order 12 and in fact it is isomorphic to the group of orientation preserving isometries of the regular tetrahedron. It clearly contains $\mathcal{U}(\mathfrak{L})$ as a subgroup but is not contained in the Lipschitz modular group $PSL(2, \mathfrak{L})$.

Definition 3.4 The *Lipschitz affine subgroup* $\mathcal{A}(\mathfrak{L})$ is the group generated by the unitary Lipschitz group $\mathcal{U}(\mathfrak{L})$ and the group of translations $\mathcal{T}_{\mathfrak{I}\mathbb{H}(\mathbb{Z})}$. The *Hurwitz affine subgroup* $\mathcal{A}(\mathfrak{H})$ is the group generated by the unitary Hurwitz group $\mathcal{U}(\mathfrak{H})$ and the group of translations $\mathcal{T}_{\mathfrak{I}\mathbb{H}(\mathbb{Z})}$. Equivalently, for $\mathfrak{K} = \mathfrak{L}$ or $\mathfrak{K} = \mathfrak{H}$,

$$\mathcal{A}(\mathfrak{K}) = \left\{ \begin{pmatrix} \mathbf{u} & \mathbf{u}b \\ 0 & \mathbf{u} \end{pmatrix} : \mathbf{u} \in \mathfrak{K}_u, \ b \in \mathfrak{I}\mathbb{H}(\mathbb{Z}), \ \mathfrak{R}(b) = 0 \right\}$$

$$= \left\{ \begin{pmatrix} \mathbf{u} & b\mathbf{u} \\ 0 & \mathbf{u} \end{pmatrix} : \mathbf{u} \in \mathfrak{K}_u, \ b \in \mathfrak{I}\mathbb{H}(\mathbb{Z}), \ \mathfrak{R}(b) = 0 \right\}.$$

The Lipschitz affine subgroup $\mathcal{A}(\mathfrak{L})$ is the maximal Lipschitz parabolic subgroup of $PSL(2, \mathfrak{L})$. Moreover $\mathcal{A}(\mathfrak{L}) \subset PSL(2, \mathfrak{L}) \cap \mathcal{A}(\mathbb{H})$. Furthermore, this subgroup leaves invariant the horizontal horospheres $\Re(\mathbf{q}) = x_0 > 0$ and also the horoball $\Re(\mathbf{q}) > x_0 > 0$.

Evidently $\mathcal{A}(\mathfrak{L})$ is a subgroup of $PSL(2, \mathfrak{L})$ and, since $\mathbf{ij} = \mathbf{k}$, it is generated by hyperbolic isometries associated with the matrices $\begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}$, $\begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}$ and $\begin{pmatrix} 1 & \mathbf{u} \\ 0 & 1 \end{pmatrix}$. In particular, since the transformation represented by the matrix $\begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$ is a rotation of angle π which keeps fixed each point of the plane $S_{\mathbf{u}}$ (the "axis of rotation"), the combination of such a rotation and the inversion leads to a transformation represented by the matrix $\begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{u} & 0 \end{pmatrix}$ with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$. For these trasformations the plane $S_{\mathbf{u}}$, with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ is invariant. Both rotations and inversion composed with a rotation of the plane leave invariant the sphere Π and have 1 as a fixed point.

We have the following properties:

(1) The inverse of a matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{A}(\mathbb{H})$ is the matrix $\begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix} \in \mathcal{A}(\mathbb{H})$.

(2) If we consider the group $\mathfrak{K}_u$ of Lipschitz or Hurwitz units, then the map

$$\mathcal{A}(\mathfrak{K}) \to \mathfrak{K}_u, \quad \begin{pmatrix} \mathbf{u} & \mathbf{u}b \\ 0 & \mathbf{u} \end{pmatrix} \mapsto \mathbf{u}$$

is an epimorphism whose kernel is $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})} = \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : \omega \in \Im\mathbb{H}(\mathbb{Z}) \right\}$.

(3) Thus we have the exact sequence

$$0 \longrightarrow \mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})} \longrightarrow \mathcal{A}(\mathfrak{K}) \longrightarrow \mathcal{U}(\mathfrak{K}) \longrightarrow 0$$

This sequence splits and the group $\mathcal{A}(\mathfrak{K})$ is the semi-direct product of $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ with $\mathcal{U}(\mathfrak{K})$.

(4) The group $\mathcal{U}(\mathfrak{L}) \subset \mathcal{U}(\mathfrak{H})$ is a normal subgroup and we have the exact sequence

$$0 \longrightarrow \mathcal{U}(\mathfrak{L}) \longrightarrow \mathcal{U}(\mathfrak{H}) \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

Definition 3.5 Let $\hat{\mathcal{U}}(\mathfrak{L})$ and $\hat{\mathcal{U}}(\mathfrak{H})$ be the maximal subgroups of $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ respectively which fix 1.

We have the following proposition since $T^2 = \mathcal{I}$ and T commutes with all of the elements of $\mathcal{U}(\mathfrak{L})$ and $\mathcal{U}(\mathfrak{L})$.

Proposition 3.6 *The groups $\hat{\mathcal{U}}(\mathfrak{L})$ and $\hat{\mathcal{U}}(\mathfrak{H})$ are the subgroups generated by T and $\mathcal{U}(\mathfrak{L})$ and T and $\mathcal{U}(\mathfrak{H})$, respectively. Moreover, $\hat{\mathcal{U}}(\mathfrak{L}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathcal{U}(\mathfrak{L})$ and $\hat{\mathcal{U}}(\mathfrak{H}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathcal{U}(\mathfrak{H})$.*

The following fundamental theorem gives the description of the quaternionic modular groups with generators and relations.

Proposition 3.7 *The Lipschitz modular group is generated by the inversion T and the translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$. The Hurwitz modular group is the group generated by the inversion T, by the translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ and by $\mathcal{U}(\mathfrak{H})$.*

Proof Let $A \in PSL(2, \mathbb{H}(\mathbb{Z}))$ satisfy Ahlfors conditions. Let $q = A(1)$ and $S \in PSL(2, \mathfrak{L})$ be such that $p := S(q) \in \mathcal{P}$. Then $(SA)(1) = p$ and by 4.6 it follows that $SA \in \mathcal{A}(\mathfrak{L})$. Hence $A \in \mathcal{A}(\mathfrak{L}) \subset PSL(2, \mathfrak{L})$. $\square$

Proposition 3.8 *The group $PSL(2, \mathfrak{L})$ is a subgroup of index three of the group $PSL(2, \mathfrak{H})$.*

Proof This is so since the order of the group of transformations induced by the diagonal matrices with entries in the Lipschitz units is of index three in the group of transformations induced by diagonal matrices with entries in the Hurwitz units. □

Remark 3.9 The groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ are discrete isometric groups of $\mathbf{H}_{\mathbb{H}}^1$ so they are *4-dimensional hyperbolic Kleinian groups* in the sense of Henri Poincaré (see [16,17] and [21]).

4 Fundamental domains

Given a group Γ_Ω acting continuously on a metric space Ω, we say that a subset $\mathcal{D}$ of Ω is a *fundamental domain* for Γ_Ω if it contains exactly one point from each of the images of a single point under the action of Γ_Ω (the so called *orbits* of Γ_Ω). Typically, a fundamental domain is required to be a convex subset with some restrictions on its boundary, for example, smooth or polyhedral. The images of a chosen fundamental domain under the group action then tessellate the space Ω.

In this paper we'll mainly deal with groups of matrices whose entries are quaternions and therefore acting on quaternionic hyperbolic spaces; we then investigate their fundamental domains and the corresponding quotient spaces.

4.1 A quaternionic kaleidoscope

We begin with the ideal convex hyperbolic polytope $\mathcal{P}$ with one vertex at infinity which is the intersection of the half-spaces which contain 2 and which are determined by the hyperbolic hyperplanes $\Pi := \{\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1 \ : \ |\mathbf{q}| = 1\}$ and $\Pi_{\pm n/2} := \{\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \ : \ x_n = \pm 1/2\}$. The only ideal vertex of $\mathcal{P}$ is the point at infinity. The (non ideal) vertices of $\mathcal{P}$ are the eight points $\frac{1}{2}(1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ which are the vertices of a cube $\mathcal{C} = \{\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbf{H}_{\mathbb{H}}^1 \ : \ |\mathbf{q}| = 1, \ |x_n| \leq 1/2, \ n = 1, \ldots, 3\}$ (Fig. 2).

The polytope $\mathcal{P}$ has seven 3-dimensional faces: one compact cube $\mathcal{C}$ and six pyramids with one ideal vertex at ∞ as their common apex and the six squares of the cube $\mathcal{C}$ as their bases. Moreover $\mathcal{P}$ has 20 2-dimensional faces (6 compact squares and 12 triangles with one ideal vertex) and 20 edges (12 compact and 8 with one ideal vertex). The convex polytope $\mathcal{P}$ satisfies the conditions of the Poincaré's polyhedron theorem, therefore the group generated by reflections on the faces of $\mathcal{P}$ is a discrete subgroup of hyperbolic isometries of $\mathbf{H}_{\mathbb{H}}^1$. We denote this subgroup by $G(3)$. The index-two subgroup generated by composition of an even number of reflections has as fundamental domain the convex polytope $\mathcal{P} \cup T(\mathcal{P})$. This subgroup of $PSL(2, \mathbb{H}(\mathbb{Z}))$ which consists of orientation-preserving isometries will be denoted by $G(3)_+$. We will see below that $\mathcal{P}$ can be tessellated by four copies of a fundamental domain of the action of $PSL(2, \mathfrak{L})$ and by twelve copies of a fundamental domain of the action of $PSL(2, \mathfrak{H})$ on $\mathbf{H}_{\mathbb{H}}^1$. The quotient space $\mathbf{H}_{\mathbb{H}}^1/G(3)$ is a quaternionic kaleidoscope which is a good non-orientable orbifold. Since the polytope $\mathcal{P}$ is of finite volume the non-orientable orbifold obtained is finite and has the same volume. If we imagine we are inside $\mathbf{H}_{\mathbb{H}}^1/G(3)$ for a moment and open our eyes we see 4-dimensional images very similar to the 3-dimensional honeycombs of Roice Nelson of the Fig. 6.

The orientable orbifold $\mathbf{H}_{\mathbb{H}}^1/G(3)_+$ is obtained from the double pyramid $\mathcal{P} \cup T(\mathcal{P})$ by identifying in pairs the faces with an ideal vertex at infinity with corresponding faces with an ideal vertex at zero. These 3-dimensional faces meet at the square faces of the cube $\mathcal{C}$ in
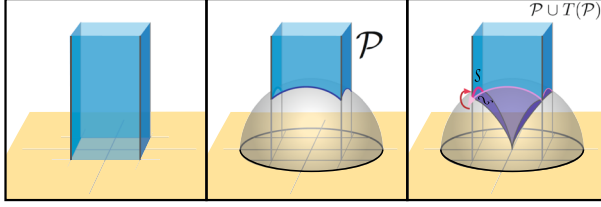
Fig. 2 Schematic picture of the chimney which is the fundamental domain of the parabolic group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ (generated by the translations $\tau_{\mathbf{i}}$, $\tau_{\mathbf{j}}$ and $\tau_{\mathbf{k}}$), the polytope $\mathcal{P}$ and the polytope $\mathcal{P}$ and its inversion $T(\mathcal{P})$. The horizontal plane represents the purely imaginary quaternions that forms the ideal boundary $\partial\mathbf{H}_{\mathbb{H}}^1$ and above it the open half-space of quaternions with positive real part $\mathbf{H}_{\mathbb{H}}^1$

Π and they are identifying in pairs by a rotation of angle $2\pi/3$ around the hyperbolic plane that contains the square faces. The underlying space is $\mathbb{R}^4$ and the singular locus of $\mathcal{O}_{G(3)_+}$ is a cube. This group is generated by the six rotations of angle $2\pi/3$ around the hyperbolic planes that contain the square faces of the cube $\mathcal{C}$ (Fig. 2).

4.2 Fundamental domains and orbifolds for translations and for the inversion

We recall that a fundamental domain of the parabolic group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ (generated by the translations $\tau_{\mathbf{i}}$, $\tau_{\mathbf{j}}$ and $\tau_{\mathbf{k}}$) is the infinite-volume convex hyperbolic chimney with one vertex at infinity which is the intersection of the half-spaces which contain 2 and which are determined by the set of six hyperbolic hyperplanes Π_n, where $n = \pm\frac{\mathbf{i}}{2}, \pm\frac{\mathbf{j}}{2}, \pm\frac{\mathbf{k}}{2}$.

The hyperbolic 4-dimensional orbifold $\mathcal{M}_{\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}}$ is a 2-cusped manifold which is an infinite volume cylinder on the 3-torus $\mathbf{T}^3$ with one cusp (an end of finite volume) and one tube (an end of infinite volume). We can write $\mathcal{M}_{\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}} = \mathbf{T}^3 \times \mathbb{R}$.

The fundamental domain of the inversion T is closed half-space whose boundary is the hyperbolic hyperplane Π.

The hyperbolic 4-dimensional orbifold $\mathcal{M}_T$ has a unique singular point and it is homeomorphic to the cone over the real projective space $\mathbf{P}_{\mathbb{R}}^3$.

4.3 Fundamental domain of $PSL(2, \mathfrak{L})$

Since the quaternionic modular group $PSL(2, \mathfrak{L})$ is generated by $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ and the inversion T, we can choose a fundamental domain which is totally contained in $\mathcal{P}$.

The finite Lipschitz unitary group $\mathcal{U}(\mathfrak{L})$ acts by rotations of angle π around the three hyperbolic 2-planes generated by 1 and $\mathbf{u}$ where $\mathbf{u} = \mathbf{i}, \mathbf{j}$ or $\mathbf{k}$. We divide the cube $\mathcal{C}$ in eight congruent cubes by cutting it along the coordinate planes. Then $\mathcal{P}$ is divided in eight congruent cubic pyramids. We label the cubes with two colors as a chessboard (see [9]).

An element of the finite unitary Lipschitz group identifies four cubes (two white cubes and two black ones) with other four cubes (two white and two black) preserving the colors.

A fundamental domain for $PSL(2, \mathfrak{L})$ can be taken to be the union of two cubic pyramids with bases two of the cubes described in the previous paragraph, one white and one black and with a common vertex at the point at infinity. We can choose adjacent cubes to obtain a convex fundamental domain but this is not necessary to have a fundamental domain.

The inversion T acts by identifying each white cube with a diametrally opposite black one in Π. Then a fundamental domain for $PSL(2, \mathfrak{L})$ is the union of two cubic pyramids in $\mathcal{P}$.

Fig. 3 Left: The action of $U(\mathfrak{L})$ on the cube $\mathcal{C}$. Right: The two hyperbolic cubes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{C}$ which are the bases of a fundamental domain $\mathcal{P}_{\mathfrak{L}}$ of $PSL(2, \mathfrak{L})$
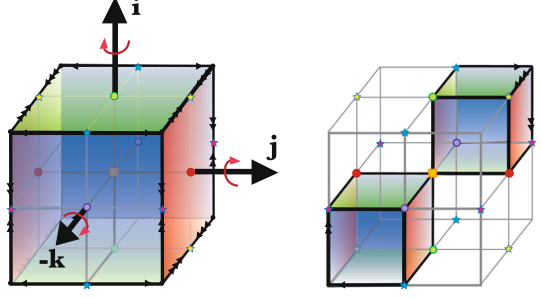
See Fig. 3. Below we describe other fundamental domains which are more suitable to study the isotropy groups and the tessellation in $\mathbf{H}^1_{\mathbb{H}}$ around singular points.

Definition 4.1 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two hyperbolic cubes in $\mathcal{C}$ which contain the vertices $\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$ and $\frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k})$, respectively.

Let $\mathcal{P}_{\mathfrak{L}}$ be the union of the two hyperbolic cubic pyramids with vertex at infinity and bases the two cubes $\mathcal{C}_1$ and $\mathcal{C}_2$.

Remark 4.2 The elements in $\mathcal{P}_{\mathfrak{L}}$ are the points $\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ in $\mathcal{P}$ such that the real numbers x_n have the same sign for all $n = 1, 2, 3$.
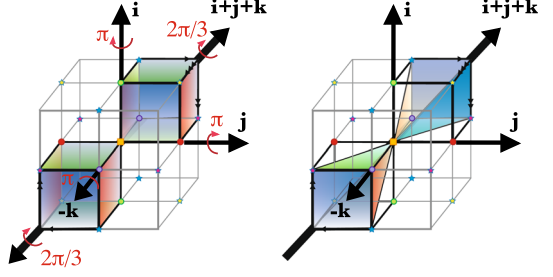
4.4 Fundamental domain of $PSL(2, \mathfrak{H})$

We recall that $PSL(2, \mathfrak{H})$ is generated by the parabolic group of translations $\mathcal{T}_{\mathfrak{I}\mathbb{H}(\mathbb{Z})}$, the inversion T and the unitary Hurwitz group $\mathcal{U}(\mathfrak{H})$. From our previous descriptions of the fundamental domains of the group of translations and the group of order 2 generated by T we know that the fundamental domain of $PSL(2, \mathfrak{H})$ is commensurable with $\mathcal{P}$, the pyramid over the cube $\mathcal{C}$. More precisely, $\mathcal{P}$ is invariant under $\mathcal{U}(\mathfrak{H})$ and therefore the fundamental domain of $PSL(2, \mathfrak{H})$ is the fundamental domain in $\mathcal{P}$ of the action of $\mathcal{U}(\mathfrak{H})$ on $\mathcal{P}$. Moreover, as $\mathcal{U}(\mathfrak{L}) \subset \mathcal{U}(\mathfrak{H})$ we have that the fundamental domain of $PSL(2, \mathfrak{H})$ is a subset of the fundamental domain $\mathcal{P}_{\mathfrak{L}}$ of the action of $PSL(2, \mathfrak{L})$. Furthermore, since $\mathcal{U}(\mathfrak{L})$ is a subgroup of $\mathcal{U}(\mathfrak{H})$ of index three then we have that $\mathcal{P}_{\mathfrak{H}}$ is a third part of $\mathcal{P}_{\mathfrak{L}}$.

Let $\mathbf{u} \in \mathfrak{H}_{\mathbf{u}}$, then $D_{\mathbf{u}} \in \mathcal{U}(\mathfrak{H})$ is induced by a diagonal matrix and acts as follows: $\mathbf{q} \mapsto \mathbf{u}\mathbf{q}\mathbf{u}^{-1} = \mathbf{u}\mathbf{q}\bar{\mathbf{u}}$. If $\mathbf{u}$ is a Hurwitz unit which is not a Lipschitz unit (*i.e.* $\mathbf{u} = 1/2(1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$) then the matrix $D_{\mathbf{u}}$ is of order three and geometrically is a rotation of angle $2\pi/3$ around the diagonal of $\mathcal{C}$ which contains $\mathbf{u}$ or $-\mathbf{u}$, but only one has a positive real part and then is in $\mathbf{H}^1_{\mathbb{H}}$. As $D_{\mathbf{u}} = D_{-\mathbf{u}}$ we can suppose that $\Re(\mathbf{u}) = 1/2$. One has $D^2_{\mathbf{u}} = D_{\mathbf{u}^2} = D_{1-\mathbf{u}}$ (since $\Re(\mathbf{u}) = 1/2 > 0$, $1 - \mathbf{u}$ is Hurwitz unit and $\Re(\mathbf{u}) = \Re(1 - \mathbf{u}) > 0$).

The group of Hurwitz units $\mathcal{U}(\mathfrak{H})$ acts transitively on the edges of $\mathcal{C}$. Therefore a fundamental domain can be determined by the choice of one edge of $\mathcal{C}$. Hence a convex fundamental domain is the pyramid with vertex at infinity with base the hyperbolic convex polyhedron with vertices the two end points of the edge, the two barycenters of the square faces that have the edge in common and 1. However we choose as a fundamental domain a non-convex polyhedron (Fig. 4).

Definition 4.3 Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two hyperbolic 3-dimensional square pyramids in $\mathcal{C}_1 \subset \mathcal{C}$ and $\mathcal{C}_2 \subset \mathcal{C}$, respectively, with apex 1 and which have as bases the squares in the boundary of $\mathcal{C}$ with sets of vertices

Fig. 4 Left: The action of $U(\mathfrak{H})$ in the cube $\mathcal{C}$. Right: the bases of the fundamental domain of $PSL(2, \mathfrak{H})$. The two hyperbolic pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{C}$ which are the bases of a fundamental domain $\mathcal{P}_\mathfrak{H}$ of $PSL(2, \mathfrak{H})$

$$S := \left\{ v_1 = \frac{1}{2}\left(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}\right), v_2 = \frac{1}{2}\left(\sqrt{2} + \mathbf{i} + \mathbf{k}\right), v_3 = \frac{1}{2}\left(\sqrt{2} + \mathbf{j} + \mathbf{k}\right), \right.$$
$$\left. v_4 = \frac{1}{2}\left(\sqrt{3} + \mathbf{k}\right) \right\}$$

and $T(\{S\}) := (T(v_1), T(v_2), T(v_3).T(v_4))$, respectively.

Let $\mathcal{P}_\mathfrak{H}$ be the union of the two hyperbolic 4-dimensional pyramids with vertex at infinity and bases the two hyperbolic 3-dimensional pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$.

4.5 Proof that $\mathcal{P}_\mathfrak{L}$ and $\mathcal{P}_\mathfrak{H}$ are fundamental domains

We start from the following important lemma:

Lemma 4.4 *Let $\gamma \in PSL(2, \mathbb{H})$ satisfy Ahlfors conditions. If $\mathbf{q} \in \mathbf{H}_\mathbb{H}^1$, then*

$$\Re(\gamma(\mathbf{q})) = \frac{\Re(\mathbf{q})}{|\mathbf{q}c + d|^2} \tag{2}$$

Proof We recall that if $\mathbf{q} \in \mathbf{H}_\mathbb{H}^1$ the action of γ in $\mathbf{H}_\mathbb{H}^1$ is given by the rule

$$\gamma(\mathbf{q}) = (a\mathbf{q} + b)(c\mathbf{q} + d)^{-1} = (a\mathbf{q} + b)(\overline{q}\overline{c} + \overline{d})\left(\frac{1}{|\mathbf{q}c + d|^2}\right).$$

Suppose $c \neq 0$, then:

$$
\begin{aligned}
c^{-1}\gamma(\mathbf{q})c &= \overline{c}(a\mathbf{q} + b)(c\mathbf{q} + d)^{-1}c/|c|^2 \\
&= (\overline{c}a\mathbf{q} + \overline{c}b)(c\mathbf{q} + d)^{-1}c/|c|^2 \\
&= (-\overline{a}c\mathbf{q} + 1 - \overline{a}d)(c\mathbf{q} + d)^{-1}c/|c|^2 \\
&= (1 - \overline{a}(c\mathbf{q} + d))(c\mathbf{q} + d)^{-1}c/|c|^2 \\
&= (c\mathbf{q} + d)^{-1}c/|c|^2 - \overline{a}c/|c|^2 \\
&= \frac{(\overline{q}\overline{c} + \overline{d})c}{|c\mathbf{q} + d|^2|c|^2} - \frac{\overline{a}c}{|c|^2} \\
&= \frac{\mathbf{q}}{|c\mathbf{q} + d|^2} + \frac{\overline{d}c}{|c\mathbf{q} + d|^2|c|^2} - \frac{\overline{a}c}{|c|^2}.
\end{aligned}
$$

Since $\Re(\mathbf{q}) = \Re(w\mathbf{q}w^{-1})$ for any invertible quaternion, and $\Re(\overline{d}c) = \Re(\overline{a}c) = 0$, we have

$$\Re(\gamma(\mathbf{q})) = \Re\left(c^{-1}\gamma(\mathbf{q})c\right) = \frac{\Re(\mathbf{q})}{|c\mathbf{q} + d|^2}.$$

12

We notice that if one restricts the entries of the matrices to the set $\mathbb{H}(\mathbb{Z})$ or $\mathbb{H}ur$, then there are only a finite number of possibilities for c and d in such a way that $|\mathbf{q}c + d|$ is less than a given number; therefore we obtain the following important

Corollary 4.5 *For every* $\mathbf{q} \in \mathbb{H}$ *one has*

$$\sup_{\gamma \in PSL(2, \mathfrak{L})} \Re(\gamma(\mathbf{q})) < \infty \quad and \quad \sup_{\gamma \in PSL(2, \mathfrak{H})} \Re(\gamma(\mathbf{q})) < \infty.$$

With the geometric tools so far introduced we can establish the following:

Theorem 4.6 *The fundamental domains* $\mathcal{P}_{\mathfrak{L}}$ *and* $\mathcal{P}_{\mathfrak{H}}$ *for the actions of the groups* $PSL(2, \mathfrak{L})$ *and* $PSL(2, \mathfrak{H})$, *respectively, have the following properties:*

(1) *for every* $\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1$ *there exists* $\gamma \in PSL(2, \mathfrak{L})$ *(resp.* $PSL(2, \mathfrak{H})$*) such that* $\gamma(\mathbf{q}) \in \mathcal{P}_{\mathfrak{L}}$ *(resp.* $\mathcal{P}_{\mathfrak{H}}$*).*
(2) *If two distinct points* $\mathbf{q}, \mathbf{q}'$ *of* $\mathcal{P}_{\mathfrak{L}}$ *(resp.* $\mathcal{P}_{\mathfrak{H}}$*) are congruent modulo* $PSL(2, \mathfrak{L})$ *(resp.* $PSL(2, \mathfrak{H})$*); i.e. if there exists* $\gamma \in PSL(2, \mathfrak{L})$ *(resp.* $PSL(2, \mathfrak{H})$*) such that* $\gamma(\mathbf{q}) = \mathbf{q}'$, *then* $\mathbf{q}, \mathbf{q}' \in \partial \mathcal{P}_{\mathfrak{L}}$ *(resp.* $\mathcal{P}_{\mathfrak{H}}$*). If* $|\mathbf{q}| > 1$ *then* $\gamma \in \mathcal{A}(\mathfrak{L})$ *(resp.* $\mathcal{A}(\mathfrak{H})$*). If* $|\mathbf{q}| = 1$ *then* $\gamma \in \mathcal{A}(\mathfrak{L})$ *(resp.* $\mathcal{A}(\mathfrak{H})$*) or* $\gamma = AT$ *where* T *is the usual inversion and* $A \in \mathcal{A}(\mathfrak{L})$ *(resp.* $\mathcal{A}(\mathfrak{H})$*).*
(3) *Let* $\mathbf{q} \in \mathcal{P}_{\mathfrak{L}}$ *(resp.* $\mathcal{P}_{\mathfrak{H}}$*) and let* $G_{\mathbf{q}} = \{g \in PSL(2, \mathfrak{L})\}$ *(resp.* $PSL(2, \mathfrak{H})$*) be the stabilizer of* $\mathbf{q}$ *in* $PSL(2, \mathfrak{L})\}$ *(resp.* $PSL(2, \mathfrak{H})$*) then* $G_{\mathbf{q}} = \{1\}$ *if* $\mathbf{q} \neq \partial \mathcal{P}_{\mathfrak{L}}$ *(resp.* $\partial \mathcal{P}_{\mathfrak{H}}$*).*

Proof This is essentially a construction of a fundamental domain from the Ford domain. The Ford domain is the exterior of all isometric spheres. In other words, it consists of all points which maximize the real part within an orbit. Of course this maximum is attained for infinitely many points, all related by the stabilizer of 1. So intersecting the Ford domain with a fundamental domain for the stabilizer of 1 gives a fundamental domain. Indeed, one can use a quaternionic version of the continued fraction algorithm to bring any quaternion with positive real part into this domain by successively applying inversions and translation maps (see [12] and [24]). Let $\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1$. By corollary 4.6 there exists $\gamma \in PSL(2, \mathfrak{L})$ (resp. $PSL(2, \mathfrak{H})$) such that $\Re(\gamma(\mathbf{q}))$ is maximum. There exists $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that the element $\mathbf{q}' = \tau_{\mathbf{i}}^{n_1} \tau_{\mathbf{j}}^{n_2} \tau_{\mathbf{k}}^{n_3} \gamma(\mathbf{q})$ is of the form $\mathbf{q}' = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ where $|x_n| \leq \frac{1}{2}, n = 1, 2, 3$. Then $\mathbf{q}'$ is an element of the fundamental domain of the parabolic group $\mathcal{T}_{\mathfrak{H}\mathbb{H}(\mathbb{Z})}$.

If $|\mathbf{q}'| < 1$, then the element $T\mathbf{q}' = (\mathbf{q}')^{-1}$ has real part strictly larger than $\Re(\mathbf{q}') = \Re(\gamma(\mathbf{q}))$, which is impossible. Then we must have $|\mathbf{q}'| \geq 1$, and $\mathbf{q}' \in \mathcal{P}$. This shows that given any $\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1$ there exists $\gamma \in PSL(2, \mathfrak{L})$ (resp. $PSL(2, \mathfrak{H})$) such that $\gamma(\mathbf{q}) \in \mathcal{P}$. We remember that the elements in $\mathcal{P}_{\mathfrak{L}}$ are the points $\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ in $\mathcal{P}$ such that the real numbers x_n have the same sign for all $n = 1, 2, 3$. The action of an element $D_{\mathbf{u}}$, with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$, in the unitary Lipschitz group $\mathcal{U}(\mathfrak{L})$ has the property of leaving invariant x_1 and x_n and changing the signs of the other two coefficients. The action of an element $D_{\mathbf{u}}$, with $\mathbf{u} = \frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$, in the unitary Hurwitz group $\mathcal{U}(\mathfrak{H})$ has the property that it rotates multiples of $2\pi/3$ the cells of $\mathbf{Y}_{\mathfrak{H}}$ around the diagonal passing through $\mathbf{u}$ and $-\mathbf{u}$ of the cube $\mathcal{C}$. Then we can use one element in $\mathcal{U}(\mathfrak{L})$ to have a point $\mathbf{q}''$ of the orbit of $\mathbf{q} \in \mathcal{P}_{\mathfrak{L}}$. In the Hurwitz case we can use one element in $\mathcal{U}(\mathfrak{H})$ to have a point $\mathbf{q}''$ of the orbit of $\mathbf{q} \in \mathcal{P}_{\mathfrak{H}}$. This proves (1). In others words, the orbit of any point $\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1$ under the action of the group $PSL(2, \mathfrak{L})$ (resp. $PSL(2, \mathfrak{H})$) has a representative in $\mathcal{P}_{\mathfrak{L}}$ (resp. $\mathcal{P}_{\mathfrak{H}}$).

Let $\mathbf{q} \in \mathcal{P}_{\mathfrak{L}}$ (resp. $\mathcal{P}_{\mathfrak{H}}$) and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathfrak{L})$ (resp. $PSL(2, \mathfrak{H})$) such that $\gamma \neq \mathcal{I}$, where $\mathcal{I}$ is the identity matrix in $PSL(2, \mathbb{H})$ and $\gamma(\mathbf{q}) \in \mathcal{P}_{\mathfrak{L}}$ (resp. $\mathcal{P}_{\mathfrak{H}}$). We can

suppose that $\Re(\gamma(\mathbf{q})) \geq \Re(\mathbf{q})$, i.e. $|c\mathbf{q}+d| \leq 1$. This is clearly impossible if $|c| \geq 1$, leaving then the cases $c = 0$ or $|c| = 1$.

(I) If $c = 0$, we have $|d| = 1$ and Ahlfors conditions imply that $a\bar{d}=1$ and $b\bar{d} + d\bar{b} = 0$. There are two cases:

(I.1) If $d = 1$, then $a = 1$ and $\Re(b) = 0$. Then

$$\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where $b = b_{\mathbf{i}}\mathbf{i} + b_{\mathbf{j}}\mathbf{j} + b_{\mathbf{k}}\mathbf{k}$. If $\mathbf{q} = x_1 + x_{\mathbf{i}}\mathbf{i} + x_{\mathbf{j}}\mathbf{j} + x_{\mathbf{k}}\mathbf{k} \in \mathcal{P}_{\mathfrak{L}}$ then

$$\gamma(\mathbf{q}) = \mathbf{q}' = x_1 + (x_{\mathbf{i}} + b_{\mathbf{i}})\mathbf{i} + (x_{\mathbf{j}} + b_{\mathbf{j}})\mathbf{j} + (x_{\mathbf{k}} + b_{\mathbf{k}})\mathbf{k} \in \mathcal{P}_{\mathfrak{L}}, \text{ and}$$

(I.1.1) If $|b| = 1$ then $b = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$; and $\mathbf{q} = r - \frac{b}{2}$, where $r \geq \frac{\sqrt{3}}{2}$. Then $\mathbf{q}$ is on the vertical geodesic that joins a barycenter of a square face of the cube $\mathcal{C}$ in the base of $\mathcal{P}$ with the point at infinity ∞ and so $\mathbf{q}, \mathbf{q}' \in \partial\mathcal{P}_{\mathfrak{L}}$.

(I.1.2) If $|b| = 2$ then $b = \pm\mathbf{i} \pm \mathbf{j}, \pm\mathbf{i} \pm \mathbf{k}, \pm\mathbf{j} \pm \mathbf{k}$; and $\mathbf{q} = r - \frac{b}{2}$, where $r \geq \frac{1}{\sqrt{2}}$. Then $\mathbf{q}$ is on the vertical geodesic that joins the middle point of an edge of the cube $\mathcal{C}$ with the point at infinity ∞ and so $\mathbf{q}, \mathbf{q}' \in \partial\mathcal{P}_{\mathfrak{L}}$.

(I.1.2) If $|b| = 3$ then $b = \pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$; and $\mathbf{q} = r - \frac{b}{2}$, where $r \geq \frac{1}{2}$. Then $\mathbf{q}$ is on the vertical geodesic that joins a vertex of the cube $\mathcal{C}$ with the point at infinity and so $\mathbf{q}, \mathbf{q}' \in \partial\mathcal{P}_{\mathfrak{L}}$.

(I.2) If $d \neq 1$ then $d = a$, $|a| = 1$ and there are two subcases:

(I.2.1) If $b = 0$ then

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Then $\mathbf{q}$ is on the hyperbolic plane generated by 1 and a and so $\mathbf{q} \in \partial\mathcal{P}_{\mathfrak{L}}$.

(I.2.2) If $b \neq 0$, then $\Re(b) = 0$, $|b| = 1$ hence $b = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, but $b \neq a$. Then

$$\gamma = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Then $\mathbf{q}$ is on the hyperbolic plane generated by 1 and a and so $\mathbf{q} \in \partial\mathcal{P}_{\mathfrak{L}}$.

(II) If $c \neq 0$, as $|\mathbf{q}| \geq 1$ then $d = 0$. As $|c\mathbf{q}| \leq 1$ then $|c| = |\mathbf{q}| = 1$. Then $\mathbf{q} \in \partial\mathcal{P}_{\mathfrak{L}}$. Ahlfors conditions imply that $\bar{b}c = 1$ and $a\bar{c} + \bar{a}c = 0$.

(II.1) If $c = 1$, then $b = 1$ and $\Re(a) = 0$. Then

$$\gamma = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

(II.2) If $c \neq 1$, then $c = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ and $b = c$. Moreover $a = 0$ or $|a| = 1$ and $\Re(a\bar{c}) = 0$. Then

$$\gamma = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix}.$$

To prove (3) suppose that $\mathbf{q} \in \text{Interior}(\mathcal{P}_{\mathfrak{L}})$. Let γ be such that $\gamma(\mathbf{q}) = \mathbf{q}$. Then there exist $\epsilon > 0$ such that $\mathbf{q} + \epsilon$ and $\gamma(\mathbf{q} + \epsilon) \in \text{Interior}(\mathcal{P}_{\mathfrak{L}})$. But then by (2) $\mathbf{q} + \epsilon$ and $\gamma(\mathbf{q} + \epsilon)$ are in $\partial\mathcal{P}_{\mathfrak{L}}$ that is a contradiction. The same proof applies to show that points in $\text{Interior}(\mathcal{P}_{\mathfrak{H}})$ have trivial isotropy group. $\qquad\square$

Fig. 5 The Coxeter decomposition into 48 tetrahedra of a cube in the Euclidean 3-space
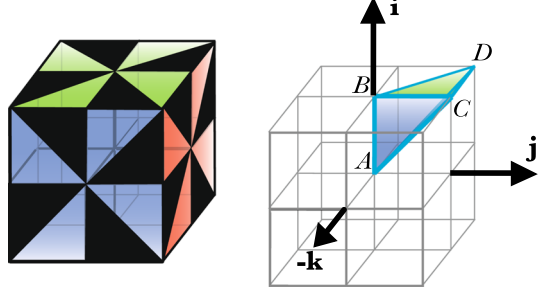
If we use the group $PSL(2, \mathfrak{L})$ to propagate $\mathcal{P}_{\mathfrak{L}}$ we obtain a tessellation of $\mathbf{H}^1_{\mathbb{H}}$ that we denote by $\mathbf{Y}_{\mathfrak{L}}$. The intersection of $\mathcal{P}_{\mathfrak{L}}$ and $\mathbf{Y}_{\mathfrak{L}}$ with each of the totally geodesic planes $S_{\mathbf{i}}$, $S_{\mathbf{j}}$, $S_{\mathbf{k}}$, where $S_{\mathbf{u}} := \{q = x_1 + x_{\mathbf{i}}\mathbf{i} + x_{\mathbf{j}}\mathbf{j} + x_{\mathbf{k}}\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} : x_{\mathbf{s}} = 0 \text{ if } \mathbf{s} \neq \mathbf{u}, 0\}$, with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$, gives a copy of the closure of a (non-convex) fundamental domain of $PSL(2, \mathbb{Z})$ and the associated tessellation in the half-space model of $\mathbf{H}^2_{\mathbb{R}}$. Indeed, it is worth noticing here that $S_{\mathbf{u}}$ is an *invariant* set for $R_{\mathbf{u}}$, $L_{\mathbf{u}}$ and $\tau_{\mathbf{u}}$ ($\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$); therefore the intersection of $\mathcal{P}_{\mathfrak{L}}$ and $\mathbf{Y}_{\mathfrak{L}}$ with each of the 3-dimensional totally geodesic hyperbolic 3-spaces $S_{\mathbf{ij}}$, $S_{\mathbf{jk}}$, $S_{\mathbf{ik}}$, where $S_{\mathbf{lm}} := \{q = x_1 + x_{\mathbf{i}}\mathbf{i} + x_{\mathbf{j}}\mathbf{j} + x_{\mathbf{k}}\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} : x_{\mathbf{s}} = 0 \text{ if } \mathbf{s} \neq \mathbf{l}, \mathbf{m}, 0\}$, with $\mathbf{l}, \mathbf{m} = \mathbf{i}, \mathbf{j}, \mathbf{k}$, gives a copy of the closure of the classical fundamental domain (and the tessellation generated by it) of the Picard group $PSL(2, \mathbb{Z}[\mathbf{i}])$ for the Gaussian integers acting on the half-space model of $\mathbf{H}^3_{\mathbb{R}}$ (see [21]).

4.6 Coxeter decomposition

Let A, B, C, D be the barycenters of the $k-$faces of a flag in the hyperbolic cube $\mathcal{C}$. For example, $A = 1$, $B = \frac{1}{2}(\sqrt{3} + \mathbf{i})$, $C = \frac{1}{\sqrt{2}} + \frac{1}{2}(\mathbf{i} + \mathbf{j})$ y $D = \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$, as in the Fig. 5. Let Δ be the non-compact hyperbolic 4-simplex whose five vertices are A, B, C, D and ∞. The orthogonal projection of Δ on the ideal boundary of $\mathbf{H}^1_{\mathbb{H}}$ is the 3-dimensional Euclidean tetrahedron $\triangle(4, 3, 4)$. Therefore we can identify Δ with the Coxeter 4-simplex $\triangle(3, 4, 3, 4)$ (see [8]).

Let [3, 4, 3, 4] be the hyperbolic Coxeter group generated by reflections on the sides of Δ. In [16] we found that $PSL(2, \mathfrak{L})$ is isomorphic to the Coxeter group $[3, 4, 3^*, 4]^+$ and $PSL(2, \mathfrak{H})$ is isomorphic to the Coxeter group $[3, 4, 3^+, 4]^+$ which is a semidirect product of $[3, 4, 3^*, 4]^+$ and the automorphism group C_3. We have the following indices: $[[3, 4, 3, 4] : PSL(2, \mathfrak{H})] = 4$, $[[3, 4, 3, 4] : PSL(2, \mathfrak{L})] = 12$ and $[PSL(2, \mathfrak{H}) : PSL(2, \mathfrak{L})] = 3$. Moreover they give presentations for these groups (see [16] and [21]).

The union of the 48 simplexes asymptotic at ∞ and isometric to Δ with bases in the cube $\mathcal{C}$ is $\mathcal{P}$. The Lipschitz fundamental domain $\mathcal{P}_{\mathfrak{L}}$ is obtained as the union of 12 simplexes asymptotic at ∞ and isometric to Δ with bases in the two cubes $\mathcal{C}_1$ and $\mathcal{C}_2$. The Hurwitz modular domain $\mathcal{P}_{\mathfrak{H}}$ is obtained as the union of 4 simplexes asymptotic at ∞ and isometric to Δ since $PSL(2, \mathfrak{L})$ is a subgroup of index 3 of $PSL(2, \mathfrak{H})$.

Finally, applying $24 \times 48 = 1152$ elements of the group [3, 4, 3, 4] to Δ we obtain an union of isometric copies of $\mathcal{P}$ that forms a 24-cell which is a cell of the regular hyperbolic honeycomb $\{3, 4, 3, 4\}$. See Fig. 6.

The group of symmetries of the 24-cell is of order $24 \times 48 = 1152$. One knows from [23] that the volume of the hyperbolic right-angled 24-cell is $4\pi^2/3$, therefore the volume of $\Delta_{\mathfrak{L}}$ is $(\pi^2/864)$. Then, the volume of $\mathcal{P}_{\mathfrak{L}}$ is $12(\pi^2/864) = \pi^2/72$ and the volume of $\mathcal{P}_{\mathfrak{H}}$ is $4(\pi^2/864) = \pi^2/216$.
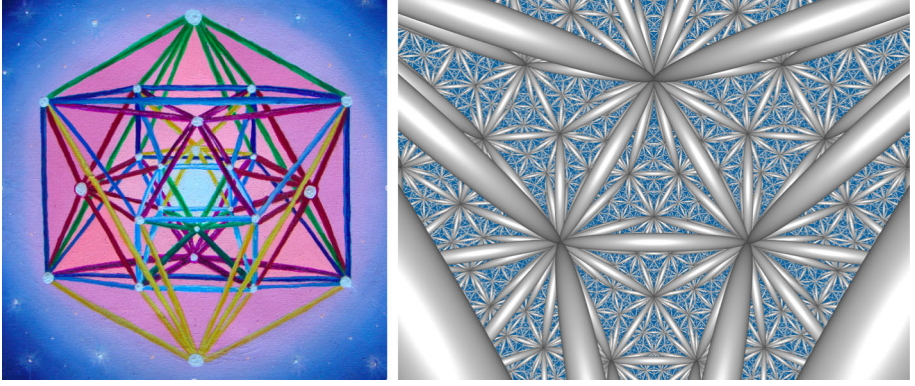
Fig. 6 The 24-cell {3, 4, 3} and the hyperbolic honeycomb {3, 4, 4}. This figure is courtesy of Roice Nelson [13]

5 The Lipschitz and Hurwitz quaternionic modular orbifolds

In this section we study the geometry of the quaternionic modular orbifolds related to the action of the quaternionic modular groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$. We describe the ends, the underlying spaces and the singular loci of the quaternionic modular orbifolds. Moreover we give the local models of these singularities and the local isotropy groups. Finally we compute their orbifold Euler characteristic.

Definition 5.1 Let $\mathcal{O}_{\mathfrak{L}}^4 := \mathbf{H}_{\mathbb{H}}^1/PSL(2, \mathfrak{L})$ and $\mathcal{O}_{\mathfrak{H}}^4 := \mathbf{H}_{\mathbb{H}}^1/PSL(2, \mathfrak{H})$ be the *Lipschitz quaternionic modular orbifold* and the *Hurwitz quaternionic modular orbifold*, respectively.

These quaternionic modular orbifolds are hyperbolic non-compact real 4-dimensional orbifolds of finite hyperbolic volume. Both have only one end and their singular loci has one connected component that accumulates to the cusp at infinity.

Moreover, these orbifolds are diffeomorphic to the quotient spaces of their fundamental domains $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$ by the action of the modular groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ on their boundaries $\partial \mathcal{P}_{\mathfrak{L}}$ and $\partial \mathcal{P}_{\mathfrak{L}}$, respectively. Then they have the same volume as $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$, respectively. These volumes are $\pi^2/72$ and $\pi^2/216$, respectively.

Each of the quaternionic modular orbifolds $\mathcal{O}_{\mathfrak{L}}^4$ and $\mathcal{O}_{\mathfrak{H}}^4$ has only one end because $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$ have each one end. We study the structure of the ends and we start by describing the sections of their ends and the thin and thick regions in the sense of Margulis thin-thick decomposition, see [22] pp. 654–665.

5.1 The sections of the ends and the thin regions of $\mathcal{O}_{\mathfrak{L}}^4$ and $\mathcal{O}_{\mathfrak{H}}^4$

For $r > 1$ we denote by $\mathcal{E}_r^3$ the *horosphere centered at the point at infinity* in $\mathbf{H}_{\mathbb{H}}^1$ which consists of the set of points in $\mathbf{H}_{\mathbb{H}}^1$ which have real part equal to r. Then $\mathcal{E}_r^3$ with the induced metric of $\mathbf{H}_{\mathbb{H}}^1$ is isometric to the Euclidean 3-space. The affine modular groups $\mathcal{A}(\mathfrak{L})$ and $\mathcal{A}(\mathfrak{H})$ are the maximal subgroups of $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ that leave invariant each horospheres $\mathcal{E}_r^3$ for any $r > 0$. Moreover $\mathcal{A}(\mathfrak{L})$ and $\mathcal{A}(\mathfrak{H})$ are isomorphic to discrete subgroups of Euclidean orientation-preserving isometries of $\mathcal{E}_r^3$.

The fact that $PSL(2, \mathfrak{L})$ is a subgroup of index 3 of $PSL(2, \mathfrak{H})$ implies the existence of an epimorphism $\pi : \mathcal{A}(\mathfrak{H}) \to \mathcal{A}(\mathfrak{L})$ with kernel $\mathbb{Z}/3\mathbb{Z}$.

We call $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$ the intersections of the fundamental domains of the quaternionic modular groups $PSL(2,\mathfrak{L})$ and $PSL(2,\mathfrak{H})$ with $\mathcal{E}^3_r$, respectively. Then $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$ are hyperbolic subsets of $\mathbf{H}^1_{\mathbb{H}}$ with finite volume which are isometric to Euclidean 3-dimensional polyhedra. In the Lipschitz case it consists of a pair of cubes which are symmetric with respect to the point r, where $r \in \mathbb{R} \cap \partial \mathcal{E}^3_{r,\mathcal{L}}$. In the Hurwitz case it consists of a pair of square pyramids in $\mathcal{E}^3_{r,\mathfrak{H}}$ symmetric with respect to the point $r \in \mathbb{R} \subset \mathbb{H}$. The orthogonal projections into the ideal boundary of $\mathbf{H}^1_{\mathbb{H}}$ of $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$ are the same as the orthogonal projections of $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$. There is covering map $\pi_{\mathcal{E}} : \mathcal{E}^3_{r,\mathfrak{L}} \to \mathcal{E}^3_{r,\mathfrak{H}}$ which is three to one.

Let $\mathcal{S}^3_{r,\mathfrak{L}} := \mathcal{E}^3_r / \mathcal{A}(\mathfrak{L})$ and $\mathcal{S}^3_{r,\mathfrak{H}} := \mathcal{E}^3_r / \mathcal{A}(\mathfrak{H})$. These are Euclidean 3-dimensional orbifolds of finite hyperbolic volume (see [7]). A pair of fundamental domains for the actions of the corresponding affine groups on $\mathcal{E}^3_r$ are the polyhedra $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$, respectively.

The actions of the restrictions of $PSL(2,\mathfrak{L})$ and $PSL(2,\mathfrak{H})$ on the boundaries $\partial \mathcal{E}^3_{r,\mathfrak{L}}$ and $\partial \mathcal{E}^3_{r,\mathfrak{H}}$, respectively give side-pairings of $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$. The quotients of the side-pairing in $\mathcal{E}^3_{r,\mathfrak{L}}$ and $\mathcal{E}^3_{r,\mathfrak{H}}$ are diffeomorphic to $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$, respectively.

There is an orbifold covering map $\pi_{\mathcal{S}} : \mathcal{S}^3_{r,\mathfrak{L}} \to \mathcal{S}^3_{r,\mathfrak{H}}$ which is three to one.

A convenient description of these Euclidean orbifolds is as follows: let $\mathbf{T}^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3| = 1\} = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the 3-torus with its standard flat metric. The group of orientation–preserving isometries of $\mathbf{T}^3$ generated by the transformations F_T, F_ω, $F_{\mathbf{i}}$, $F_{\mathbf{j}}$, $F_{\mathbf{k}}$ given by the formulas: $F_T(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$, $F_\omega(z_1, z_2, z_3) = (z_2, z_3, z_1)$, $F_{\mathbf{i}}(z_1, z_2, z_3) = (z_1, \overline{z_2}, \overline{z_3})$, $F_{\mathbf{j}}(z_1, z_2, z_3) = (\overline{z_1}, z_2, \overline{z_3})$ and $F_{\mathbf{k}} := F_{\mathbf{j}} F_{\mathbf{i}}$, is isomorphic to the group $\hat{\mathcal{U}}(\mathfrak{H})$ generated by T and $\mathcal{U}(\mathfrak{H})$. The group $\hat{\mathcal{U}}(\mathfrak{H})$ has as subgroups $\hat{\mathcal{U}}(\mathfrak{L})$, $\mathcal{U}(\mathfrak{H})$ and $\mathcal{U}(\mathfrak{L})$. These subgroups are generated by the sets of transformations $\{F_T, F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}}\}$, $\{F_\omega, F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}}\}$, $\{F_\omega, F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}}\}$ and $\{F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}}\}$, respectively.

For $r > 0$, $\mathbf{T}^3 \times \{r\} / \langle F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}} \rangle$ is homeomorphic to the Euclidean 3-orbifold $\mathcal{S}^3_{r,\mathfrak{L}}$. As a topological space it is homeomorphic to the 3-sphere $\mathbb{S}^3$. On the other hand $\mathbf{T}^3 \times \{0\} / \langle F_T, F_{\mathbf{i}}, F_{\mathbf{j}}, F_{\mathbf{k}} \rangle$ is homeomorphic to the closed 3-ball $\mathbf{B}^3$.

Let $[(z_1, z_2, z_3)]$ denote the equivalence class of orbits under the transformations F_T, F_ω, $F_{\mathbf{i}}$, $F_{\mathbf{j}}$, $F_{\mathbf{k}}$.

There exists a strong deformation retract of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ to the Euclidean 3-orbifolds $\mathcal{S}^3_{2,\mathfrak{L}}$ and $\mathcal{S}^3_{2,\mathfrak{H}}$, repectively. In fact, as a topological space $\mathcal{O}^4_{\mathfrak{L}} := \mathbf{H}^1_{\mathbb{H}} / PSL(2,\mathfrak{L})$ is homeomorphic to $\mathbf{T}^3 \times [0, \infty) / \sim$, where $\mathbf{T}^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3| = 1\}$ and $\sim$ is the equivalence relation given by the orbits of the action of some groups of diffeomorphisms of $\mathbf{T}^3$ generated by the set and subsets of elements F_T, $F_{\mathbf{i}}$, $F_{\mathbf{j}}$, $F_{\mathbf{k}}$. Moreover $\mathbf{T}^3 \times \{r\} / \Gamma$ is homeomorphic to $\mathbb{S}^3$ for $r > 0$ and $\mathbf{T}^3 \times \{0\} / \Gamma$ is homeomorphic to $\mathbf{B}^3$.

The underlying spaces of the 3-dimensional Euclidean orbifolds $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$ are homeomorphic to the 3-sphere $\mathbb{S}^3$ because they are obtained by pasting two 3-dimensional balls along their boundaries which are 2-dimensional spheres.

The singular loci of the 3-dimensional Euclidean orbifolds $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$ are the 1-skeletons of their fundamental domains divided by the actions of the corresponding groups. Thus, their singular loci are the two graphs which are the 1-skeleton of a cube and the graph in the Fig. 7, respectively. All edges of the singular locus of $\mathcal{S}^3_{r,\mathfrak{L}}$ are labeled by 2. The labels of the edges of the singular locus of $\mathcal{S}^3_{r,\mathfrak{H}}$ are showed in the Fig. 7.

Fig. 7 The singular locus of the Hurwitz cusp section $\mathcal{S}^3_{r,\mathfrak{H}}$
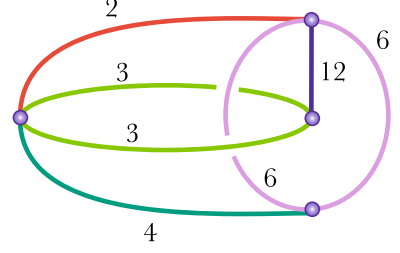
5.1.1 The singular locus of the Lipschitz cusp section

All the isotropy groups of the vertices in the fundamental domain of $\mathcal{S}^3_{r,\mathfrak{L}}$ are isomorphic to $\mathcal{U}(\mathfrak{L})$. All the isotropy groups of points in the edges of the fundamental domain of $\mathcal{S}^3_{r,\mathfrak{L}}$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The isotropy groups of the 6 open 2-dimensional faces and two open 3-dimensional faces are trivial. The orbifold $\mathcal{S}^3_{r,\mathfrak{L}}$ has 8 vertices, 12 edges, 6 square faces and two cubic 3-dimensional faces.

The orbifold Euler characteristic of $\mathcal{S}^3_{r,\mathfrak{L}}$ is $8(\frac{1}{4}) - 12(\frac{1}{2}) + 6 - 2 = 0$. For the definition of orbifold Euler characteristic we refer to [11].

5.1.2 The singular locus of the Hurwitz cusp section

For $\mathcal{S}^3_{r,\mathfrak{H}}$ there are vertices in the fundamental domain $\mathcal{E}^3_{r,\mathfrak{H}}$ of $\mathcal{S}^3_{r,\mathfrak{H}}$ with different isotropy groups : $\mathcal{U}(\mathfrak{L})$ of order four and $\mathcal{U}(\mathfrak{H})$ of order 12. Also the edges have three types of isotropy groups: the trivial group, the group $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. The center r and the vertices $r + \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}$ and $r + \frac{-\mathbf{i}-\mathbf{j}-\mathbf{k}}{2}$ of the cubes $\mathcal{E}^3_{r,\mathfrak{H}}$ have isotropy groups isomorphic to $\mathcal{U}(\mathfrak{H})$. The points $r + \frac{\mathbf{k}}{2}$, $r - \frac{\mathbf{k}}{2}$ and $r + \frac{\mathbf{i}+\mathbf{k}}{2}$, $r + \frac{-\mathbf{i}-\mathbf{k}}{2}$ and $r + \frac{\mathbf{j}+\mathbf{k}}{2}$ and $r + \frac{-\mathbf{j}-\mathbf{k}}{2}$ have isotropy groups isomorphic to $\mathcal{U}(\mathfrak{L})$.

The points in the edges of $\mathcal{E}^3_{r,\mathfrak{H}}$ with have r as a vertex and the points $r + \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}$ and $r + \frac{-\mathbf{i}-\mathbf{j}-\mathbf{k}}{2}$ as second vertex have isotropy groups isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The points in the edges of $\mathcal{E}^3_{r,\mathfrak{H}}$ with have r as a vertex and the points $r + \frac{\mathbf{j}-\mathbf{k}}{2}, r + \frac{\mathbf{i}-\mathbf{k}}{2}, r + \frac{-\mathbf{j}+\mathbf{k}}{2}, r + \frac{-\mathbf{i}+\mathbf{k}}{2}$ as second vertex have trivial isotropy groups. All the points in the other edges of $\mathcal{E}^3_{r,\mathfrak{H}}$ have isotropy groups isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The isotropy groups of the 5 open 2-dimensional faces and of the two open 3-dimensional faces are trivial. The orbifold $\mathcal{S}^3_{r,\mathfrak{L}}$ has 4 vertices, 7 edges, 4 triangular faces and one square face and two 3-dimensional faces.

The orbifold Euler characteristic of $\mathcal{S}^3_{r,\mathfrak{H}}$ is $2(\frac{1}{12}) + 2(\frac{1}{4}) - 1 - 2(\frac{1}{3}) - 4(\frac{1}{2}) + 5 - 2 = 0$.

Remark 5.2 As it is expected, the orbifold Euler characteristics of both $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$ vanishes since both orbifolds are compact and Euclidean.

5.1.3 The structure of the ends

The family of Euclidean orbifolds $\mathcal{S}^3_{r,\mathfrak{H}}$ consists of orbifolds which are homothetic for all $r > 1$. The Euclidean volume $V_{\mathfrak{H}}(r)$ decreases exponentially to 0 as $r \to \infty$. The same is true for the family $\mathcal{S}^3_{r,\mathfrak{L}}, r > 1$ and the corresponding volume $V_{\mathfrak{L}}(r)$, since $V_{\mathfrak{L}}(r) = 3V_{\mathfrak{H}}(r)$.

The thin parts are open cylinders on the sections $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$, respectively. More precisely, $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$ separate $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$, respectively, into two connected components with

boundaries $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$, respectively. One of the components is compact and the other is non-compact but with finite hyperbolic volume. Using Margulis notation the compact part is the *thick* region and the non-compact part is the *thin* region of the corresponding orbifolds. The thin regions are the non-compact orbifolds diffeomorphic to half-open cylinders $\mathcal{Z}^4_{r,\mathfrak{L}} := \mathcal{S}^3_{r,\mathfrak{L}} \times [0, 1)$ and $\mathcal{Z}^4_{r,\mathfrak{H}} := \mathcal{S}^3_{r,\mathfrak{H}} \times [0, 1)$, respectively. There is an orbifold cover $\pi_{\mathcal{Z}} : \mathcal{Z}^4_{r,\mathfrak{L}} \to \mathcal{Z}^4_{r,\mathfrak{H}}$ which is three to one.

5.2 The thick regions and underlying spaces of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$

Each of the underlying spaces of the 4-dimensional orbifolds $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ has only one end. The sections of the ends are 3-dimensional Euclidean orbifolds $\mathcal{S}^3_{r,\mathfrak{L}}$ and $\mathcal{S}^3_{r,\mathfrak{H}}$ which each of their underlying spaces is homeomorphic to the 3-sphere $\mathbb{S}^3$. Then the thin regions of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ are homeomorphic to the 4-ball $\mathbf{D}^4$ minus one point for example in its center. Moreover each of the thick regions of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ is homeomorphic to the 4-ball $\mathbf{D}^4$.

Then each of the underlying spaces of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ is homeomorphic to the 4-sphere $\mathbb{S}^4$ minus one point thus each of the underlying spaces is homeomorphic to $\mathbb{R}^4$.

5.3 The singular locus of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$

The 3-dimensional faces of $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$ are identified in pairs by the action of the generators of the Lipschitz and Hurwitz modular groups, respectively. We denote by $\Sigma_{\mathfrak{L}}$ and $\Sigma_{\mathfrak{H}}$ the singular loci of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$, respectively. They are the 2-dimensional skeletons of their fundamental domains $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$. Then each singular locus is non-compact with one connected component.

The Lipschitz singular locus $\Sigma^2_{\mathfrak{L}}$ is the union of a 2-dimensional cube $\mathcal{C}_{\Sigma}$ which is obtained by identifying the boundaries of the cubes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{C} \subset \Pi$ by the action of the group $\hat{\mathcal{U}}(\mathfrak{L})$ and the non-compact cone over its 1-skeleton of the 2-dimensional sides of $\mathcal{P}_{\mathfrak{L}}$ which are asymptotic to the point at infinity.

The Hurwitz singular locus $\Sigma^2_{\mathfrak{H}}$ is the union of the 2-dimensional pyramid $\mathcal{P}_{\Sigma}$ which is obtained by identifying the boundaries of the union of the two pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C} \subset \Pi$ by the action of the group $\hat{\mathcal{U}}(\mathfrak{H})$ and the non-compact cone over its 1-skeleton of the 2-dimensional sides of $\mathcal{P}_{\mathfrak{H}}$ that are asymptotic to the point at infinity.

5.4 Local models of the modular orbifolds singularities

In this section we study the local models of the isolated singularities of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$.

The local models of the isolated singularities of $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$ are obtained as quotients of a hyperbolic 4-ball $\mathbf{B}^4$ by the action of a discrete subgroup of hyperbolic isometries which fix its center and its boundary which is a 3-sphere. Let $\mathbb{F}_0 \subset \mathbb{F}_{\mathbf{B}}$ be the subgroup of hyperbolic isometries of the hyperbolic ball which fix the center of $\mathbf{B}$. Then $\mathbb{F}_0$ is the group of orientation-preserving isometries of the 3-sphere $\mathbb{S}^3$. The group $\mathbb{F}_0$ is isomorphic to SO(4).

Let $\mathbf{B}^4 \subset \mathbb{H}$ denote, as before, the disk model of $\mathbf{H}^1_{\mathbb{H}}$.

The ideal boundary of $\mathbf{B}^4$ is the unitary 3-sphere $\mathbb{S}^3 = \{\mathbf{q} = \alpha + \beta\mathbf{j} \in \mathbf{H}^1_{\mathbb{H}} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1\}$.

Let $SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$ be the *unitary special group*. The

3-sphere is a Lie group isomorphic to SU(2). An element $\mathbf{q} = \alpha + \beta \mathbf{j} \in \mathbb{S}^3$ corresponds to

the element $\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$ in SU(2).

We define

$$f_{(u,v)} : \mathbf{B}^4 \to \mathbf{B}^4;$$
$$f_{(u,v)}(\mathbf{q}) \mapsto u \mathbf{q} v$$

where $|u| = |v| = 1$. We observe that $f_{(u,v)} \in$ SO(4) and $(u, v) \in \mathbb{S}^3 \times \mathbb{S}^3 = SU(2) \times SU(2)$. We introduce

$$\phi : SU(2) \times SU(2) \to SO(4);$$
$$(u, v) \mapsto f_{(u,v)}.$$

Since $SU(2) \times SU(2)$ is simply connected but the fundamental group of SO(4) is $\mathbb{Z}/2\mathbb{Z}$ then the kernel of ϕ is the group with two elements consisting of $(1, 1)$ and $(\overline{1}, \overline{1})$.

$(q_1, q_2) \in \mathbb{S}^3 \times \mathbb{S}^3$. This defines a homomorphism of Lie groups $\mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ with kernel $\mathbb{Z}/2\mathbb{Z}$ generated by $(-1, -1)$. Then SO(4) is isomorphic to the central product $\mathbb{S}^3 \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^3$. The finite subgroups of $SO(4)$ are, up to conjugation, exactly the finite subgroups of the central products of two binary polyhedral groups G_1 and G_2

$$G_1 \times_{\mathbb{Z}/2\mathbb{Z}} G_2 \subset \mathbb{S}^3 \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^3.$$

The finite subgroups of $SU(2)$ have been classified by Felix Klein in [18] and they are the cyclic groups of order n ($n > 1$)), the binary dihedral groups $\langle 2, 2, n \rangle$ of order $4n$, the binary tetrahedral group $\langle 2, 3, 3 \rangle$ of order 24, the binary octahedral group $\langle 2, 3, 4 \rangle$ of order 48 and the binary icosahedral group $\langle 2, 3, 5 \rangle$ of order 120. These are the *binary polyhedral groups*. Let Γ be a finite subgroup of $\mathbb{F}_0$. Let $r > 0$ and $\mathbf{B}_r^4$ be the hyperbolic ball centered at the origin of radius r. The ball $\mathbf{B}_r^4$ is invariant under the action of Γ. Let $\mathcal{O}^4(\Gamma, r) = \mathbf{B}_r^4 / \Gamma$. For every $r > 0$ the orbifold $\mathcal{O}^4(\Gamma, r) = \mathbf{B}_r^4 / \Gamma$ is equivalent, up to rescaling the orbifold metric, to a fixed $\mathbf{B}_\epsilon^4 / \Gamma$ for ϵ sufficiently small. Let $\mathcal{O}^4(\Gamma) = \mathbf{B}_\epsilon^4 / \Gamma$.

Definition 5.3 Let p and q be two integers. Let $\Gamma(p, q) \subset SU(2)$ be the abelian subgroup generated by the map $T_{p,q}(\mathbf{z}_1, \mathbf{z}_2) = (e^{2\pi i/p} \mathbf{z}_1, e^{2\pi i/q} \mathbf{z}_2)$. The group $\Gamma(p, q)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$. Let us denote by $\mathcal{O}(p, q)$ the orbifold $\mathcal{O}^4(\Gamma(p, q), \epsilon) = \mathbf{B}_\epsilon^4 / \Gamma(p, q)$. If $\Gamma(G_1, G_2) \subset SO(4)$ is a finite subgroup isomorphic to $G_1 \times_{\mathbb{Z}/2\mathbb{Z}} G_2$, where G_1 and G_2 are the binary polyhedral groups then we denote by $\mathcal{O}(G_1, G_2)$ the orbifold $\mathbf{B}_\epsilon^4 / \Gamma(G_1, G_2)$. If $G_k = \mathbb{Z}/p\mathbb{Z}$, where $k = 1, 2$, then we write p in the place of G_k in the notation $\mathcal{O}(G_1, G_2)$. If $G_k = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$, where $k = 1, 2$, then we write (p, q) in the place of G_k in the notation $\mathcal{O}(G_1, G_2)$.

5.5 Singular locus of the Lipschitz and Hurwitz modular orbifolds

We give the local groups and the local models of the singular points in the locus of the Lipschitz and Hurwitz modular orbifolds $\mathcal{O}_{\mathfrak{L}}^4$ and $\mathcal{O}_{\mathfrak{H}}^4$, respectively. We also give a presentation, a fundamental domain, a Cayley graph and a thorough study of its spherical 3-orbifold link for each group related to singular points in the singular loci of $\mathcal{O}_{\mathfrak{L}}^4$ and $\mathcal{O}_{\mathfrak{H}}^4$, respectively.

We can describe orbifold stratifications of the set of singular points of $\mathcal{O}_{\mathfrak{L}}^4$ and $\mathcal{O}_{\mathfrak{H}}^4$ according to their isotropy groups. The lists of types of singular points and their corresponding

isotropy groups can be divided into two types of strata: the compact and the non compact. Also in these two groups we can divide the strata according to the dimension of the corresponding stratum in the stratification. We give a list of points in 11 strata in the Lipschitz singular locus $\Sigma_{\mathfrak{L}}^2$ of $\mathcal{O}_{\mathfrak{L}}^4$ and 15 strata in the Hurwitz singular locus $\Sigma_{\mathfrak{H}}^2$ of $\mathcal{O}_{\mathfrak{H}}^4$ which have isomorphic isotropy groups and denote these groups by Γ_k and Λ_m, where $k = 1, \ldots, 11; m = 1, \ldots, 15$.

The isotropy group of a point in a non-compact stratum in the singular loci $\Sigma_{\mathfrak{L}}^2$ or $\Sigma_{\mathfrak{H}}^2$ is the isotropy group of the action of $\mathcal{A}(\mathfrak{L})$ and $\mathcal{A}(\mathfrak{H})$, respectively. The singular points of the orbifold $\mathcal{O}_{\mathfrak{L}}^4$ which are in compact strata are in the boundary of the cubes $\mathcal{C}_1$ and $\mathcal{C}_2$. The singular points of the orbifold $\mathcal{O}_{\mathfrak{H}}^4$ which are in compact strata are in the boundary of the squared pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$. The strata can be characterized by the dimension of the corresponding stratum in the stratification and whether the stratum contains the point 1 or not.

The point $\mathbf{q} = \frac{1}{3}(\sqrt{3}+\mathbf{i}+\mathbf{j}+\mathbf{k}) \in \mathcal{P}_{\mathfrak{H}} \subset \mathcal{P}_{\mathfrak{L}}$ is a regular point for the orbits of $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$. We had considered previously $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{P}_{\mathfrak{H}}$ as unions of the non-compact cones over the cubes $\mathcal{C}_1$ and $\mathcal{C}_2$ and the pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ with apices the point at infinity, respectively. However, to obtain the isotropy groups it is better to define new fundamental regions $\tilde{\mathcal{P}}_{\mathfrak{L}}$ and $\tilde{\mathcal{P}}_{\mathfrak{H}}$ as follows:

(1) Let $\tilde{\mathcal{P}}_{\mathfrak{L}}$ be the non-compact bicone over $\mathcal{C}_1$ with apices the ideal vertices at 0 and the point at infinity.
(2) Let $\tilde{\mathcal{P}}_{\mathfrak{H}}$ be the non-compact bicone over $\mathcal{P}_1$ with apices the ideal vertices at 0 and the point at infinity.

These are convex bicones over the cube $\mathcal{C}_1$ and the pyramid $\mathcal{P}_1$, each with two ends and they are fundamental domains for $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$, respectively.

Remark 5.4 The action of the groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ on their new fundamental regions $\tilde{\mathcal{P}}_{\mathfrak{L}}$ and $\tilde{\mathcal{P}}_{\mathfrak{H}}$ is equivalent to the action of $G(3)$ on $\mathcal{P}$ described before in Sect. 4.1. The groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ act on $\tilde{\mathcal{P}}_{\mathfrak{L}}$ and $\tilde{\mathcal{P}}_{\mathfrak{H}}$ by rotations around the 2-faces of the cube $\mathcal{C}_1$ and the pyramid $\mathcal{P}_1$, respectively.

For the strata in $\Sigma_{\mathfrak{L}}^2$ or $\Sigma_{\mathfrak{H}}^2$ that contain 1 it is easy to calculate the isotropy group as a subgroup of $\hat{\mathcal{U}}(\mathfrak{L})$ or $\hat{\mathcal{U}}(\mathfrak{H})$. For the strata $\Sigma_{\mathfrak{L}}^2$ or $\Sigma_{\mathfrak{H}}^2$ which do not contain 1 we consider the orbit of 1 by the action of points in their isotropy groups. For each point $\mathbf{p}$ in one of these strata 1 is a regular point for the action of its isotropy group on a 3-sphere $\mathbb{S}_{r_0}^3(\mathbf{p})$, where r_0 is the distance from $\mathbf{p}$ to 1. Then the orbit of 1 is in correspondence 1–1 with fundamental regions of the correspondent isotropy group acting in a 3-sphere $\mathbb{S}^3$.

The fundamental regions on $\mathbb{S}^3$ of the isotropy groups of each stratum in the singular loci $\Sigma_{\mathfrak{L}}$ and $\Sigma_{\mathfrak{H}}$ are formed by two 4-simplices. Each 4-simplex has 5 3-dimensional faces. Therefore the isotropy groups have presentations with at most 4 generators.

5.6 The stratification of the Lipschitz singular locus

We give the list of 11 strata in the Lipschitz singular locus $\Sigma_{\mathfrak{L}}^2$. For each stratum we enlist its isotropy group $\Gamma_k, k = 1, \ldots 11$, determine a fundamental domain of the action of its isotropy group on $\mathbb{S}^3$, and give a geometrical description of its spherical 3-orbifold (or spherical link).

In the following list we consider the canonical projection $\mathfrak{p} : \mathcal{C} \to \mathcal{C}_\Sigma$. It is important to see Fig. 3 in each case.

Non compact strata

Γ_1 **Eight 1-cells** The 1-skeleton of the non-compact cone over the cubes $\mathcal{C}_1$ and $\mathcal{C}_2 \subset \mathcal{P}_{\mathfrak{L}}$ is a set of eight open lines in $\Sigma_{\mathfrak{L}}$ which are represented in $\mathcal{P}_{\mathfrak{L}}$ as

(a) the half-line $\{\mathbf{q} \; : \; \mathbf{q} = r, r \in \mathbb{R}, r > 1\}$,

(b) the three lines $\mathbf{q} = r_1 \pm \mathbf{i}/2, r_1 \pm \mathbf{j}/2, r_1 \pm \mathbf{k}/2$, where $r_1 > \frac{\sqrt{3}}{2}$,

(c) the three lines $\mathbf{q} = r_2 \pm (\mathbf{i}/2 + \mathbf{j}/2), r_2 \pm (\mathbf{i}/2 + \mathbf{k}/2), r_2 \pm (\mathbf{j}/2 + \mathbf{k}/2)$, where $r_2 > \frac{\sqrt{2}}{2}$, and finally

(d) the line $\mathbf{q} = r_3 \pm (\mathbf{i}/2 + \mathbf{j}/2 + \mathbf{k}/2)$, where $r_3 > \frac{1}{2}$.

These eight half-lines orthogonally project, under the natural projection $\mathcal{P}_{\mathfrak{L}} \to \mathcal{C}$ by geodesics asymptotic to the point at infinity, to the barycenter of the cube $\mathcal{C} \subset \mathcal{P}_{\mathfrak{L}}$, the barycenters of the square faces of $\mathcal{C}$, the half of its edges, and two of its vertices, respectively. These 8 open half-lines in $\mathcal{P}_{\mathfrak{L}}$ project to 8 open half-lines in $\mathcal{O}_{\mathfrak{L}}^4$. Their local isotropy groups are isomorphic to the group of order 4 isomorphic to $\mathcal{U}(\mathfrak{L})$. We define $\Gamma_1 = \mathcal{U}(\mathfrak{L}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as the local isotropy group of the quaternions in these 8 open half-lines. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 2)$.

Γ_2 *Twelve 2-cells* The 2-skeleton of the non-compact cone over the cubes $\mathcal{C}_1$ and $\mathcal{C}_2$ is a set of twelve triangles with one vertex at the point at infinity which are the quaternions in $\mathcal{P}_{\mathfrak{L}}$ that orthogonally projects over the quaternions which are the edges of $\mathcal{C}_1$ and $\mathcal{C}_2$. Their isotropy groups are isomorphic to the cyclic group of order 2 isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and we define $\Gamma_2 = \mathbb{Z}/2\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2)$.

Compact strata

- **0-dimensional**

  Γ_3 **One 0-cell** The common vertex $v_1 = 1$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ which is the barycenter of the cube $\mathcal{C}$. Its isotropy group is the abelian group $\Gamma_3 = \mathbb{Z}/2\mathbb{Z} \times \mathcal{U}(\mathfrak{L}) \cong \hat{\mathcal{U}}(\mathfrak{L}) \cong (\mathbb{Z}/2\mathbb{Z})^3$ of order 8 generated by the involution T and the elements in the Lipschitz unitary group. If we take a round hyperbolic ball $\mathbf{B}_r(1)$ with center at $\mathbf{q} = 1$ and small radius r we obtain that its boundary $\mathbb{S}_r^3(1)$ intersects the tessellation $\mathbf{Y}_{\mathfrak{L}}$ of $PSL(2, \mathfrak{L})$ in a spherical tessellation by sixteen Coxeter spherical right-angled regular tetrahedra. These tetrahedra are the faces of a 4-dimensional regular convex polytope which is known as the 16-cell, it is the dual polytope of the hypercube known as the 8-cell. These polytopes are two of the six Platonic polytopes of dimension 4. The Cayley graph of Γ_3 is the 1-skeleton of a truncated octahedron in $\mathbb{S}_r^3(1)$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, (2, 2))$.

  Γ_4 *Three 0-cells* The vertices v_2, v_3, v_4 of $\mathcal{C}_\Sigma$ which are the images under $\mathfrak{p}$ of the 6 barycenters of the square faces of the cube $\mathcal{C}$. The isotropy group of these vertices is isomorphic to the group $\Gamma_4 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of order 12. It is the group generated by $\tau_{\mathbf{u}}T$ and $TD_{\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ and $\mathbf{u} \neq \mathbf{v}$. The group Γ_4 leaves invariant two orthogonal hyperbolic planes meeting at the barycenter of the square face of the cube; one is the plane containing the face and Γ_4 acts on it as the group of order 4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (the group generated by $(\tau_{\mathbf{u}}T)^3$ and $TD_{\mathbf{v}}$ restricted to this plane), and the other is its orthogonal complement, and Γ_4 acts on it as a rotation of order 3 (the group generated by $(\tau_{\mathbf{u}}T)^2$ restricted to this plane). The 12 middle points of the edges are identified in groups of four points by translations with three singular vertices of $\mathcal{C}_\Sigma \subset \mathcal{O}_{\mathfrak{L}}^4$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 6)$.

  Γ_5 *Three 0-cells* The three vertices v_5, v_6, v_7 of $\mathcal{C}_\Sigma$ which are the images under $\mathfrak{p}$ of the 12 middle points of the edges of $\mathcal{C}$. The isotropy group of these vertices is isomorphic to the group $\Gamma_5 = \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ of order 24. It is the

group generated by $T D_{\mathbf{v}}$, $\tau_{\mathbf{u}+\mathbf{v}} T$, $(\tau_{\mathbf{v}} T)^2$ and $(\tau_{\mathbf{u}} T)^2$ where $\mathbf{u}, \mathbf{v} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, and $\mathbf{u} \neq \mathbf{v}$. The group Γ_5 leaves invariant two orthogonal hyperbolic planes meeting at the middle point of the edge of the cube; one is the hyperbolic plane whose ideal boundary is the line generated by $\mathbf{u} + \mathbf{v}$ and Γ_5 acts on it as a rotation of order 4 (the group generated by $\tau_{\mathbf{u}+\mathbf{v}} T$ restricted to this plane), and the other is its orthogonal complement, and Γ_5 acts as on it as a rotation of order 6 (the group generated by $D_{\mathbf{uv}} T$ and $(\tau_{\mathbf{u}} T)^2$ restricted to this plane). The 12 middle points of the edges of the cube are identified in groups of four points by translations with three singular vertices of $\mathcal{C}_\Sigma \subset \mathcal{O}_{\mathfrak{L}}^4$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(4, 6)$.

Γ_6 *One 0-cell* The vertex v_8 of $\mathcal{C}_\Sigma$ which is the image under $\mathfrak{p}$ of the 8 vertices of $\mathcal{C}$. The isotropy group of this vertex is isomorphic to the group $\Gamma_6 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \langle 2, 3, 3 \rangle$ of order 96 where $\langle 2, 3, 3 \rangle$ is the binary tetrahedral group of order 24. It is the group generated by $\tau_{\mathbf{i}+\mathbf{j}+\mathbf{k}} T$, $(\tau_{\mathbf{i}+\mathbf{j}} T)^2$, $(\tau_{\mathbf{i}+\mathbf{k}} T)^2$, $(\tau_{\mathbf{j}+\mathbf{k}} T)^2$, $(\tau_{\mathbf{u}} T)^2$ where $\mathbf{u} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}((2, 2), \langle 2, 3, 3 \rangle)$.

- **1-dimensional**

Γ_7 **Three 1-cells** The points of the three edges of $\mathcal{C}_\Sigma$ that are incident with the barycenter of $\mathcal{C}$ have isotropy group $\Gamma_7 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If the point is contained in the edge of $\mathcal{C}_1$ which contains 1 and $\sqrt{3} + \mathbf{u}/2$, then Γ_7 is the group generated by $(\tau_{\mathbf{u}} T)^3$ where $\mathbf{u} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. The group Γ_7 leaves invariant the hyperbolic 2-plane generated by 1 and $\mathbf{u}$ and the hyperbolic hyperplane Π. Moreover Γ_7 acts on it as a rotation of order 2 in Π. The points of these 6 edges in $\mathcal{C}_1$ and $\mathcal{C}_2$ are identified in groups of two and form 3 singular edges in $\mathcal{O}_{\mathfrak{L}}^4$ incident with 1. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 2)$.

Γ_8 *Three 1-cells* The points of the six edges of $\mathcal{C}_\Sigma$ which have one vertex at the barycenter of the square faces of $\mathcal{C}$ and the other vertex is the middle point of an edge of the cube $\mathcal{C}$. Its isotropy group $\Gamma_8 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is the group generated by $(\tau_{\mathbf{u}} T)^2$ and $D_{\mathbf{v}} T$ where $\mathbf{u}, \mathbf{v} = \pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$, $\mathbf{u} \neq \mathbf{v}$. The group Γ_8 leaves invariant two orthogonal hyperbolic planes meeting at the vertex of the cube. Γ_8 acts on one plane as a rotation of order 6 and as a rotation of order 4 on the other plane. The points of the 6 edges of $\mathcal{C}$ are identified and form 3 singular edges in $\mathcal{C}_\Sigma \subset \mathcal{O}_{\mathfrak{L}}^4$ which are incident with $1 \in \mathcal{C}_\Sigma$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 3)$.

Γ_9 *Six 1-cells* The points of the three edges of $\mathcal{C}_\Sigma$ which are incident with a vertex of $\mathcal{C}$. Its isotropy group $\Gamma_9 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is the group generated by $(\tau_{\mathbf{u}} T)^3$ where $\mathbf{u} = \pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$. The group Γ_9 leaves invariant two orthogonal hyperbolic planes meeting at the vertex of the cube. Γ_9 acts on one plane as a rotation of order 6 and as a rotation of order 2 in the other. The points of the 12 edges of $\mathcal{C}$ are identified and form six singular edges in $\mathcal{O}_{\mathfrak{L}}^4$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 6)$.

- **2-dimensional**

Γ_{10} **Three 2-cells** The points of the interior of a square face of $\mathcal{C}_\Sigma$ that is incident with the barycenter of $\mathcal{C}$. Its isotropy group $\Gamma_{10} = \mathbb{Z}/2\mathbb{Z}$ is the group generated by $D_{\mathbf{u}} T$, where $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$. The group Γ_{10} acts on one plane as a rotation of order 2 around a hyperbolic 2-plane in Π. The points in the 6 squares are identified in groups

of two with 3 singular 2-cells in $\mathcal{O}_{\mathfrak{L}}^4$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2)$.

Γ_{11} *Three 2-cells* The points of the interior of a square face of $\mathcal{C}_\Sigma$ that is not incident with the barycenter of $\mathcal{C}$. Its isotropy group $\Gamma_{11} = \mathbb{Z}/3\mathbb{Z}$ is the group generated by $(\tau_u T)^2$, where $u = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. The group Γ_{11} leaves invariant two orthogonal hyperbolic planes meeting at the point in the square face of the cube; one is the plane containing the face and Γ_{11} acts on it as a rotation of order 3, and the other is its orthogonal complement and Γ_{11} acts on it as the identity. The points in the 6 squares are identified in groups of two with 3 singular 2-cells in $\mathcal{O}_{\mathfrak{L}}^4$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(3)$.

5.7 The stratification of the Hurwitz singular locus

Now we describe the orbifold stratification of the set of singular points of the Hurwitz modular orbifold $\mathcal{O}_{\mathfrak{H}}^4$ according to their isotropy groups.

We give the list of 15 strata in the Hurwitz singular locus $\Sigma_{\mathfrak{H}}^2$. For each stratum we enlist the isotropy group Λ_k, $k = 1, \ldots 15$, determine a fundamental domain of its action on $\mathbb{S}^3$, and study in detail the corresponding spherical 3-orbifold (or spherical link).

In the following list we consider the canonical projection $\mathfrak{p} : \mathcal{P}_d \to \mathcal{P}_\Sigma$ where $d = 1, 2$.

In the same way as in the case of the Lipschitz singular locus the list of types of singular points and their corresponding isotropy groups can be divided by the dimension and the compactness of the corresponding stratum in the stratification. The non-compact strata have the same isotropy groups that the respective Euclidean 3-orbifold which is the intersection of $\mathcal{O}_{\mathfrak{H}}^4$ with a horosphere. The list is the following

Non compact strata

Λ_1, Λ_2 **Five 1-cells** The 1-skeleton of the non-compact cone over the pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ is a set of five open half-lines which are represented in $\mathcal{P}_{\mathfrak{H}}$ as

(a) the half-line $\{\mathbf{q} \in \mathbf{H}_{\mathbb{H}}^1 : \mathbf{q} = r, r \in \mathbb{R}, r > 1\}$,

(b) the line $\mathbf{q} = r_1 \pm \mathbf{k}/2$, where $r_1 > \frac{\sqrt{3}}{2}$,

(c) the two lines $\mathbf{q} = r_2 \pm (\mathbf{i}/2 + \mathbf{j}/2), r_2 \pm (\mathbf{j}/2 + \mathbf{k}/2)$, where $r_2 > \frac{\sqrt{2}}{2}$ and finally

(d) the line $\mathbf{q} = r_3 \pm (\mathbf{i}/2 + \mathbf{j}/2 + \mathbf{k}/2)$, where $r_3 > \frac{1}{2}$.

These five half-lines orthogonally project, under the natural projection $\mathcal{P}_{\mathfrak{H}} \to \mathcal{C}$ by geodesics asymptotic to the point at infinity, to the vertices of $\mathcal{P}_1$: the barycenter of the cube $\mathcal{C}$, the barycenters of any square face of $\mathcal{C}$, the half of two of its edges, and two of its vertices, respectively. These 5 open half-lines in $\mathcal{P}_{\mathfrak{H}}$ project to 4 open lines in $\mathcal{O}_{\mathfrak{H}}^4$. Their isotropy groups are isomorphic for (a) and (d) to the abelian group of order 8 isomorphic to $\mathcal{U}(\mathfrak{H})$ and for (b) and (c) to the dihedral group of order 4 isomorphic to $\mathcal{U}(\mathfrak{L})$. We obtain $\Lambda_1 := \mathcal{U}(\mathfrak{L}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\Lambda_2 := \mathcal{U}(\mathfrak{H}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as the isotropy groups of the quaternions in these 5 open half-lines. The local models for the singular points in these strata are isometric to the orbifolds $\mathcal{O}(2, 2)$ and $\mathcal{O}(2, (2, 2))$, respectively.

Λ_3, Λ_4 *Eight 2-cells* The 2-skeleton of the non-compact cone over the pyramid $\mathcal{P}_1 \subset \mathcal{P}_{\mathfrak{H}}$ is a set of eight triangles with one vertex at infinity which are orthogonally projected over the quaternions which are the edges of $\mathcal{P}_1$ and $\mathcal{P}_2$. The isotropy groups of points in the five triangles with base the edges of the square base of $\mathcal{P}_1$ and the edge that joins 1 with the barycenter of a squared face of $\mathcal{C}$ are isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$. For

points in the diagonal edge of $\mathcal{P}_1$ which joins 1 with a vertex of $\mathcal{C}$ their isotropy groups are isomorphic to $\mathbb{Z}/3\mathbb{Z}$. For points in the two edges which joins 1 with middle points of the edges of $\mathcal{C}$ their isotropy groups are isomorphic to the trivial group, then these points are not singular. We define for these six strata $\Lambda_3 = \mathbb{Z}/2\mathbb{Z}$ and $\Lambda_4 = \mathbb{Z}/3\mathbb{Z}$. The local models for the singular points in these strata are isometric to the orbifolds $\mathcal{O}(2)$ and $\mathcal{O}(3)$, respectively.

Compact strata

- **0-dimensional**

    Λ_5 **One 0-cell** The common vertex $v_1 = 1$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ which is the barycenter of the cube $\mathcal{C}$. Its isotropy group is the abelian group $\Lambda_5 = \hat{\mathcal{U}}(\mathfrak{H})$ of order 24 generated by the involution T and the elements in the Hurwitz unitary group. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, \langle 2, 3, 3 \rangle)$.

    Λ_6 **One 0-cell** The vertex v_2 of $\mathcal{P}_\Sigma$ which is the image under $\mathfrak{p}$ of 2 opposite barycenters of the square faces of the cube $\mathcal{C}$. The isotropy group of this vertex is isomorphic to the group $\Lambda_6 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of order 12. It is the group generated by $\tau_{\mathbf{u}}T$ and $TD_{\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$ and $\mathbf{u} \neq \mathbf{v}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 6)$.

    Λ_7 *One 0-cell* The vertex v_3 of $\mathcal{P}_\Sigma$ which is the images under $\mathfrak{p}$ of the 12 middle points of the edges of $\mathcal{C}$. The isotropy group of this vertex is isomorphic to the group $\Lambda_7 = \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ of order 24. It is the group generated by $TD_{\mathbf{v}}$, $\tau_{\mathbf{u}+\mathbf{v}}T$, $(\tau_{\mathbf{v}}T)^2$ and $(\tau_{\mathbf{u}}T)^2$ where $\mathbf{u}, \mathbf{v} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, and $\mathbf{u} \neq \mathbf{v}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(4, 6)$.

    Λ_8 *One 0-cell* The vertex v_4 of $\mathcal{P}_\Sigma$ which is the image under $\mathfrak{p}$ of the 8 vertices of $\mathcal{C}$. The isotropy group of this vertex is isomorphic to the group $\Lambda_8 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \times \langle 2, 3, 3 \rangle$ of order 288 where $\langle 2, 3, 3 \rangle$ is the binary tetrahedral group of order 24. It is the group generated by D_ω, $\tau_{\mathbf{i}+\mathbf{j}+\mathbf{k}}T$, $(\tau_{\mathbf{i}+\mathbf{j}}T)^2$, $(\tau_{\mathbf{i}+\mathbf{k}}T)^2$, $(\tau_{\mathbf{j}+\mathbf{k}}T)^2$, $(\tau_{\mathbf{u}}T)^2$ where $\mathbf{u} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}((2, 6), \langle 2, 3, 3 \rangle)$.

- **1-dimensional**

    Λ_9 **One 1-cell** The points of the edge of $\mathcal{P}_\Sigma$ which is incident with the barycenter of $\mathcal{C}$ and the barycenter of a square face of $\mathcal{C}$. Their isotropy groups are isomotphic to $\Lambda_9 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which is the group generated by D_iT and D_jT. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 2)$.

    Λ_{10} *Two 1-cells* The points of the two edges of $\mathcal{P}_\Sigma$ which are incident with the barycenter of $\mathcal{C}$ and the middle points of edges of $\mathcal{C}$. Their isotropy groups are isomorphic to $\Lambda_{10} = \mathbb{Z}/2\mathbb{Z}$ which is the group generated by D_iT. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2)$.

    Λ_{11} *One 1-cell* The points of the edge of $\mathcal{P}_\Sigma$ which is incident with the barycenter of $\mathcal{C}$ and the vertex of $\mathcal{C}$. Their isotropy groups are isomorphic to $\Lambda_{11} = \mathbb{Z}/3\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(3)$.

    Λ_{12} *Two 1-cells* The points of the two edges of $\mathcal{P}_\Sigma$ which are incident with the barycenter of a square face of $\mathcal{C}$ and a middle point of its edges. Their isotropy groups are isomorphic to $\Lambda_{12} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 3)$.

    Λ_{13} *One 1-cell* The points of the edge of $\mathcal{P}_\Sigma$ which are incident with a vertex of $\mathcal{C}$ and a middle point of its edges. Their isotropy groups are isomorphic to $\Lambda_{13} =$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2, 6)$.

- **2-dimensional**

Λ_{14} **One 2-cell** The isotropy groups of the points of the interior of the square face of $\mathcal{P}_\Sigma$ are isomorphic to $\Lambda_{14} = \mathbb{Z}/3\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(3)$.

Λ_{15} *Two 2-cell* The isotropy groups of the points in the interiors of the two triangle faces of $\mathcal{P}_\Sigma$ which contain 1 and the barycenter of a square face of $\mathcal{C}$ are isomorphic to $\Lambda_{15} = \mathbb{Z}/2\mathbb{Z}$. The local model for the singular points in this stratum is isometric to the orbifold $\mathcal{O}(2)$.

5.8 The Euler orbifold-characteristic of the Lipschitz and Hurwitz modular orbifolds

We use our previous computations on the order of the local groups of the strata in the singular loci of the Lipschitz and Hurwitz modular orbifolds to obtain the following:

Theorem 5.5 *The Euler orbifold–characteristic of the Lipschitz and Hurwitz modular orbifolds are*

$$\chi^{orb}(\mathcal{O}_{\mathfrak{L}}^4) = \frac{1}{96} \quad \text{and} \quad \chi^{orb}(\mathcal{O}_{\mathfrak{L}}^4) = \frac{1}{288},$$

respectively.

Proof The Euler orbifold–characteristic of the Lipschitz and Hurwitz modular orbifolds can be computed by the alternate sums of the number of strata for each dimension in $\Sigma_{\mathfrak{L}}^2$ and in $\Sigma_{\mathfrak{H}}^2$ divided for the order of the isotropy group of a point in the stratum.

The Lipschitz modular orbifold has a stratification as CW complex with one vertex with isotropy group of order 8, another vertex with isotropy group of order 96, three vertices of order 12, and three vertices of order 24. It has three edges with isotropy group of order 4, six edges with isotropy group of order 6, three edges with isotropy group of order 12, and eight edges with isotropy group of order 4. It has three 2-cells with isotropy group of order 2, three 2-cells with isotropy group of order 3 and twelve 2-cells with isotropy group of order 2. Finally it has six 3-cells and one 4-cell with isotropy groups of orders 1.

The Hurwitz modular orbifold has a stratification as CW complex with one vertex with isotropy group of order 12, one vertex with isotropy group of order 288, and two vertices of order 24. It has one edge with isotropy group of order 2, one edge with isotropy group of order 3, three edges with isotropy group of order 4, two edges with isotropy group of order 6, and three edges with isotropy group of order 12. It has two 2-cells with trivial isotropy group, six 2-cells with isotropy group of order 2 and three 2-cells with isotropy group of order 3. Finally it has five 3-cells and one 4-cell with isotropy groups of orders 1. □

Remark 5.6 Since there is an orbifold cover $\mathfrak{p}_{\mathfrak{L},\mathfrak{H}} : \mathcal{O}_{\mathfrak{L}}^4 \to \mathcal{O}_{\mathfrak{H}}^4$ of order 3 we obtain, as expected: $\chi^{orb}(\mathcal{O}_{\mathfrak{L}}^4) = 3\chi^{orb}(\mathcal{O}_{\mathfrak{H}}^4)$.

The volumes and the orbifold Euler characteristic are related by the Gauss–Bonnet–Euler formula for orbifolds (see [23]).

6 Lorentz transformations

We are mostly interested in the half-space model and in the hyperboloid model of Lorentz and Minkowski hyperbolic models. Thus we will study the Cayley transformations that give us isometries of the hyperbolic models. In particular we study the representation of $PSL(2, \mathbb{H})$ as Lorentz transformations. A *Lorentz–Minkowski matrix M* is any 5×5 matrix such that $M^t J M = J$ where M^t is the transpose matrix of M and J is the matrix

$$
J = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

We observe that the determinant of any Lorentz–Minkowski matrix M is ± 1.

We now describe two 4-hyperbolic models as subsets in $\mathbb{R}^5$: the hyperboloid model

$$
\mathbf{Lor} := \left\{ (x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \ : \ x_0 > 0, \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = -1 + x_0^2 \right\} \quad (3)
$$

and the half-space model

$$
\mathbf{H}^+ := \left\{ (1, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 \ : \ x_4 > 0 \right\}.
$$

Each of these models has its corresponding complete metric of constant curvature -1 and one can pass from one to the other by explicit projections called *Cayley transformations* (see [6]).

Indeed, consider the function $\Phi_{\mathbf{Lor}, \mathbf{H}^+} : \mathbf{Lor} \to \mathbf{H}^+$

$$
\Phi_{\mathbf{Lor}, \mathbf{H}^+} (x_0, x_1, x_2, x_3, x_4) = \left(1, \frac{x_1}{x_0 + x_4}, \frac{x_2}{x_0 + x_4}, \frac{x_3}{x_0 + x_4}, \frac{1}{x_0 + x_4} \right)
$$

In fact, $x_0 + x_4$ is positive since, $x_4^2 - x_0^2 = (x_4 - x_0)(x_4 + x_0) = -(1 + x_1^2 + x_2^2 + x_3^2) < 0$ and hence either $x_0 + x_4 > 0$ or $x_4 - x_0 > 0$, but in this second case this is equivalent to $x_4 > x_0$ and since x_0 is positive, then $x_0 + x_4$ is positive.

In order to prove that the function $\Phi_{\mathbf{Lor}}$ is an one-to-one function, we show that it is invertible. Therefore, given $(1, \mathbf{y}) = (1, y_1, y_2, y_3, y_4) \in \mathbf{H}^+$, if $|\mathbf{y}|^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$, then it is readily seen that the inverse of $\Phi_{\mathbf{Lor}, \mathbf{H}^+}$ is given by the formula:

$$
\Phi^{-1}_{\mathbf{Lor}, \mathbf{H}^+} ((1, y_1, y_2, y_3, y_4)) = \left(\frac{1 + |\mathbf{y}|^2}{2y_4}, \frac{y_1}{y_4}, \frac{y_2}{y_4}, \frac{y_3}{y_4}, \frac{1 - |\mathbf{y}|^2}{2y_4} \right) \in \mathbf{Lor}. \quad (4)
$$

For a matrix M the condition $M^t J M = J$ is equivalent to Ahlfors conditions, therefore

Proposition 6.1 *Any Lorentz–Minkovski matrix is in one to one correspondence with a matrix of $PSL(2, \mathbb{H})$ which satisfies Ahlfors conditions.*

For the matrix associated with the general translation $\tau_{x,y,z} = \begin{pmatrix} 1 & x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ 0 & 1 \end{pmatrix}$ we have the following Lorentz representation

$$\mathfrak{T}(x, y, z) := \begin{pmatrix} 1 + \frac{(x^2+y^2+z^2)}{2} & x & y & z & \frac{(x^2+y^2+z^2)}{2} \\ x & 1 & 0 & 0 & x \\ y & 0 & 1 & 0 & y \\ z & 0 & 0 & 1 & z \\ -\frac{(x^2+y^2+z^2)}{2} & -x & -y & -z & 1 - \frac{(x^2+y^2+z^2)}{2} \end{pmatrix}$$

We notice that $\mathfrak{T}(x, y, z)\mathfrak{T}(x', y', z') = \mathfrak{T}(x + x', y + y', z + z')$.

The Lorentz transformation corresponding to $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the matrix $-J$.

6.1 The Hurwitz modular group $PSL(2, \mathfrak{H})$ in the Lorentz model

The algebra of the quaternions $\mathbb{H}$ is isomorphic to the real algebra of 4×4 matrices generated by I_4, $S_\mathbf{i}$, $S_\mathbf{j}$, $S_\mathbf{k}$, where I_4 is the identity 4×4 matrix and

$$S_\mathbf{i} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_\mathbf{j} := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_\mathbf{k} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This follows from the fact that $S_\mathbf{i}^2 = S_\mathbf{j}^2 = S_\mathbf{k}^2 = -I_4$ and $S_\mathbf{i}S_\mathbf{j} = S_\mathbf{k}$, $S_\mathbf{j}S_\mathbf{k} = S_\mathbf{i}$ and $S_\mathbf{k}S_\mathbf{i} = S_\mathbf{j}$. In particular the group of Hurwitz units $\mathcal{U}(\mathfrak{H})$ consists of the 24 special orthogonal matrices: $\pm I_4, \pm S_\mathbf{i}, \pm S_\mathbf{j}, \pm S_\mathbf{k}, \frac{1}{2}(\pm I_4 \pm S_\mathbf{i} \pm S_\mathbf{j} \pm S_\mathbf{k})$ (all possible 16 combinations of signs are allowed). We remark that this group is isomorphic to the binary tetrahedral group.

Definition 6.2 Let $\mathcal{U}(\mathfrak{H}, \mathbf{Lor}) \subset SO_+(4, 1)$ be the finite group of order 24 given by the Lorentz matrices:

$$\pm\widehat{I_4}, \ \pm\widehat{S_\mathbf{i}}, \ \pm\widehat{S_\mathbf{j}}, \ \pm\widehat{S_\mathbf{k}}, \ \frac{1}{2}(\pm\widehat{I_4} \pm \widehat{S_\mathbf{i}} \pm \widehat{S_\mathbf{j}} \pm \widehat{S_\mathbf{k}}).$$

We remark that the inversion T corresponds to $-I_4 \in \mathcal{U}(\mathfrak{H}, \mathbf{Lor})$ i.e. the matrix $-J$.

Proposition 6.3 *In the Lorentz model the group $PSL(2, \mathfrak{H})$ corresponds to the subgroup of $SO_+(4,1)$ denoted by $\Gamma_\mathfrak{H}$ generated by $\mathcal{U}(\mathfrak{H}, \mathbf{Lor})$ and the translations $\mathfrak{T}(n, m, p)$ where $n, m, p \in \mathbb{Z}$.*

Since $PSL(2, \mathfrak{L}) \subset PSL(2, \mathfrak{H})$ we have a corresponding subgroup $\Gamma_\mathfrak{L} \subset \Gamma_\mathfrak{H}$ of the Lorentz group.

The fundamental domain of $\Gamma_\mathfrak{H}$ is contained in the fundamental domain of $\Gamma_\mathfrak{L}$ and therefore as we seen before the group $PSL(2, \mathfrak{L})$ leaves invariant the hyperbolic honeycomb whose cell is the 24-cell.

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